We obtain Chern-Osserman’s inequality of a complete properly immersed minimal surface in hyperbolic $n$-space, provided the $L^2$-norm of the second fundamental form of the surface is finite.

1. Introduction

Let $M$ be a complete minimal surface in Euclidean space $\mathbb{R}^n$ with finite total Gaussian curvature. Then the total Gaussian curvature of $M$ satisfies the Chern-Osserman inequality ([2], [6])

$$-\chi(M) \leq \frac{-1}{2\pi} \int_M K - k,$$

where $K$ is the Gaussian curvature of $M$, $\chi(M)$ is the Euler characteristic of $M$ and $k$ is the number of ends of $M$. The explicit expression of the total Gaussian curvature was obtained by Jorge and Meeks:

$$-\chi(M) = \frac{-1}{2\pi} \int_M K - \sup \frac{\text{area} M \cap B(t)}{\pi t^2}$$

$$= \frac{1}{4\pi} \int_M |A|^2 - \sup \frac{\text{area} M \cap B(t)}{\pi t^2},$$

where $A$ is the second fundamental form of $M$ and $B(t)$ is the extrinsic distance ball of radius $t$ from a fixed point.

In the paper we present an analogue of the Chern-Osserman inequality of complete minimal surfaces in hyperbolic $n$-space $\mathbb{H}^n$ of constant curvature $-1$, namely

**Theorem.** Let $M$ be an oriented immersed complete minimal surface in $\mathbb{H}^n$, $A$ the second fundamental form of $M$, $r$ the distance of $\mathbb{H}^n$ from a fixed point and $M_t = \{ x \in M : r(x) < t \}$. Suppose $\int_M |A|^2(x) < \infty$; then

1. Suppose $\sup_{t \in \mathbb{H}^n} \frac{\text{area}(M_t)}{\cosh t} < +\infty$;

2. $-\chi(M) \leq \frac{1}{4\pi} \int_M |A|^2 - \sup \frac{\text{area}(M_t)}{\cosh t}$,

where $\chi(M)$ is the Euler characteristic of $M$. Consequently, $M$ has finite topological type.

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The minimal surfaces in $H^n$ have some properties similar to minimal surfaces in $R^n$, such as the monotonicity formula (see Proposition 2.2 below). But there are many differences between minimal surfaces in $H^n$ and those in $R^n$, one of the differences is that the total Gaussian curvature of a complete minimal surface in $H^n$ is always infinite (this can be seen from the correspondent Gauss equation). Another important difference comes from the fundamental work of M. T. Anderson [1] for the existence of minimal varieties in $H^n$. By his results, minimal surfaces in the hyperbolic space are much richer than those in Euclidean space, and the asymptotic behavior of the surfaces is not “regular” in general; in particular, the Bernstein Theorem (a complete minimal graph in $R^3$ is flat) does not hold in $H^3$.

In [5] De Oliveira proved that if $M$ is an immersed complete minimal surface in $H^n$ with $\int_M |A|^2 < \infty$, then $M$ is properly immersed and is conformally equivalent to a compact surface with a finite number of disks removed. M. Kokubu [4] established the Weierstrass type representation formula for minimal surfaces in hyperbolic space. By his result, the Gauss maps of minimal surfaces in hyperbolic space are neither holomorphic nor anti-holomorphic. So the method employed in [2] and [6] is not valid in our case.

