PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 8, Pages 2337–2345
S 0002-9939(99)05368-X
Article electronically published on November 29, 1999

MAXIMAL ESTIMATES FOR THE (C,α) MEANS OF d-DIMENSIONAL WALSH-FOURIER SERIES

FERENC WEISZ

(Communicated by Christopher D. Sogge)

Abstract. The d-dimensional dyadic martingale Hardy spaces H_p are introduced and it is proved that the maximal operator of the (C,α) $(\alpha = (\alpha_1,\dots,\alpha_d))$ means of a Walsh-Fourier series is bounded from H_p to L_p $(1/(\alpha_k+1) < p < \infty)$ and is of weak type (L_1,L_1), provided that the supremum in the maximal operator is taken over a positive cone. As a consequence we obtain that the (C,α) means of a function $f \in L_1$ converge a.e. to the function in question. Moreover, we prove that the (C,α) means are uniformly bounded on H_p whenever $1/(\alpha_k+1) < p < \infty$. Thus, in case $f \in H_p$, the (C,α) means converge to f in H_p norm. The same results are proved for the conjugate (C,α) means, too.

1. Introduction

The Hardy-Lorentz spaces $H_{p,q}$ of dyadic martingales on $[0,1)^d$ are introduced with the $L_{p,q}$ Lorentz norms of the diagonal maximal function. Of course, $H_p = H_{p,p}$ are the usual Hardy spaces $(0 < p \le \infty)$.

For multi-dimensional trigonometric-Fourier series Marcinkiewicz and Zygmund [5] proved that the Fejér means $\sigma_n^1 f$ of a function $f \in L_1(\mathbf{T}^d)$ converge a.e. to f as $\min(n_1,\dots,n_d) \to \infty$ provided that n is in a positive cone, i.e., provided that $2^{-\tau} \le n_i/n_j \le 2^\tau$ for every $i,j = 1,\dots,d$ and for some $\tau \ge 0$ $(n = (n_1,\dots,n_d) \in \mathbf{N}^d)$. Recently the author [15] extended this result to the (C,α) means.

Here we investigate the (C,α) means σ_n^α of d-dimensional Walsh-Fourier series and the maximal operator $\sigma_*^\alpha := \sup_{\substack{2^{-\tau} \le n_i/n_j \le 2^\tau \\ i,j=1,\dots,d}} |\sigma_n^\alpha|$, where $\alpha = (\alpha_1,\dots,\alpha_d)$ and $0 < \alpha_j \le 1$. We speak about Fejér means if $\alpha_j = 1$ $(j = 1,\dots,d)$. In the one-dimensional case the above convergence result for (C,α) means is due to Fine [2] (for Fejér means see also Schipp [9]). The author [16] obtained the analogue of the Marcinkiewicz–Zygmund result for the Fejér means of two-dimensional Walsh-Fourier series by proving the weak (L_1,L_1) inequality $\sup_{\rho>0} \rho\lambda(\sigma_*^1 f > \rho) \le C\|f\|_1$. Moreover, the author [16] verified that σ_*^1 is bounded from $H_{p,q}$ to $L_{p,q}$ if $1/2 < p \le \infty$ and $0 < q \le \infty$.

Received by the editors September 16, 1998.
1991 *Mathematics Subject Classification.* Primary 42C10, 43A75; Secondary 60G42, 42B30.
Key words and phrases. Martingale Hardy spaces, p-atom, atomic decomposition, p-quasi-local operator, interpolation, Walsh functions, (C,α) summability.
This research was done while the author was visiting the Humboldt University in Berlin and was supported by the Alexander von Humboldt Foundation.

In this paper we generalize these results for the (C,α) means and for d dimensions. We introduce the conjugate transforms $\tilde{f}^{(t)}$ $(t \in [0,1))$, the conjugate (C,α) means $\tilde{\sigma}_n^{(t);\alpha}$ and the conjugate maximal operator $\tilde{\sigma}_*^{(t);\alpha}$. It will be shown that σ_*^α and $\tilde{\sigma}_*^{(t);\alpha}$ are bounded from $H_{p,q}$ to $L_{p,q}$ for $1/(\alpha_k+1) < p < \infty$ and $0 < q \le \infty$ and are of weak type (L_1, L_1).

