

COLLAPSIBILITY OF $\Delta(\Pi_n)/\mathcal{S}_n$ AND SOME RELATED CW COMPLEXES

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ABSTRACT. Let $\Delta(\Pi_n)$ denote the order complex of the partition lattice. The natural \mathcal{S}_n -action on the set $[n]$ induces an \mathcal{S}_n -action on $\Delta(\Pi_n)$. We show that the regular CW complex $\Delta(\Pi_n)/\mathcal{S}_n$ is collapsible. Even more, we show that $\Delta(\Pi_n)/\mathcal{S}_n$ is collapsible, where Π_Δ is a suitable type selection of the partition lattice. This allows us to generalize and reprove in a conceptual way several previous results regarding the multiplicity of the trivial character in the \mathcal{S}_n -representation on $H_*(\Delta(\Pi_n))$.

1. INTRODUCTION

Assume P is a poset equipped with an order-preserving action of a group G . This paper grew out of the author's attempts to understand the relation between the topology of $\Delta(P)/G$ and $\Delta(P/G)$, where Δ denotes the order complex construction and P/G is a category defined as a colimit of a suitable functor (see [4, Definition 2.2]). In [4] E. Babson and the author have given certain conditions on the action which guarantee that $\Delta(P)/G$ is homeomorphic to $\Delta(P/G)$.

In this paper we are concerned with the case when $P = \Pi_n$ or a suitable subposet of it, G is a symmetric group \mathcal{S}_n , and the action is induced from the \mathcal{S}_n action on the ground set $[n]$.

The partition lattice Π_n is an extensively studied object in algebraic combinatorics. Two important appearances of it are:

- as an intersection lattice of the braid arrangement;
- in studying the complexes of disconnected graphs in connection with Vassiliev knot invariants (see [16, 1]).

It has been proved many times by several means that the order complex of Π_n is Cohen-Macaulay (see for instance [2, 3]).

It is easy to see that Π_n/\mathcal{S}_n is a category with a terminal object, hence $\Delta(\Pi_n)/\mathcal{S}_n$ is contractible. However, since the conditions from [4] are not satisfied in this case we cannot immediately conclude that $\Delta(\Pi_n)/\mathcal{S}_n$ is contractible. The purpose of this paper is to show that this is in fact the case. $\Delta(\Pi_n)/\mathcal{S}_n$ inherits the cell structure from $\Delta(\Pi_n)$ and we show that, with regard to this cell structure, $\Delta(\Pi_n)/\mathcal{S}_n$ is collapsible.

More generally, for any Λ —a set of number partitions of n —we define Π_Λ to be the subposet of Π_n consisting of all partitions of the type from Λ . We show that

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under certain conditions $\Delta(\Pi_n)/\mathcal{S}_n$ is collapsible. These conditions can be best understood as follows. Choose k such that $2 \leq k \leq n$ (for example $k = 2!$). Then the conditions of Theorem 4.1 can be reformulated as: if $\lambda \in \Lambda$, then the maximal number partition of the type $(k^m, 1^{n-km})$, which refines λ , is also in Λ .

When X is a finite simplicial complex and G is a discrete group acting on X , it is known that, for any i , the Betti number $\beta_i(X/G, \mathbf{k})$ is equal to the multiplicity of the trivial character in the induced representation of G on the homology group $H_i(X, \mathbf{k})$, where X/G is the quotient space and \mathbf{k} is the field of characteristic 0 or coprime to $|G|$ (see for instance [6, Theorem 1] or [5, page 120, Theorem 2.4]). This allows us in Corollary 4.2 to translate (a weaker version of) our result into the language of the representation theory. Questions of such nature have been studied before, most notably by R. Stanley in [13], P. Hanlon in [10] and S. Sundaram in [14]. We give more conceptual proofs for [13, Proposition 7.8], [10, Theorem 3.1] and [14, Theorem 4.2] as the special cases of our result (the authors were working with the case when Λ is coming from a rank selection and $k = 2$). We hope that this sheds some light on the role which the number partitions of the type $(k^m, 1^{n-km})$ play in the fact that for certain rank selections the multiplicity of the trivial character in the induced representation on the homotopy groups is 0.

