RELATIONS BETWEEN THE TAYLOR SPECTRUM
AND THE XIA SPECTRUM

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Dedicated to Professor Jyunji Inoue on his sixtieth birthday

Abstract. Let $T = (T_1, T_2, \ldots, T_n)$ be a doubly commuting $n$-tuple of $p$-hyponormal operators $T_j$ with unitary operators $U_j$ from the polar decompositions $T_j = U_j |T_j|$ ($j = 1, \ldots, n$). Let $U = (U_1, \ldots, U_n)$ and $A = |T_1| \cdots |T_n|$. In this paper, we will show relations between the Taylor spectrum $\sigma_T(T)$ and the Xia spectrum $\sigma_X(U, A)$.

1. Introduction

In [12], D. Xia introduced a class of semi-hyponormal tuples and a notion of spectrum for such tuples. We call this spectrum the Xia spectrum. Xia proved Putnam’s inequality for semi-hyponormal tuples. In [3], M. Chô and T. Huruya generalized Putnam’s inequality to $p$-hyponormal tuples. Also, in [9], B. P. Duggal showed a very interesting inequality of doubly commuting $n$-tuples of $p$-hyponormal operators. In this paper, we show that the Xia spectrum of a doubly commuting $n$-tuple $T = (T_1, \ldots, T_n)$ of $p$-hyponormal operators $T_j$ with unitary operators $U_j$ from the polar decompositions $T_j = U_j |T_j|$ ($j = 1, \ldots, n$) essentially coincides with its Taylor spectrum.

Let $\mathcal{H}$ be a complex separable Hilbert space and $B(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, let $\sigma(T)$ be the spectrum of $T$. An operator $T \in B(\mathcal{H})$ is called $p$-hyponormal if $(T^*T)^p \geq (TT^*)^p$. If $p = \frac{1}{2}$, then $T$ is called semi-hyponormal. Let $W$ be a unitary operator and $A \in B(\mathcal{H})$. If

$$S_W^\pm(A) = \lim_{n \to \pm \infty} (W^{-n}AW^n)$$

exist, then the operators $S_W^\pm(A)$ are called the polar symbols of $A$ (with respect to $W$). Let $T = U |T|$ be the polar decomposition of $T$. If $T$ is semi-hyponormal and $U$ is unitary, then $S_U^\pm(|T|)$ exist (cf. [13]). In [13], D. Xia proved the following theorem:

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Theorem A (Theorem IV.4.1 of [13]). Let $T = U|T|$ be a semi-hyponormal operator with $U$ unitary. Then
\[ \sigma(T) = \bigcup_{0 \leq k \leq 1} \sigma(T_k), \]
where $T_k = kS_U^+(T) + (1-k)S_U^-(T)$.

2. Generalized polar symbols

Throughout this paper let $p$ be such that $0 < p < \frac{1}{2}$. Let $T = U|T|$ be a $p$-hyponormal operator with $U$ unitary. Since $U|T|^{2p}$ is semi-hyponormal, there exist $S_U^+(|T|^{2p})$. For $0 \leq k \leq 1$, we denote
\[ T_k = U\{kS_U^+(|T|^{2p}) + (1-k)S_U^-(|T|^{2p})\}^{\frac{1}{2}}; \]
we call the operators $T_k$ the generalized polar symbols of $T$. Note that if an operator $T = U|T|$ is a semi-hyponormal operator with $U$ unitary, then $T_k = T_k$ for every $0 \leq k \leq 1$. It is easy to check that $T_k$ is a normal operator for every $0 \leq k \leq 1$. For $T \in B(\mathcal{H})$, let $\sigma_{na}(T)$ denote the normal approximate point spectrum of $T$, i.e., the set of all complex numbers $z$ which satisfy the following condition: there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that
\[ \lim_{n \to \infty} \| (T - z)x_n \| = \lim_{n \to \infty} \| (T - z^*)x_n \| = 0. \]
If $T$ is a normal operator, then $\sigma(T) = \sigma_{na}(T)$. In [12] this spectrum is called the joint approximate point spectrum, but we use this term for $n$-tuple of operators. The following theorem holds.