A usual density argument implies then that the (C,α) means $\sigma_n^\alpha f$ (resp. the conjugate (C,α) means $\tilde{\sigma}_n^{(t);\alpha} f$) of $f \in L_1$ converge a.e. to f (resp. to $\tilde{f}^{(t)}$) as $n \to \infty$ and $2^{-\tau} \le n_k/n_j \le 2^\tau$. Note that $\tilde{f}^{(t)}$ is not necessarily integrable whenever f is.

We obtain also that the operators σ_n^α and $\tilde{\sigma}_n^{(t);\alpha}$ $(n \in \mathbf{N}^d)$ are uniformly bounded from $H_{p,q}$ to $H_{p,q}$ if $1/(\alpha_k+1) < p < \infty, 0 < q \le \infty$. Hence $\sigma_n^\alpha f \to f$ and $\tilde{\sigma}_n^{(t);\alpha} f \to \tilde{f}^{(t)}$ in $H_{p,q}$ norm as $n \to \infty$ whenever $f \in H_{p,q}$.

2. Dyadic Hardy spaces and conjugate transforms

For a set $\mathbf{X} \ne \emptyset$ let $\mathbf{X}^d$ be its Cartesian product taken with itself d times $(d \in \mathbf{N})$. By a *dyadic interval* we mean one of the form $[k2^{-n}, (k+1)2^{-n})$ for some $k, n \in \mathbf{N}$, $0 \le k < 2^n$. If $I_1, \ldots, I_d$ are dyadic intervals and $\lambda(I_1) = \ldots = \lambda(I_d)$, then the set $I := I_1 \times \ldots \times I_d$ is called a *dyadic cube*. The σ-algebra generated by the dyadic cubes of length $2^{-n} \times \ldots \times 2^{-n}$ will be denoted by $\mathcal{F}_n$ $(n \in \mathbf{N})$.

The expectation and the conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E and E_n, respectively. We briefly write $L_{p,q}$ instead of the Lorentz space $L_{p,q}([0,1)^d, \lambda)$, where λ is the Lebesgue measure (for the exact definition see e.g. Weisz [16]).

We investigate the class of one-parameter martingales $f = (f_n, n \in \mathbf{N})$ with respect to $(\mathcal{F}_n, n \in \mathbf{N})$. The *(diagonal) maximal function* and the *quadratic variation* of a martingale f is defined by

$$f^* := \sup_{n \in \mathbf{N}} |f_n|, \qquad S(f) := (\sum_{n=0}^{\infty} |f_n - f_{n-1}|^2)^{1/2},$$

respectively. For $0 < p, q \le \infty$ the *martingale Hardy-Lorentz space* $H_{p,q}$ consists of all martingales for which

$$\|f\|_{H_{p,q}} := \|f^*\|_{p,q} < \infty.$$

Note that in case $p = q$, the usual definition of Hardy space $H_{p,p} = H_p$ is obtained.

Recall that $L_1 \subset H_{1,\infty}$, more exactly,

$$\|f\|_{H_{1,\infty}} := \sup_{\rho > 0} \rho \lambda(f^* > \rho) \le C \|f\|_1 \qquad (f \in L_1) \tag{1}$$

(see Neveu [7]). It was verified in Weisz [14] that

$$\|f^*\|_{p,q} \sim \|S(f)\|_{p,q} \qquad (0 < p < \infty, 0 < q \le \infty),$$

$$H_{p,q} \sim L_{p,q} \qquad (1 < p < \infty, 0 < q \le \infty),$$

where $\sim$ denotes the equivalence of the norms and spaces.

The following interpolation result concerning Hardy-Lorentz spaces will be used several times in this paper (see Weisz [14]).

Theorem A. *If a sublinear (resp. linear) operator T is bounded from H_{p_0} to L_{p_0} (resp. to H_{p_0}) and from L_{p_1} to L_{p_1} $(p_0 \le 1 < p_1 \le \infty)$, then it is also bounded from $H_{p,q}$ to $L_{p,q}$ (resp. to $H_{p,q}$) if $p_0 < p < p_1$ and $0 < q \le \infty$.*

Every point $x \in [0,1)$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}}, \qquad 0 \le x_k < 2,\ x_k \in \mathbf{N}.$$

In case there are two different forms, we choose the one for which $\lim_{k\to\infty} x_k = 0$.

The functions

$$r_n(x) := \exp(\pi x_n \sqrt{-1}) \qquad (n \in \mathbf{N})$$

are called *Rademacher functions*.