2. Preliminaries

Let $H^n$ be a hyperbolic $n$-space of constant curvature $-1$, and $M$ a properly immersed complete minimal surface in $H^n$. Denote the covariant derivative of $H^n$ and $M$ by $D$ and $\nabla$ respectively; the second fundamental form of $M$ is defined by

$$ A : TM \otimes TM \to T^\perp M, $$

(2.1)  
$$ A(X,Y) = D_X Y - \nabla_X Y, \text{ for } X,Y \in C^\infty(TM). $$

For a smooth function $f$ on $H^n$, and any two tangent vector fields $X,Y \in C^\infty(TM)$,

$$ (D^2 f)(X,Y) = (Df)(X,Y) $$

$$ = X(df(Y)) - df(D_X Y) $$

$$ = X(df(Y)) - df(\nabla_X Y + A(X,Y)) $$

$$ = \nabla^2 f(X,Y) - \langle A(X,Y), Df \rangle, $$

which, together with the fact that $(D^2 r)(X,X) = \coth r(\langle X,X \rangle - \langle X, Dr \rangle^2)$, implies

**Proposition 2.1.** For any unit tangent vector $e$ of $M$,

$$ (\nabla^2 r)(e,e) = \coth r(1 - \langle e, \nabla r \rangle^2) + \langle A(e,e), \nabla^\perp r \rangle, $$

where $\nabla^\perp r$ is the projection of $Dr$ onto the normal of $M$.

Let $r$ be the distance function of $H^n$ from a fixed point. By Sard’s theorem, for almost all $t > 0$, $M_t = \{ x \in M : r(x) < t \}$ is a related compact open subset of $M$ with the boundary $\partial M_t$ being a closed immersed curve of $M$. $\{M_t\}$ is a family of exhaustion of $M$. We will consider the growth of the area of $M_t$, and make use of the following notations for convenience:

$$ v(t) = \text{area} M_t, \text{ and } R(t) = \int_{M_t} |A|^2. $$

**Proposition 2.2** ([1], Theorem 1). \( \frac{v(t)}{\cosh t-1} \) is monotone nondecreasing in $t$, i.e., \( v'(t) \cosh t - v(t) \sinh t \geq v'(t) \).
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Proposition 2.3. Suppose $M$ is an oriented and properly immersed complete minimal surface in $H^n$. Then for almost all $t > 0$,

$$v(t) + \frac{1}{2} R(t) + 2\pi \chi(M_t) = v'(t) \coth t - \int_{\partial M_t} \langle A(\frac{\nabla r}{|\nabla r|}, \frac{\nabla r}{|\nabla r|}, \frac{\nabla r}{|\nabla r|}), \frac{\nabla r}{|\nabla r|} \rangle.$$

Proof. Denote the geodesic curvature of $\partial M_t$ in $M$ by $k^t_g$, and by $K$ the Gaussian curvature of $M$. Then the Gauss-Bonnet formula reads

$$\int_{\partial M_t} k^t_g + \int_{M_t} K = 2\pi \chi(M_t),$$

where $\chi(M_t)$ is the Euler characteristic of $M_t$, i.e. $\chi(M_t) = 2 - 2g - k$ with $g$ being the genus of $M_t$ and $k$ being the number of components of $\partial M_t$.

Substituting the Gauss equation $K = -\frac{1}{2} |A|^2$ into (2.2) we have

$$v(t) + \frac{1}{2} R(t) + 2\pi \chi(M_t) = \int_{\partial M_t} k^t_g.$$ (2.3)

Suppose $e$ is the unit tangent vector of $\partial M_t$. Since the normal of $\partial M_t$ in $M$ is $\frac{\nabla r}{|\nabla r|}$,

$$k^t_g = -\langle \nabla e, \frac{\nabla r}{|\nabla r|} \rangle$$

$$= \frac{1}{|\nabla r|} (\nabla^2 r)(e, e)$$

$$= \frac{1}{|\nabla r|} \left( \coth r + \langle A(e, e), \nabla \frac{r}{|\nabla r|} \rangle \right),$$

where the last equality follows by Proposition 2.1.

Substituting (2.4) into (2.3), and using the fact that $v'(t) = \int_{\partial M_t} \frac{1}{|\nabla r|}$, and that $A(e, e) + A(\frac{\nabla r}{|\nabla r|}, \frac{\nabla r}{|\nabla r|}) = 0$, we complete the proof.