Theorem B (Lemma I.2.4 of [13]). Let $T \in B(\mathcal{H})$ and let $T = U|T|$ be the polar decomposition of $T$. Let $r > 0$. Then $re^{i\theta} \in \sigma_{na}(T)$ if and only if there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that
\[ \lim_{n \to \infty} \| (|T| - r)x_n \| = \lim_{n \to \infty} \| (U - e^{i\theta})x_n \| = 0. \]
Therefore, for a semi-hyponormal operator $T = U|T|$ with $U$ unitary and a non-zero $re^{i\theta} \in \mathbb{C}$, it follows that $re^{i\theta} \in \sigma_{na}(T_k)$ if $re^{i\theta} \in \sigma(T_k)$, because each $T_k$ is a normal operator ($0 \leq k \leq 1$).

The following result was proved in [5]. For the sake of completeness, we will give a simple proof.

Theorem 1 (Theorem of [5]). Let $T = U|T|$ be a p-hyponormal operator with $U$ unitary. Then
\[ \sigma(T) = \bigcup_{0 \leq k \leq 1} \sigma(T_k). \]

For the proof of this theorem, we need the following result.

Theorem C (Theorem 3 of [1]). Let $T = U|T|$ be a p-hyponormal operator with $U$ unitary. Then
\[ \sigma(U|T|^{2p}) = \{ r2pe^{i\theta} \mid re^{i\theta} \in \sigma(T) \}. \]

Proof of Theorem 1. Note that $S_U^-(|T|^{2p}) \leq |T|^{2p} \leq S_U^+(|T|^{2p})$ (cf. Th.II.2.7 of [13]). If $0 \in \sigma(T)$, then $0 \in \sigma(|T|)$ and hence $0 \in \sigma(T_0)$, $T_0 = U\{S_U^+(|T|^{2p})\}^{\frac{1}{2}}$.
Conversely, let $0 \in \bigcup_{0 \leq k \leq 1} \sigma(T_k)$. Since $T_k$ is normal, we have $0 \in \sigma(S_U^+(|T|^{2p}))$ and
hence $0 \in \sigma(|T|)$ (cf. Th.II.1.5 of [12]). Therefore, we have $0 \in \sigma(T)$. Next we prove that, for a non-zero $z = re^{i\theta} \in \mathbb{C}$, $z \in \sigma(T)$ if and only if $z \in \bigcup_{0 \leq k \leq 1} \sigma(T_k)$. Let $S = |T|^{2p}$. Then $S$ is semi-hyponormal and from Theorem we have

$$z \in \sigma(T) \iff r^{2p}e^{i\theta} \in \sigma(S) \iff \exists k (0 \leq k \leq 1) ; r^{2p}e^{i\theta} \in \sigma(S_{(k)}) \quad \text{from Theorem A}$$

$$\iff \exists k (0 \leq k \leq 1) ; r^{2p}e^{i\theta} \in \sigma_{na}(S_{(k)}) \quad \text{from Theorem B}$$

$$\iff \exists k (0 \leq k \leq 1) ; re^{i\theta} \in \sigma_{na}(T_k)$$

$$\iff z \in \bigcup_{0 \leq k \leq 1} \sigma(T_k).$$

The proof is now complete.

3. THE TAYLOR SPECTRUM AND THE XIA SPECTRUM

For a commuting $n$-tuple $T = (T_1, \ldots, T_n)$, the Taylor spectrum and the joint approximate point spectrum of $T$ are denoted by $\sigma_T(T)$ and $\sigma_{ja}(T)$, respectively. It is well known that $\sigma_T(T) = \sigma_{ja}(T)$ if $T$ is a commuting $n$-tuple of normal operators. If $T = (T_1, \ldots, T_n)$ is a doubly commuting $n$-tuple of $p$-hyponormal operators, then, by Theorem 7 of [3], it follows that $\sigma_T(T) = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n : (z_1, \ldots, z_n) \in \sigma_{ja}(T^*) \}$, where $T^* = (T_1^*, \ldots, T_n^*)$. Let $U = (U_1, \ldots, U_n)$ be a commuting $n$-tuple of unitary operators. Let $Q_j (j = 1, \ldots, n)$ on $B(H)$ be defined by

$$Q_j A = A - U_j AU_j^* \quad (A \in B(H)).$$

Let $A \in B(H)$ and $A \geq 0$. An $(n + 1)$-tuple $(U, A)$ is called $p$-hyponormal if

$$Q_{j_1} \cdots Q_{j_m} A^{2p} \geq 0$$

for all $1 \leq j_1 < \cdots < j_m \leq n$. We simply denote $S^+_{U_j}(A)$ by $S^+_{j}(A)$ for every $j = 1, \ldots, n$. Let $(U, A)$ be a $p$-hyponormal tuple and $0 \leq k \leq 1$. We denote

$$(kS_j^+ + (1 - k)S_j^-)_p A = \{ kS_j^+ (A^{2p}) + (1 - k)S_j^- (A^{2p}) \}_p.$$