For a martingale $f \sim \sum_{n=0}^{\infty}(f_n - f_{n-1})$ the conjugate transforms are defined by the martingale

$$\tilde{f}^{(t)} \sim \sum_{n=0}^{\infty} r_n(t)(f_n - f_{n-1}),$$

where $t \in [0,1)$ is fixed. Note that $\tilde{f}^{(0)} = f$. As is well known, if f is an integrable function, then the conjugate transforms $\tilde{f}^{(t)}$ do exist almost everywhere, but they are not integrable in general.

Using the quadratic variation we can easily see that

$$\|\tilde{f}^{(t)}\|_{H_p} = \|f\|_{H_p} \qquad (0 < p < \infty; t \in [0,1)). \tag{2}$$

Furthermore, Khintchine's inequality (see e.g. Paley [8] or Weisz [14]) implies that

$$\|f\|_{H_p}^p \sim \int_0^1 \|\tilde{f}^{(t)}\|_p^p \, dt \qquad (0 < p < \infty). \tag{3}$$

3. (C,α) SUMMABILITY

The product system generated by the Rademacher functions is the *one-dimensional Walsh system*

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k},$$

where $n = \sum_{k=0}^{\infty} n_k 2^k$, $0 \le n_k < 2$ and $n_k \in \mathbf{N}$.

The Kronecker product $(w_n; n \in \mathbf{N}^d)$ of d Walsh systems is said to be the *d-dimensional Walsh system*. Thus

$$w_n(x) := w_{n_1}(x_1) \cdots w_{n_d}(x_d),$$

where $n = (n_1, \ldots, n_d) \in \mathbf{N}^d$, $x = (x_1, \ldots, x_d) \in [0,1)^d$.

Recall that the *Walsh-Dirichlet kernels* $D_k := \sum_{j=0}^{k-1} w_j$ satisfy

$$D_{2^k}(x) = \begin{cases} 2^k & \text{if } x \in [0, 2^{-k}), \\ 0 & \text{if } x \in [2^{-k}, 1) \end{cases} \tag{4}$$

for $k \in \mathbf{N}$ (see Fine [1]).

If $f \in L_1$, then the number $\hat{f}(n) := E(f w_n)$ $(n \in \mathbf{N}^d)$ is said to be the nth *Walsh-Fourier coefficient* of f. We can extend this definition to martingales in the usual way (see Weisz [16]).

Denote by $s_n f$ the nth partial sum of the Walsh-Fourier series of a martingale f, namely,

$$s_n f := \sum_{j=1}^{d} \sum_{k_j=0}^{n_j-1} \hat{f}(k) w_k.$$

It is easy to see that $s_{2^m,\dots,2^m} f = f_m$. For $k, n \in \mathbf{N}^d$ we say that $k \leq n$ if $k_1 \leq n_1, \dots, k_d \leq n_d$. If $k_1 < n_1, \dots, k_d < n_d$, then we write $k \ll n$. Let

$$\rho_0 := r_0, \qquad \rho_k := r_j \quad \text{if} \quad (2^{j-1}, \dots, 2^{j-1}) \not\gg k \ll (2^j, \dots, 2^j),$$

where $k \in \mathbf{N}^d$, $j \in \mathbf{N}$. Then the nth partial sum of the conjugate transforms is given by

$$\tilde{s}_n^{(t)} f := \sum_{j=1}^{d} \sum_{k_j=0}^{n_j-1} \rho_k(t) \hat{f}(k) w_k = s_n \tilde{f}^{(t)}.$$

Let $\alpha = (\alpha_1, \dots, \alpha_d)$ with $0 < \alpha_k \leq 1$ $(k = 1, \dots, d)$ and let

$$A_j^\gamma := \binom{j+\gamma}{j} = \frac{(\gamma+1)(\gamma+2)\dots(\gamma+j)}{j!} \qquad (j \in \mathbf{N}; \gamma \neq -1, -2, \dots).$$

It is known that $A_j^\gamma \sim O(j^\gamma)$ $(j \in \mathbf{N})$ (see Zygmund [18]). The (C, α) *means* of a martingale f are defined by

$$\sigma_n^\alpha f := \frac{1}{\prod_{i=1}^d A_{n_i-1}^{\alpha_i}} \sum_{j=1}^{d} \sum_{k_j=1}^{n_j} A_{n_j-k_j}^{\alpha_j-1} s_k f.$$

It is simple to show that

$$\sigma_n^\alpha f(x) = \int_{[0,1)^d} f(t) (K_{n_1}^{\alpha_1}(x_1 \dot{+} t_1) \cdots K_{n_d}^{\alpha_d}(x_d \dot{+} t_d)) \, dt \qquad (n \in \mathbf{N}^d) \tag{5}$$

if $f \in L_1$, where the (C, γ) kernel K_m^γ $(m \in \mathbf{N}, \gamma > 0)$ is defined by

$$K_m^\gamma(y) = \frac{1}{A_{m-1}^\gamma} \sum_{k=1}^{m} A_{m-k}^{\gamma-1} D_k(y) \qquad (y \in [0,1)). \tag{6}$$

Note that $\dot{+}$ denotes the dyadic addition; for the definition, see e.g. Schipp, Wade, Simon, Pál [11].