For the sake of brevity, we have chosen to formulate our argument in terms of Discrete Morse Theory from [8]. In Section 3 we give a simple combinatorial argument for the statement of Discrete Morse Theory that we need (see also [1, Proposition 3.7]).

2. FRAMEWORK

Let Δ denote the nerve functor $\Delta : \mathbf{Cat} \rightarrow \text{Simplicial Sets}$, mapping a category K to its nerve $\Delta(K)$ (see [11, 12]). In the combinatorial context Δ has been mostly studied when K is a poset, in which case $\Delta(K)$ has elements of K as vertices and a simplex for each chain (a set of comparable elements). When K is a poset, $\Delta(K)$ is a simplicial complex, which is often called the *order complex*.

When X is a cell complex, we denote by $P(X)$ its face poset, i.e., the poset whose elements are cells of X and the order relation is inclusion. A *triangulated space* (cf. [9]) is defined to be a cell complex X such that the lower intervals of $P(X)$ are Boolean algebras. We denote by $|X|$ the geometric realization of X . The triangulated spaces make appearance in the topological combinatorics in the following important examples:

- If K is a category, then $\Delta(K)$ is a triangulated space iff K has no inverses. In particular, if G acts on a poset P in an order-preserving way, then $\Delta(P/G)$ is a triangulated space, where P/G is defined as a colimit (cf. [4]).

- If a group G acts on a triangulated space X in such a way that: *if a cell σ is fixed by $g \in G$, then σ is fixed by g pointwise* (cf. [5, Condition (A), p. 115]), then $|X|/G$ is homeomorphic to $|X/G|$, where X/G is a certain triangulated space. The cells of X/G are orbits of the cells of X and $p : X \rightarrow X/G$ is a cell map. In particular, if G acts on a poset P in an order-preserving way, then $|\Delta(P)|/G$ is homeomorphic to $|\Delta(P)/G|$ (see [4]). The reader may wish to compare this to the simplicial case (see [5, page 117]).

The language of the elementary collapses will be indispensable for our argument.

Definition 2.1. Let X be a regular CW complex. Assume that F_1 and F_2 are cells of X such that F_2 is a maximal cell which contains F_1 , and there is no other maximal

cell containing F_1 . A **collapse** is the replacement of X with $X \setminus \{F|F_1 \subseteq F\}$. A collapse is called **elementary** if $\dim F_1 + 1 = \dim F_2$.

Clearly, a collapse is a strong deformation retract, hence it preserves the homotopy type of the space.

3. DISCRETE MORSE THEORY

Our main tool is the result proved by R. Forman as part of his Discrete Morse theory (see [8]). Although contained in his paper, it is convenient for us to reformulate Forman’s result in different notations and give a simple and direct argument.

Let Δ be a regular CW complex. A *matching* W on $P = P(\Delta)$ (cf. [8, Definition 9.1]) is a set of disjoint pairs (σ, τ) such that $\tau, \sigma \in P, \tau \succ \sigma$ (“ \succ ” denotes the covering relation). We set

$$\begin{aligned} \overrightarrow{W} &= \{\sigma \in P \mid \text{there exists } \tau \text{ such that } (\sigma, \tau) \in W\}, \\ \overleftarrow{W} &= \{\tau \in P \mid \text{there exists } \sigma \text{ such that } (\sigma, \tau) \in W\}. \end{aligned}$$

If $(\sigma, \tau) \in W$, then we set $W(\sigma) = \tau$. If $\sigma \notin \overleftarrow{W} \cup \overrightarrow{W}$, then we call the cell σ *critical*. Let $m_i(W)$ denote the number of critical i -cells.

Definition 3.1 (cf. [8, Definition 9.2]). A matching is called *acyclic* if it is impossible to find a sequence $\sigma_0, \dots, \sigma_t \in \overrightarrow{W}$ such that $\sigma_0 \neq \sigma_1, \sigma_0 = \sigma_t$ and $W(\sigma_i) \succ \sigma_{i+1}$ for $0 \leq i \leq t - 1$.

The following result is a direct consequence of [8, Theorem 9.3] and [8, Corollary 3.5]. We give a short explicit argument.