For $k = (k_1, \ldots, k_n) \in [0, 1]^n$, the general polar symbols $A_k$ of $A$ are defined by

$$A_k = \prod_{j=1}^n (k_j S_j^+ + (1 - k_j)S_j^-)_p A.$$

Then, by [6], $(U, A, A_k)$ is a commuting $(n + 1)$-tuple of normal operators for every $k \in [0, 1]^n$. We define the Xia spectrum $\sigma_X(U, A)$ of $(U, A)$ by

$$\sigma_X(U, A) = \bigcup_{k \in [0, 1]^n} \sigma_{ja}(U, A_k).$$

By Theorem 2 of [6] it follows that, for a $p$-hyponormal tuple $(U, A)$,

$$\|Q_1 \cdots Q_n A^{2p}\| \leq \frac{2^n}{(2\pi)^n} \int \cdots \int_{\sigma_X(U, A)} r^{2p-1}d\theta_1 \cdots d\theta_n dr.$$

We now have the following

**Lemma 2.** Let $T = (T_1, \ldots, T_n)$ be a doubly commuting $n$-tuple of $p$-hyponormal operators $T_j = U_j T_j$ with $U_j$ unitary operators $(j = 1, \ldots, n)$, and let $U = (U_1, \ldots, U_n)$ and $A = |T_1| \cdots |T_n|$. Then $(U, A)$ is $p$-hyponormal.
Proof. Since $A^{2p} = |T_1|^{2p} \cdots |T_n|^{2p}$, we have
\[ Q_jA^{2p} = \left( \prod_{i \neq j} |T_i|^{2p} \right) (|T_j|^{2p} - U_j|T_j|^{2p}U_j^*) \]
for every $j (j = 1, \ldots, n)$. Hence $(U, A)$ is $p$-hyponormal.

With the above notations (Lemma [2]), we also have, using the above,
\[ \| \prod_{j=1}^n (|T_j|^{2p} - |T_j|^*|T_j|^{2p}) \| \leq \frac{2p}{(2\pi)^{2n}} \int \cdots \int_{\sigma_X(U, A)} r^{2p-1} d\theta_1 \cdots d\theta_n \, dr \]
(this inequality is due to Duggal [3]), and
\[ (k_jS_j^+ + (1 - k_j)S_j^-)A = \left( \prod_{i \neq j} (T_i) \right) \left( k_jS_j^+ + (1 - k_j)S_j^- \right) \left( |T_j|^{2p} \right) \frac{1}{T_j} \]
Hence, for every $k = (k_1, \ldots, k_n) \in [0, 1]^n$, it follows that
\[ A_k = \prod_{j=1}^n A_j, \]
where $A_j = \left( k_jS_j^+ + (1 - k_j)S_j^- \right) \left( |T_j|^{2p} \right) \frac{1}{T_j} (j = 1, \ldots, n)$. We prove the following

**Theorem 3.** Let $T = (T_1, \ldots, T_n)$ be a doubly commuting $n$-tuple of $p$-hyponormal operators with unitary operators $U_j$ from the polar decompositions $T_j = U_j|T_j|$ ($j = 1, \ldots, n$). Let $U = (U_1, \ldots, U_n)$ and $A = |T_1| \cdots |T_n|$. If $(z_1, \ldots, z_n, a) \in \sigma_X(U, A)$, then there exist non-negative numbers $a_1, \ldots, a_n$ such that $(z_1a_1, \ldots, z_na_n) \in \sigma_T(T)$ and $a = a_1 \cdots a_n$.

Conversely, if $(z_1a_1, \ldots, z_na_n) \in \sigma_T(T)$, then $(z_1, \ldots, z_n, a_1 \cdots a_n) \in \sigma_X(U, A)$, where $|z_j| = 1$ and $a_j \geq 0$ for every $j (j = 1, \ldots, n)$.