Every $m \in \mathbf{N}$ can be written in the form $m = 2^{m_1} + 2^{m_2} + \dots + 2^{m_r}$ with $m_1 > m_2 > \dots > m_r \geq 0$. Using Yano's [17] estimate for the (C, γ) kernel and (8) from Weisz [16] we can show that

(7)

$$|K_m^\gamma(x)| \leq C m^{-\gamma} \sum_{k=1}^{r} \sum_{j=0}^{m_k-1} \sum_{i=j}^{m_k-1} 2^{i(\gamma-1)} 2^j D_{2^i}(x \dot{+} 2^{-j-1}) + C m^{-\gamma} \sum_{k=1}^{r} 2^{m_k \gamma} D_{2^{m_k}}(x)$$

for $0 < \gamma \leq 1$ (cf. e.g. Weisz [16, pp. 66-67]). From this and (4) it follows immediately that

$$\int_0^1 |K_m^\gamma| \, d\lambda \leq C \qquad (m \in \mathbf{N}; 0 < \gamma \leq 1). \tag{8}$$

In this paper the constants C depend only on α and the constants C_p (resp. $C_{p,q}$) depend only on p and α (resp. p, q and α) and may denote different constants in different contexts.

The *conjugate* (C,α) *means* of a martingale f are introduced by

$$\tilde{\sigma}_n^{(t);\alpha} f := \frac{1}{\prod_{i=1}^d A_{n_i-1}^{\alpha_i}} \sum_{j=1}^d \sum_{k_j=1}^{n_j} A_{n_j-k_j}^{\alpha_j-1} \tilde{s}_k^{(t)} f \qquad (t \in [0,1); n \in \mathbf{N}^d).$$

For a given $\tau \geq 0$ the *maximal operator* and the *conjugate maximal operator* are defined by

$$\sigma_*^\alpha f := \sup_{\substack{2^{-\tau} \leq n_i/n_j \leq 2^\tau \\ i,j=1,\ldots,d}} |\sigma_n^\alpha f|, \qquad \tilde{\sigma}_*^{(t);\alpha} f := \sup_{\substack{2^{-\tau} \leq n_i/n_j \leq 2^\tau \\ i,j=1,\ldots,d}} |\tilde{\sigma}_n^{(t);\alpha} f|.$$

4. The boundedness of the maximal (C,α) operator on H_p

A bounded measurable function a is a *p-atom* if there exists a dyadic cube I such that

(i) $\int_I a \, d\lambda = 0$,
(ii) $\|a\|_\infty \leq \lambda(I)^{-1/p}$,
(iii) $\{a \neq 0\} \subset I$.

For each dyadic interval I let I^s $(s \in \mathbf{N})$ be the dyadic interval for which $I \subset I^s$ and $\lambda(I^s) = 2^s \lambda(I)$. If $I := I_1 \times \ldots \times I_d$ is a dyadic cube, then set $I^s := I_1^s \times \ldots \times I_d^s$.

An operator T which maps the set of martingales into the collection of measurable functions will be called *p-quasi-local* if there exists $s \in \mathbf{N}$ such that

$$\int_{[0,1)^d \setminus I^s} |Ta|^p \, d\lambda \leq C_p$$

for every p-atom a where I is the support of the atom. The quasi-local operators were defined first in Móricz, Schipp and Wade [6] only for $p = 1$ and for L_1 functions instead of atoms. The following result can be found in Weisz [16]:

Theorem B. *Suppose that the operator T is sublinear and p-quasi-local for some $0 < p \leq 1$. If T is bounded from L_{p_1} to L_{p_1} for a fixed $1 < p_1 \leq \infty$, then*

$$\|Tf\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p).$$

Now we can formulate our main result.