Lemma 2.4. For $t > s > 0$,

$$\int_{M_t} \cosh r \cosh t - \int_{M_s} \cosh r \cosh s = \int_{M_t - M(s)} 1 + |\nabla \frac{r}{|\nabla r|}|^2 \sinh^2 r \cosh^3 r.$$

Proof. By the minimality of $M$ and Proposition 2.1, we observe that

$$\Delta r = \coth r (2 - |\nabla r|^2),$$

where $\Delta$ is the Laplacian of $M$. This yields

$$\Delta \cosh r = 2 \cosh r.$$ (2.5)

Integrating (2.2) over $M_t$, and using Green’s formula,

$$2 \int_{M_t} \cosh r = \int_{\partial M_t} |\nabla r| \sinh r.$$ (2.6)
By using the co-area formula (7), we have
\[
\frac{d}{dt} \left( \int_{M_t} \cosh r \cosh^2 t \right) = \frac{1}{\cosh^3 t} \left( \cosh t \frac{d}{dt} \int_{M_t} \cosh r - 2 \sinh t \int_{\partial M_t} \cosh r \right)
\]
\[
= \frac{1}{\cosh^3 t} \left( \cosh t \int_{\partial M_t} \frac{\cosh r}{|\nabla r|} - \sinh t \int_{\partial M_t} |\nabla r| \sinh r \right)
\]
\[
= \frac{1}{\cosh^3 t} \int_{\partial M_t} \frac{1}{|\nabla r|} (1 + \sinh^2 r |\nabla r|^2).
\]
(2.4)

The proposition is then proved by integrating (2.6) from \( s \) to \( t \) and the co-area formula.

**Lemma 2.5.**
\[
\int_0^t v'(s) \cosh s ds \geq \frac{\cosh t + 1}{2} v(t).
\]

**Proof.** By Proposition 2.2, \( v'(t) \geq \frac{v(t) \sinh t}{\cosh t - 1} \), so we have
\[
\int_0^t v'(s) \cosh s ds = v(t) \cosh t - \int_0^t v(s) \sinh s ds
\]
\[
\geq v(t) \cosh t - \int_0^t v'(s)(\cosh s - 1) ds
\]
\[
= v(t)(\cosh t + 1) - \int_0^t v'(s) \cosh s ds,
\]
which proves the lemma.

### 3. Proof of the Theorem

Suppose \( M \) is as in the Theorem. By the result of De Oliveira [5], \( M \) is properly immersed.

(1) By Proposition 2.3 we have
\[
(3.1) \quad \frac{v'(t) \cosh t - v(t) \sinh t}{\sinh t} = \frac{1}{2} R(t) + 2\pi \chi(M_t) + \int_{\partial M_t} \langle A(\frac{\nabla r}{|\nabla r|}, \frac{\nabla r}{|\nabla r|}), \frac{\nabla r}{|\nabla r|} \rangle;
\]
hence
\[
\frac{d}{dt} \frac{v(t)}{\cosh t} \leq \frac{\sinh t}{\cosh^2 t} \left( \frac{1}{2} R(t) + 2\pi \chi(M_t) \right) + \int_{\partial M_t} \frac{|A|}{|\nabla r|} \frac{|\nabla r|^2 \sinh t}{\cosh^2 r}.
\]

Since \( \chi(M_t) \leq 1 \), integrating the above inequality from 0 to \( t \) and by using the co-area formula, we have
\[
(3.2) \quad \frac{v(t)}{\cosh t} \leq 2 \int_0^t \left( \frac{1}{2} R(s) + 2\pi e^{-s} ds + \int_{\partial M_t} |A| \frac{|\nabla r|^2 \sinh r}{\cosh^2 r} \right).
\]
By using the Cauchy-Schwarz inequality,
\[
\frac{v(t)}{\cosh t} \leq C_1 + \left( \int_{M_t} |A|^2 \cosh r \right)^{\frac{1}{2}} \left( \int_{M_t} (\nabla^+ r)^2 \sinh^2 r \right)^{\frac{1}{2}} \\
\leq C_1 + C_2 \left( \int_{M_t} \frac{\cosh r}{\cosh^2 t} \right)^{\frac{1}{2}} \quad \text{(by Proposition 2.3)} \\
\leq C_1 + C_2 \left( \frac{v(t)}{\cosh t} \right)^{\frac{1}{2}},
\]