For the proof of this theorem, we need the following Berberian extension theorem.

**Theorem D** (Theorem 1 of [1]). Let $B(H)$ be the algebra of all bounded operators on $H$. Then there exist an extension space $K$ of $H$ and a faithful *-representation of $B(H)$ into $B(K)$ such that
\[ \sigma_{ja}(T_1, \ldots, T_n) = \sigma_{ja}(T_1^*, \ldots, T_n^*) = \sigma_p(T_1^*, \ldots, T_n^*), \]
where $\sigma_p(T_1, \ldots, T_n)$ is the joint point spectrum of $(T_1, \ldots, T_n)$. Moreover, if $T$ is $p$-hyponormal, then $T^*$ is also $p$-hyponormal.

**Proof of Theorem** First we assume that $(z_1, \ldots, z_n, a) \in \sigma_X(U, A)$. We show by induction that there exist $a_1, \ldots, a_n$ ($\forall a_j \geq 0$) such that
\[ (z_1a_1, \ldots, z_na_n) \in \sigma_T(T) \text{ and } a = a_1 \cdots a_n. \]
If $n = 1$, Theorem 3 holds by Theorem 3 of [6]. By inductive hypothesis, there exist $k = (k_1, \ldots, k_n) \in [0, 1]^n$ and a sequence $\{x_m\}$ of unit vectors such that
\[ (U_j - z_j)x_m \rightarrow 0 \quad (j = 1, \ldots, n) \text{ and } (A_k - a)x_m \rightarrow 0, \]
where $A_k = \prod_{j=1}^n (k_jS_j^+ + (1 - k_j)S_j^-)A$. By Lemma 2 we have
\[ A_k = \prod_{j=1}^n A_j, \]
where $A_j = \{ k_j S_j^+ (|T_j|^2p) + (1 - k_j) S_j^- (|T_j|^2p) \} \frac{1}{p}$. By Theorem $D$ let $K$ be the extension space of $H$. Then

$$M = \text{Ker}(U_1^\circ - z_1) \cap \cdots \cap \text{Ker}(U_n^\circ - z_n) \cap \text{Ker}(A_n^\circ - a)$$

is a non-zero subspace of $K$. Since $(U_1^\circ, ..., U_n^\circ, A_1^\circ, ..., A_n^\circ)$ is a commuting 2n-tuple, $M$ is an invariant subspace for $A_1^\circ, ..., A_n^\circ$. Also since $a \in \sigma(A_n^\circ |M)$, there exist $a_1, ..., a_n$ and a non-zero vector $x^0 \in M$ such that

$$(A_j^\circ - a_j)x^0 = 0 \text{ for every } j (j = 1, ..., n) \text{ and } a = a_1 \cdots a_n,$$

by Theorem $D$ and the spectral mapping theorem for the joint spectrum. Let

$$N = \text{Ker}(U_n^\circ - z_n) \cap \text{Ker}(A_n^\circ - a_n).$$

Then $$(z_1, ..., z_{n-1}, a_1 \cdots a_{n-1}) \in \sigma_X(U', A')$$

where $U' = (U_1, ..., U_{n-1})$ and $A' = \prod_{j=1}^{n-1} A_j$. By Theorem $D$ and the inductive hypothesis, we have

$$(z_1 a_1, ..., z_{n-1} a_{n-1}) \in \sigma_T(T_1, ..., T_{n-1}).$$

Since $S = (T_1^\circ |N, ..., T_{n-1}^\circ |N)$ is a doubly commuting $(n-1)$-tuple of $p$-hyponormal operators on $N$ and $(z_1 a_1, ..., z_{n-1} a_{n-1}) \in \sigma_T(S)$, Theorem 7 of $[3]$ and Theorem $D$ imply that there exists a non-zero vector $y^0$ in $N$ such that

$$(T_j^\circ - z_j a_j)^* y^0 = 0 \text{ for every } j (j = 1, ..., n-1).$$

Let

$$L = \bigcap_{j=1}^{n-1} \text{Ker}((T_j^\circ - z_j a_j)^*).$$

Then $N \cap L$ is a non-zero subspace of $K$. Hence we have $(z_n, a_n) \in \sigma_{fp}(U_n^\circ |L, A_n^\circ |L)$ and $(z_n, a_n) \in \sigma_X(U_n^\circ |L, T_n^\circ |L)^\circ$. Also by the induction we have