Theorem 1. *Suppose that* $\max\{1/(\alpha_j+1), j = 1, \ldots, d\} =: p_0 < p < \infty$, $0 < q \leq \infty$ *and* $0 < \alpha_j \leq 1$ $(j = 1, \ldots, d)$. *Then*

$$\|\sigma_*^\alpha f\|_{p,q} \leq C_{p,q} \|f\|_{H_{p,q}} \qquad (f \in H_{p,q}). \tag{9}$$

In particular, if $f \in L_1$, then

$$\lambda(\sigma_*^\alpha f > \rho) \leq \frac{C}{\rho} \|f\|_1 \qquad (\rho > 0). \tag{10}$$

For the proof we need the following lemma.

Lemma. *If $1 \leq s \leq K$, $0 < \gamma \leq 1$ and $1/(\gamma+1) < p \leq 1$, then*

$$\int_{2^{-K+s}}^1 \sup_{m \geq 2^{K-s}} \left(\int_0^{2^{-K}} |K_m^\gamma(x \dot{+} t)| \, dt\right)^p dx \leq C_p 2^{-K},$$

where C_p depends on s, p and γ.

Proof. If $j \geq K-s$ and $x \notin [0, 2^{-K+s})$, then $x \dot{+} 2^{-j-1} \notin [0, 2^{-K+s})$. Thus

$$\int_0^{2^{-K}} D_{2^j}(x \dot{+} t)\, dt = \int_0^{2^{-K}} D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1})\, dt = 0$$

for $x \notin [0, 2^{-K+s})$ and $i \geq j \geq K-s$. Applying (7) we conclude

$$\begin{aligned}
\int_0^{2^{-K}} |K_m^\gamma(x \dot{+} t)|\, dt &\leq C m^{-\gamma} \sum_{\substack{k=1 \\ m_k < K-s}}^{r} \sum_{j=0}^{m_k-1} \sum_{i=j}^{m_k-1} 2^{i(\gamma-1)} 2^j \int_0^{2^{-K}} D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1})\, dt \\
&\quad + C m^{-\gamma} \sum_{\substack{k=1 \\ m_k \geq K-s}}^{r} \sum_{j=0}^{K-s-1} \sum_{i=j}^{K-1} 2^{i(\gamma-1)} 2^j \int_0^{2^{-K}} D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1})\, dt \\
&\quad + C m^{-\gamma} \sum_{\substack{k=1 \\ m_k \geq K-s}}^{r} \sum_{j=0}^{K-s-1} \sum_{i=K}^{\infty} 2^{i(\gamma-1)} 2^j \int_0^{2^{-K}} D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1})\, dt \\
&\quad + C m^{-\gamma} \sum_{\substack{k=1 \\ m_k < K-s}}^{r} 2^{m_k \gamma} \int_0^{2^{-K}} D_{2^{m_k}}(x \dot{+} t)\, dt \\
&= (A_m) + (B_m) + (C_m) + (D_m).
\end{aligned}$$

The equality

$$\int_0^{2^{-K}} D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1})\, dt = 2^{i-K} 1_{[2^{-j-1}, 2^{-j-1} \dot{+} 2^{-i})}(x) \qquad (j \leq i \leq K-1)$$

(see e.g. Weisz [16]) implies that

$$(A_m) \leq C 2^{(-K+s)\gamma} \sum_{l=1}^{K-s-1} \sum_{j=0}^{l-1} \sum_{i=j}^{K-1} 2^{i(\gamma-1)} 2^j 2^{i-K} 1_{[2^{-j-1}, 2^{-j-1} \dot{+} 2^{-i})}(x).$$

Consequently, if $p > 1/(\gamma+1)$, then

$$\begin{aligned}
\int_{2^{-K+s}}^1 \sup_{m \geq 2^{K-s}} (A_m)^p\, d\lambda &\leq C_p 2^{-K\gamma p - Kp} \sum_{l=1}^{K-s-1} \sum_{j=0}^{l-1} \sum_{i=j}^{K-1} 2^{i(\gamma p - 1)} 2^{jp} \\
&\leq C_p 2^{-K\gamma p - Kp} \sum_{l=1}^{K-s-1} \sum_{j=0}^{l-1} 2^{j(\gamma p + p - 1)} \leq C_p 2^{-K}.
\end{aligned}$$

Using the fact that the function $f(x) := x 2^{-x}$ is bounded for $x \geq 1$, we obtain

$$\begin{aligned}
(B_m) &\leq C 2^{-m_1 \gamma}(m_1 + s + 1 - K) \sum_{j=0}^{K-s-1} \sum_{i=j}^{K-1} 2^{i(\gamma-1)} 2^j \int_0^1 D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1})\, dt \\
&\leq C 2^{-K\gamma} \sum_{j=0}^{K-s-1} \sum_{i=j}^{K-1} 2^{i(\gamma-1)} 2^j 2^{i-K} 1_{[2^{-j-1}, 2^{-j-1} \dot{+} 2^{-i})}(x).
\end{aligned}$$

The estimation can be finished in the same way as for (A_m) above.