Theorem 3.2. *Let Δ be a regular CW complex of dimension d , and let $P = P(\Delta)$ be its face poset. Let W be an acyclic matching on $P(\Delta)$.*

(1) *Δ is homotopy equivalent to a CW complex Δ^M , which has $m_i(W)$ cells of dimension i .*

(2) *If the critical cells form a subcomplex Δ^C of Δ , then $\Delta^C = \Delta^M$ and there is a sequence of elementary collapses leading from Δ to Δ^M .*

Proof. A formal proof by induction can be given. However, to underline the simplicity of the situation, we give a direct gluing argument. More precisely, we describe a certain procedure of gluing Δ from its cells so that the critical cells are glued over their entire boundary and the matched cells are glued in pairs over the entire boundary of their union.

Start with an empty cell complex. At every step, let x be one of the cells of minimal dimension along those which are not yet glued on. Let $\dim x = i$; then the complete $(i - 1)$ -skeleton is already glued on. If x is critical, then simply glue x to the complex. If x is not critical, then we must have $x \in \overrightarrow{W}$. If all of the boundary cells of $W(x)$ (except for x) are glued on, then just glue x and $W(x)$ to the complex. Otherwise, since the matching W is acyclic, there exists $x_1 < W(x)$, such that $x_1 \neq x$ and x_1 is not yet glued on. Repeat the argument for x_1 . Either we glue on something or, since W is acyclic, we find $x_2 \neq x, x_2 \neq x_1, W(x_1) > x_2$ such that x_2 is not yet glued, etc.

Since there are only finitely many cells, the process will terminate and some gluing will occur. Since gluing on a pair of two matched cells over the entire boundary does not change the homotopy type (the inverse of such gluing is an elementary collapse) we prove (1).

To prove (2) simply observe that if the critical cells form a subcomplex, then we can glue them all on in the beginning of the process described above. \square

4. RANK SELECTIONS OF THE PARTITION LATTICE

Let Π_n be the partition lattice with the ground set $[n]$. Let Λ be a set of some number partitions of n , such that $(1, \dots, 1) \notin \Lambda$, $(n) \notin \Lambda$. Denote by Π_Λ the subposet of Π_n consisting of all partitions of type from Λ . In Figure 1, we have pictured the triangulated space $\Delta(\overline{\Pi}_5)/\mathcal{S}_5$, where $\overline{\Pi}_5$ is Π_5 with maximal and minimal elements removed. It consists of five triangles, four in the background, and the fifth is filled with diagonal lines. It is readily seen that $(\overline{\Pi}_5)/\mathcal{S}_5$ is collapsible.

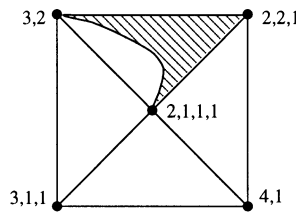


FIGURE 1.

When λ is a number partition of n , $\lambda = (\lambda_1, \dots, \lambda_t)$, we define $s_k(\lambda) = \sum_{i=1}^t \lfloor \frac{\lambda_i}{k} \rfloor$ and $\mu_k(\lambda) = (k^{s_k(\lambda)}, 1^{n-k \cdot s_k(\lambda)})$. Clearly $\mu_k(\lambda)$ refines λ . The next theorem is the main result of this paper. It gives a condition on Λ which guarantees that $\Delta(\Pi_\Lambda)/\mathcal{S}_n$ is collapsible.

Theorem 4.1. *Let $n > k \geq 2$. Assume that for each partition $\lambda \in \Lambda$ we have $\mu_k(\lambda) \in \Lambda$. Then $X_\Lambda = \Delta(\Pi_\Lambda)/\mathcal{S}_n$ is collapsible.*

Proof. As mentioned before, X_Λ is a triangulated space and $p : \Delta(\Pi_\Lambda) \rightarrow X_\Lambda$ is a cell map. The vertices of X_Λ are indexed by the set Λ . Elements of Λ are partially ordered by refinement and this induces an order on the vertices of each simplex of X_Λ . We call the vertices which have the form $(k^1, 1^{n-ka})$, for some a , *special*.