$$z_n a_n \in \sigma(T_n^\circ |L).$$

Since $T_n^\circ |L$ is a $p$-hyponormal operator on $L$, there exists a non-zero vector $w^0 \in L$ such that

$$(T_n^\circ - z_n a_n)^* w^0 = 0.$$ 

Therefore, there exists a sequence $\{ x_m \}$ of unit vectors such that

$$(T_j - z_j a_j)^* x_m \rightarrow 0 \text{ for every } j (j = 1, ..., n).$$

Hence we have $(z_1 a_1, ..., z_n a_n) \in \sigma_T(T)$. 

Conversely, we assume that $(z_1 a_1, ..., z_n a_n) \in \sigma_T(T)$. Also assume that the hypothesis holds for doubly commuting $(n-1)$-tuples of $p$-hyponormal operators. By Theorem 7 of $[3]$ there exists a sequence $\{ x_m \}$ of unit vectors such that

$$(T_j - z_j a_j)^* x_m \rightarrow 0 \text{ for every } j (j = 1, ..., n).$$

Consider the extension space $K$ of $H$ and let

$$U = \text{Ker}((T_n^\circ - z_n a_n)^*).$$

By Theorem $D$ and $[1]$ there exists $z^0 \in U$ such that

$$(T_j^\circ - z_j a_j)^* z^0 = 0 \text{ for every } j (j = 1, ..., n-1).$$
Since \( (T_1^0, \ldots, T_{n-1}^0) \) is a commuting \((n-1)\)-tuple of \(p\)-hyponormal operators on \( \mathcal{U} \), it holds that \((z_1a_1, \ldots, z_{n-1}a_{n-1}) \in \sigma_{\mathcal{U}}(T_1^0, \ldots, T_{n-1}^0)\). By the inductive hypothesis
\[
(z_1, \ldots, z_{n-1}, a_1 \cdots a_{n-1}) \in \sigma_X(U', A'),
\]
where \( U' = (U_1^0, \ldots, U_{n-1}^0) \) and \( A' = [T_1^0, \ldots, T_{n-1}^0] \). Hence there exist \((m_1, \ldots, m_{n-1}) \in \mathbb{N}^{n-1} \) and a non-zero vector \( u^o \in \mathcal{U} \) such that
\[
(U_j^o - z_j)u^o = (A_1^o \cdots A_{n-1}^o - a_1 \cdots a_{n-1})u^o = 0,
\]
where \( A_j = \{m_jS^+_j(T_j^{2p}) + (1 - m_j)S^-_j(T_j^{2p})\} \) for every \( j = 1, \ldots, n-1 \).

Next let
\[
V = \bigcap_{j=1}^{n-1} \ker(U_j^o - z_j) \cap \ker(A_1^o \cdots A_{n-1}^o - a_1 \cdots a_{n-1}).
\]

Since \( \mathcal{U} \cap V \) is a non-zero subspace, we have
\[
z_an \in \sigma(T_n^o|V).
\]

Hence by Theorem 1 there exists \( 0 \leq n \leq 1 \) such that \( z_an \in \sigma(U_nA_n) \), where \( A_n = \{m_nS^+_n(T_n^{2p}) + (1 - m_n)S^-_n(T_n^{2p})\} \). Since \( U_nA_n \) is a normal operator, by Theorem D there exists \( v^o \in V \) such that
\[
(U_n^o - z_n)v^o = (A_n^o - a_n)v^o = 0.
\]

Let \( m = (m_1, \ldots, m_n) \) and \( A_m = \prod_{j=1}^n A_j \). By Theorem D we have
\[
(z_1, \ldots, z_n, a_1 \cdots a_n) \in \sigma_p(U_1^o, \ldots, U_n^o, A_m^o),
\]
and hence \((z_1, \ldots, z_n, a_1 \cdots a_n) \in \sigma(a(U, A_m)). \)

Using the definition of the Xia spectrum, we obtain
\[
(z_1, \ldots, z_n, a_1 \cdots a_n) \in \sigma_X(U, A).
\]

The proof is now complete.

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