Consider (C_m) and now use the equality

$$\int_0^{2^{-K}} D_{2^i}(x\dot{+}t\dot{+}2^{-j-1})\,dt = 1_{[2^{-j-1},2^{-j-1}\dot{+}2^{-K})}(x) \qquad (i \geq K).$$

Then

$$(C_m) \leq C2^{-K\gamma} \sum_{j=0}^{K-s-1} \sum_{i=K}^{\infty} 2^{i(\gamma-1)} 2^j 1_{[2^{-j-1},2^{-j-1}\dot{+}2^{-K})}(x)$$

and

$$\int_{2^{-K+s}}^1 \sup_{m\geq 2^{K-s}} (C_m)^p\,d\lambda \leq C_p 2^{-K\gamma p} \sum_{j=0}^{K-s-1} \sum_{i=K}^{\infty} 2^{i(\gamma-1)p} 2^{jp} 2^{-K} \leq C_p 2^{-K}$$

whenever $\gamma < 1$. For $\gamma = 1$ the proof can be found in [16].

Finally, by

$$\int_0^{2^{-K}} D_{2^l}(x\dot{+}t)\,dt = 2^{l-K} 1_{[2^{-K},2^{-l})}(x) \qquad (l \in \mathbf{N})$$

we have

$$(D_m) \leq C2^{-K\gamma} \sum_{l=0}^{K-s-1} 2^{l\gamma} \int_0^{2^{-K}} D_{2^l}(x\dot{+}t)\,dt \leq C2^{-K\gamma} \sum_{l=0}^{K-s-1} 2^{l\gamma} 2^{l-K} 1_{[2^{-K},2^{-l})}(x).$$

Hence

$$\int_{2^{-K+s}}^1 \sup_{m\geq 2^{K-s}} (D_m)^p\,d\lambda \leq C_p 2^{-K\gamma p - Kp} \sum_{l=0}^{K-s-1} 2^{lp(\gamma+1)} 2^{-l} \leq C_p 2^{-K},$$

which shows the lemma. □

Proof of Theorem 1. By Theorems A and B the proof of Theorem 1 will be complete if we show that the operator σ_*^α is p-quasi-local for each $p_0 < p \leq 1$ and is bounded from L_∞ to L_∞.

The boundedness follows from (8). Let a be an arbitrary p-atom with support $I = I_1 \times \ldots \times I_d$ and $\lambda(I_j) = 2^{-K}$ $(j = 1, \ldots, d; K \in \mathbf{N})$. We can assume that $I_j = [0, 2^{-K})$ $(j = 1, \ldots, d)$. It is easy to see that $\hat{a}(n) = 0$ if $n_j < 2^K$ for all $j = 1, \ldots, d$. In this case $\sigma_n^\alpha a = 0$. Therefore we can suppose that $n_j \geq 2^K$ for at least one j. Choose $s \in \mathbf{N}$ such that $s - 1 < \tau \leq s$. If e.g. $n_1 \geq 2^K$, then, by the hypothesis, $n_j \geq 2^{-\tau} n_1 \geq 2^{K-s}$ $(j = 1, \ldots, d)$.

To prove the quasi-locality of σ_*^α we have to integrate $|\sigma_*^\alpha a|^p$ over $[0,1)^d \setminus I^s$. Obviously, it is enough to integrate over

$$([0,1) \setminus I_1^s) \times \ldots ([0,1) \setminus I_k^s) \times I_{k+1}^s \times \ldots \times I_d^s \quad \text{for} \quad k = 1, \ldots, d.$$

Using (5), (8) and the definition of the atom we can see that

$$\begin{aligned} |\sigma_n^\alpha a(x)| &\leq \int_{I_1\times\ldots\times I_d} |a(t)|(|K_{n_1}^{\alpha_1}(x_1\dot{+}t_1)| \times \cdots \times |K_{n_d}^{\alpha_d}(x_d\dot{+}t_d)|)\,dt \\ &\leq C2^{dK/p} \prod_{j=1}^{k} \int_{I_j} |K_{n_j}^{\alpha_j}(x_j\dot{+}t_j)|\,dt_j. \end{aligned}$$