Let us now describe an acyclic matching on $P = P(X_\Lambda)$. Let $F \in P$. If all vertices of F are special, then we put F in the set of critical simplices. If not, define $\lambda(F)$ to be the smallest not special vertex of F . By the assumption of the theorem we have $\mu_k(\lambda) \in \Lambda$. If $\mu_k(\lambda)$ is a vertex of F , then we put F in \overleftarrow{W} , otherwise we put F in \overrightarrow{W} . This defines a partition of simplices of X_Λ into three sets: the critical simplices, \overleftarrow{W} , and \overrightarrow{W} .

Claim. For each $F \in \overrightarrow{W}$, there exists a unique $\tilde{F} \in \overleftarrow{W}$ such that \tilde{F} covers F , and $\mu_k(\lambda(F))$ is a vertex of \tilde{F} .

Proof. For brevity, we write λ , resp. μ , instead of $\lambda(F)$, resp. $\mu_k(\lambda(F))$, in the rest of the proof.

Existence. Let H be a simplex of $\Delta(\Pi_\Lambda)$ such that $p(H) = F$. Then H is a chain in Π_Λ of the form $(a_1, \dots, a_t, b, c_1, \dots, c_q)$, where a_i is a set partition of the type $(k^{n_i}, 1^{n-kn_i})$, b has type λ , and $n_t < s_k(\lambda)$. By an obvious combinatorial argument there exists a chain $\tilde{H} = (a_1, \dots, a_t, a, b, c_1, \dots, c_q)$ such that a has type μ . Clearly, $\tilde{F} = p(\tilde{H})$ belongs to \overleftarrow{W} , it covers F , and μ is a vertex of \tilde{F} .

Uniqueness. Assume that $\tilde{F}, \tilde{F}' \in \overline{W}$ both cover F , and μ is a vertex of both \tilde{F} and \tilde{F}' . Choose $\tilde{H}, \tilde{H}' \in \Delta(\Pi_\Lambda)$ such that $p(\tilde{H}) = \tilde{F}$ and $p(\tilde{H}') = \tilde{F}'$. Because of the conditions on \tilde{F} and \tilde{F}' and the definition of p we can choose \tilde{H} and \tilde{H}' so that $\tilde{H} = (a_1, \dots, a_t, a, b, c_1, \dots, c_q)$, $\tilde{H}' = (a_1, \dots, a_t, a', b, c_1, \dots, c_q)$, where, as before, a_i has type $(k^{n_i}, 1^{n-kn_i})$, a, a' have the type μ and b has type λ .

a' is constructed by cutting each block of b into at most $k - 1$ singletons and some blocks of size k . Since there is essentially a unique way to refine b in this way, we can replace \tilde{H}' with $g\tilde{H}'$ for $g \in \mathcal{S}_n$ such that

- $ga' = a$;
- each block of the set partition b is fixed;
- each block of size k of a_t is fixed.

Then the chain $(a_1, \dots, a_t, b, c_1, \dots, c_q)$ is fixed and hence $g\tilde{H}' = \tilde{H}$. This proves that $p(\tilde{H}) = p(\tilde{H}')$, hence $\tilde{F} = \tilde{F}'$. □

For $F \in \overline{W}^{\vec{}}$ define $W(F) = \tilde{F}$, where \tilde{F} is as in the claim above. Since F is obtained from \tilde{F} by removing a vertex, we conclude that W is a bijection.

Assume there exists a sequence $\sigma_0, \dots, \sigma_t \in \overline{W}^{\vec{}}$ such that $\sigma_0 \neq \sigma_1, \sigma_0 = \sigma_t$ and $W(\sigma_i) \succ \sigma_{i+1}$ for $0 \leq i \leq t - 1$. Assume $W(\sigma_0) = (a_1, \dots, a_\alpha, b_1, \dots, b_\beta)$, where a_i 's are special and b_1 is not. Then $\sigma_0 = (a_1, \dots, a_{\alpha-1}, b_1, \dots, b_\beta)$. Since $\sigma_1 \in \overline{W}^{\vec{}}$, and $W(\sigma_0) \neq W(\sigma_1)$, we have $\sigma_1 = (a_1, \dots, a_\alpha, b_2, \dots, b_\beta)$. We see that the number of special vertices in σ_1 is larger by 1 than in σ_0 . Repeating the argument, we see that σ_t has t special vertices more than σ_0 , therefore $\sigma_0 \neq \sigma_t$. This leads to the conclusion that W is an acyclic matching.