(3.3)

where $C_1$ and $C_2$ are two constants independent of $t$. By Proposition 2.2, $\frac{v(t)}{\cosh t}$ is monotone non-decreasing in $t$; therefore, either $\sup v(t) \cosh t \leq C_2$ or
\[
\frac{v(t)}{\cosh t} \leq \left( \frac{v(t)}{\cosh t} \right)^{\frac{1}{2}} + C_2 \left( \frac{v(t)}{\cosh t} \right)^{\frac{1}{2}}
\]
when $t$ is large enough. It follows that
\[
\sup \frac{v(t)}{\cosh t} \leq \max \{C_2^2, (1 + C_2)^2\}.
\]

This proves (1) of the Theorem.

(2) By the arithmetic geometric mean inequality, (3.1) implies
\[
\frac{v'(t) \cosh t - v(t) \sinh t}{\sinh t} \leq \frac{1}{2} R(t) + 2\pi \chi(M_t) + \frac{1}{2} \int_{\partial M_t} \left( \frac{|A|^2}{|\nabla r|} + \frac{|\nabla^+ r|^2}{|\nabla r|} \right) \leq \frac{1}{2} R(t) + 2\pi \chi(M_t) + \frac{1}{2} R'(t) + \frac{1}{2} \int_{\partial M_t} \frac{|\nabla^+ r|^2}{|\nabla r|}.
\]

(3.4)

Since by Proposition 2.1, $\Delta \cosh r = 2 \cosh r$, Green’s formula gives
\[
\int_{\partial M_t} |\nabla r| \sinh r = \int_{M_t} 2 \cosh r;
\]

then by the co-area formula we have
\[
\int_{\partial M_t} \frac{|\nabla^+ r|^2}{|\nabla r|} = \int_{\partial M_t} \frac{1}{|\nabla r|} - |\nabla r| = v'(t) - \frac{1}{\sinh t} \int_{\partial M_t} |\nabla r| \sinh r
\]
\[
= v'(t) - \frac{1}{\sinh t} \int_{M_t} 2 \cosh r
\]
\[
= v'(t) - \frac{2}{\sinh t} \int_0^t v'(s) \cosh s ds.
\]

(3.5)

By Lemma 2.5 we obtain
\[
\int_{\partial M_t} \frac{|\nabla^+ r|^2}{|\nabla r|} \leq v'(t) - \frac{\cosh t + 1}{\sinh t} v(t).
\]

(3.6)
Substituting (3.6) into (3.4) we have
\[
\frac{v'(t) \cosh t - v(t) \sinh t}{\sinh t} \leq \frac{1}{2} (R(t) + R'(t)) + 2\pi \chi(M_t) + \frac{v'(t) \sinh t - v(t) \cosh t - v(t)}{\sinh t}.
\]
This implies
\[
-2\pi \chi(M_t) \leq \frac{1}{2} (R(t) + R'(t)) + \frac{(v'(t) + v(t))(\sinh t - \cosh t) - v(t)}{\sinh t}
\]
(3.7)
\[
\leq \frac{1}{2} (R(t) + R'(t)) - \frac{v(t)}{\sinh t}
\]
Since \(\int_{t_0}^{\infty} R'(t) dt \leq \infty\), there is a monotone increasing sequence \(\{t_i\}\) diverging to infinity such that \(R'(t_i) \rightarrow 0\) as \(i \rightarrow \infty\). Taking \(t = t_i\) in (3.7) and letting \(i\) tend to infinity, we prove the theorem.

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References