The Lemma implies that

$$\int_{([0,1)\setminus I_1^s)\times\ldots([0,1)\setminus I_k^s)\times I_{k+1}^s\times\ldots\times I_d^s} |\sigma_*^\alpha a(x)|^p\,dx \le C_p 2^{dK} 2^{-kK} 2^{-(d-k)K} = C_p,$$

which verifies that σ_*^α is p-quasi-local for each $p_0 < p \le 1$. Hence (9) for $p = q$ follows from Theorem B. Applying Theorem A we obtain (9). Let us point out this result for $p = 1$ and $q = \infty$. If $f \in L_1$, then (1) implies

$$\|\sigma_*^\alpha f\|_{1,\infty} = \sup_{\rho>0} \rho\lambda(\sigma_*^\alpha f > \rho) \le C\|f\|_{H_{1,\infty}} \le C\|f\|_1,$$

which shows (10). The proof of the theorem is complete. □

This theorem for $\alpha_1 = \alpha_2 = 1$ and for two-dimensional functions is due to the author [16]. We recall that in the one-dimensional case (9) was shown by Fujii [3] for $\alpha = p = q = 1$ (see also Simon [12] and Schipp, Simon [10]) and (10) by Schipp [9] for $\alpha = 1$.

We can state the same for the conjugate maximal operator of the (C, α) means.

Theorem 2. *Assume that $t \in [0, 1)$, $p_0 < p < \infty$, $0 < q \le \infty$ and $0 < \alpha_j \le 1$ $(j = 1, \ldots, d)$. Then*

$$\|\tilde\sigma_*^{(t);\alpha} f\|_{p,q} \le C_{p,q}\|f\|_{H_{p,q}} \qquad (f \in H_{p,q})$$

for every $p_0 < p < \infty$ and $0 < q \le \infty$. In particular, if $f \in L_1$, then

$$\lambda(\tilde\sigma_*^{(t);\alpha} f > \rho) \le \frac{C}{\rho}\|f\|_1 \qquad (\rho > 0).$$

Proof. By Theorem 1 for $p = q$ and (2) we obtain

$$\|\tilde\sigma_*^{(t);\alpha} f\|_p = \|\sigma_*^\alpha \tilde f^{(t)}\|_p \le C_p\|\tilde f^{(t)}\|_{H_p} = C_p\|f\|_{H_p} \qquad (f \in H_p)$$

for every $p_0 < p < \infty$. Now Theorem 2 follows from Theorem A. □

Since the set of the Walsh polynomials is dense in L_1, the weak type inequalities of Theorems 1 and 2 and the usual density argument (see Marcinkiewicz, Zygmund [5]) imply

Corollary 1. *If $0 < \alpha_j \le 1$ $(j = 1, \ldots, d)$ and $f \in L_1$, then $\sigma_n^\alpha f \to f$ a.e. and $\tilde\sigma_n^{(t);\alpha} f \to \tilde f^{(t)}$ a.e. $(t \in [0, 1))$ as $n \to \infty$ whenever $2^{-\tau} \le n_i/n_j \le 2^\tau$ $(i, j = 1, \ldots, d)$.*

Note that $\tilde f^{(t)}$ is not necessarily integrable whenever f is. The first convergence result for Fejér means and for two-dimensional functions is due to the author [16] (see also Gát [4]) and in the one-dimensional case it is due to Fine [2].

Now we consider the norm convergence of $\sigma_n^\alpha f$. It follows from (9) that $\sigma_n^\alpha f \to f$ in L_p norm as $n \to \infty$ if $f \in L_p$ $(1 < p < \infty)$ and $2^{-\tau} \le n_i/n_j \le 2^\tau$ (for Fejér means see also Wade [13]). We are going to generalize this result.

Theorem 3. *If $t \in [0, 1)$, $0 < \alpha_j \le 1$ and $2^{-\tau} \le n_i/n_j \le 2^\tau$ $(i, j = 1, \ldots, d)$, then*

$$\|\tilde\sigma_n^{(t);\alpha} f\|_{H_{p,q}}, \le C_{p,q}\|f\|_{H_{p,q}} \qquad (f \in H_{p,q})$$

whenever $p_0 < p < \infty$ and $0 < q \le \infty$.