The critical simplices form a subcomplex of X_Λ , which we call X_Λ^C . By Theorem 3.2 there is a sequence of elementary collapses from X_Λ to X_Λ^C . Observe that if $A = (a_1, \dots, a_t)$ and $B = (b_1, \dots, b_t)$ are two simplices of $\Delta(\Pi_\Lambda)$ such that for $i = 1, \dots, t$, both a_i and b_i are of the type $(k^{\alpha_i}, 1^{n-k\alpha_i})$ for some α_i , then there exists $g \in \mathcal{S}_n$ such that $gA = B$, i.e., $p(A) = p(B)$. This implies that X_Λ^C is a simplex, so we can conclude that X_Λ is collapsible. □

Let \mathbf{k} be a field such that either $\text{char } \mathbf{k} = 0$ or $\text{char } \mathbf{k} > n$. Following a conjecture of R. Stanley [13, page 151], P. Hanlon has proved in [10, Theorem 3.1] that if Π_n^t is the $\{1, \dots, t\}$ rank selection of the partition lattice, then the multiplicity of the trivial character in the representation $\mathcal{S}_n \rightarrow GL(H_i(\Delta(\Pi_n^t), \mathbf{k}))$ is 0 for all i and t . The following corollary generalizes his result.

Corollary 4.2. *Assume Λ is as in Theorem 4.1. Then the multiplicity of the trivial character in the representation $\mathcal{S}_n \rightarrow GL(H_i(\Delta(\Pi_\Lambda), \mathbf{k}))$ is 0 for all i .*

Proof. Again let $X_\Lambda = \Delta(\Pi_\Lambda)/\mathcal{S}_n$. By [6, Theorem 1], $\beta_i(X_\Lambda, \mathbf{k})$ is equal to the multiplicity of the trivial character in $\mathcal{S}_n \rightarrow GL(H_i(\Delta(\Pi_\Lambda), \mathbf{k}))$. Thus the statement of the corollary is equivalent to saying that X_Λ is \mathbf{k} -acyclic, which is immediate from Theorem 4.1. □

When λ is a number partition of n , Π_λ denotes the subposet of Π_n consisting of all elements which are joins of elements of type λ . One can view Π_λ as an intersection lattice of the corresponding subspace arrangement. A special interesting case of Corollary 4.2 which is new is a $\{1, \dots, t\}$ rank selection of Π_λ for $\lambda = (k^m, 1^{n-km})$ for some k and m .

It also follows from Theorem 4.1 that if Λ is a rank selection such that the ranks $1, \dots, \lfloor \frac{n}{2} \rfloor$ are among the selected ones, then the multiplicity of the trivial character as above is 0. Up to a few border cases, this is [14, Theorem 4.2]. We refer the interested reader to [14] for a thorough account of known results and open questions about the multiplicity of the trivial character in the representation of \mathcal{S}_n on the homology groups of rank selections of the partition lattice, as well as for an extensive amount of computational data.

In general, in our terminology, all the questions and results of this kind can be reformulated as studying the Betti numbers of the subcomplexes of one large complex $\Delta(\Pi_n)/\mathcal{S}_n$, which are obtained by deleting a certain set of vertices (and their open stars). As we have seen, sometimes this point of view may lead to a more structural understanding of the situation.

We finish this section with yet another application of Theorem 4.1. Let Δ_n denote the simplicial complex whose k -faces are all disconnected graphs on n vertices having k edges and the face structure of Δ_n is induced by the inclusions of the graphs. These complexes have been introduced and studied by V. Vassiliev in [16] in connection with his construction of knot invariants. It follows from [15, Theorem 1] that Δ_n is \mathcal{S}_n -homotopy equivalent to $\Delta(\Pi_n)$ (cf. [16], [1, Proposition 2.1(i)]). This implies that $\Delta_n/\mathcal{S}_n \simeq \Delta(\Pi_n)/\mathcal{S}_n$. Therefore by Theorem 4.1 we have

Corollary 4.3. Δ_n/\mathcal{S}_n is contractible. □

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