Proof. We have by Theorems 1 and 2 that

$$\|(\sigma_n^\alpha f)^{\sim(t)}\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p).$$

It follows from (3) that

$$\|\tilde\sigma_n^{(t);\alpha} f\|_{H_p} \le C_p \|f\|_{H_p} \qquad (f \in H_p).$$

Now Theorem A proves Theorem 3. ☐

Corollary 2. *Suppose that $t \in [0,1)$, $p_0 < p < \infty$, $0 < q \le \infty$ and $0 < \alpha_j \le 1$ $(j = 1,\dots,d)$. If $f \in H_{p,q}$, then $\tilde\sigma_n^{(t);\alpha} f \to \tilde f^{(t)}$ in $H_{p,q}$ norm as $n \to \infty$ and $2^{-\tau} \le n_i/n_j \le 2^\tau$ $(i,j = 1,\dots,d)$.*

We suspect that Theorems 1, 2 and 3 for $p \le p_0$ are not true, although we could not find any counterexample.

References

1. N J Fine, *On the Walsh functions*, Trans. Amer. Math. Soc. **65** (1949), 372–414. MR **11:**3526
2. ———, *Cesàro summability of Walsh-Fourier series*, Proc. Nat. Acad. Sci. USA **41** (1955), 558–591. MR **17:**31f
3. N Fujii, *A maximal inequality for H^1-functions on a generalized Walsh-Paley group*, Proc. Amer. Math. Soc. **77** (1979), 111–116. MR **81b:**42070
4. Gy Gát, *Pointwise convergence of the Cesàro means of double walsh series*, Ann. Univ. Sci. Budapest Sect. Comp. **16** (1996), 173–184. MR **99b:**42033
5. J Marcinkiewicz and A Zygmund, *On the summability of double Fourier series*, Fund. Math. **32** (1939), 122–132.
6. F Móricz, F Schipp, and W R Wade, *Cesàro summability of double Walsh-Fourier series*, Trans. Amer. Math. Soc. **329** (1992), 131–140. MR **92j:**42028
7. J Neveu, *Discrete-parameter martingales*, North-Holland, 1971. MR **53:**6729
8. R E A C Paley, *A remarkable sytem of orthogonal functions*, Proc. Lond. Math. Soc. **34** (1932), 241–279.
9. F Schipp, *Über gewissen Maximaloperatoren*, Ann. Univ. Sci. Budapest Sect. Math. **18** (1975), 189–195. MR **55:**3670
10. F Schipp and P Simon, *On some (H, L_1)-type maximal inequalities with respect to the Walsh-paley sytem*, Functions, Series, Operators, Proc. Conf. in Budapest, 1980, North Holland, Amsterdam, 1981, pp. 1039–1045. MR **86a:**42032
11. F Schipp, W R Wade, P Simon, and J Pál, *Walsh series: An introduction to dyadic harmonic analysis*, Akadémiai Kiadó, Budapest, 1990. MR **92g:**42001
12. P Simon, *Investigations with respect to the Vilenkin system*, Ann. Univ. Sci. Budapest Sect. Math. **27** (1985), 87–101. MR **87b:**42032
13. W R Wade, *A growth estimate for Cesàro partial sums of multiple Walsh-Fourier series*, Coll. Math. Soc. J. Bolyai 49, Alfred Haar Memorial Conference, Budapest (Hungary), 1985, North-Holland, Amsterdam, 1986, pp. 975–991. MR **89f:**42026
14. F Weisz, *Martingale Hardy spaces and their applications in Fourier-analysis*, Lecture Notes in Math., vol. 1568, Springer, Berlin, 1994. MR **96m:**60108
15. ———, *Cesàro summability of multi-dimensional trigonometric-Fourier series*, J. Math. Anal. Appl. **204** (1996), 419–431. MR **98g:**42022
16. ———, *Cesàro summability of two-dimensional Walsh-Fourier series*, Trans. Amer. Math. Soc. **348** (1996), 2169–2181. MR **96i:**42004
17. S Yano, *On approximations by Walsh functions*, Proc. Amer. Math. Soc. **2** (1951), 962–967. MR **13:**549g
18. A Zygmund, *Trigonometric series*, Cambridge Press, London, 1959. MR **21:**6498

Department of Numerical Analysis, Eötvös L. University, H-1117 Budapest, Pázmány P. sétány 1/D, Hungary

E-mail address: `weisz@ludens.elte.hu`

Current address: Department of Mathematics, Humboldt University, D-10099 Berlin, Unter den Linden 6, Germany