

## ADJACENCY PRESERVING MAPPINGS OF INVARIANT SUBSPACES OF A NULL SYSTEM

WEN-LING HUANG

(Communicated by Christopher Croke)

**ABSTRACT.** In the space  $I_r$  of invariant  $r$ -dimensional subspaces of a null system in  $(2r + 1)$ -dimensional projective space, W.L. Chow characterized the basic group of transformations as all the bijections  $\varphi : I_r \rightarrow I_r$ , for which both  $\varphi$  and  $\varphi^{-1}$  preserve adjacency. In the present paper we show that the two conditions  $\varphi : I_r \rightarrow I_r$  is a surjection and  $\varphi$  preserves adjacency are sufficient to characterize the basic group. At the end of this paper we give an application to Lie geometry.

### 1. INTRODUCTION

Let  $n, r$  be positive integers,  $3 \leq n = 2r + 1$ . Let  $\Pi$  be an arbitrary  $n$ -dimensional Pappian projective space. A *null system*  $\delta$  on  $\Pi$  is a polarity on  $\Pi$  which satisfies  $x \in x^\delta$  for every point  $x$  of  $\Pi$ . The space of the  $r$ -dimensional subspaces of  $\Pi$  which are invariant under a fixed null system  $\delta$  will be denoted by  $I_r := \{a \in [r] \mid a^\delta = a\}$ , where  $[k]$ ,  $-1 \leq k \leq n$ , is the set of all  $k$ -dimensional subspaces of  $\Pi$ .

The *basic group of transformations* in the space  $I_r$  (also called the group of *semi-symplectic* transformations) consists of the transformations induced by all the collineations  $f$  of  $\Pi$  which satisfy  $\delta f = f\delta$ . Two invariant  $r$ -dimensional subspaces  $a, b$  are *at distance*  $d$ , if their intersection is  $(r - d)$ -dimensional. If  $d = 1$ , then they are called *adjacent*.

W.L. Chow [4] has shown that any bijection  $\varphi : I_r \rightarrow I_r$  for which both  $\varphi$  and  $\varphi^{-1}$  preserve adjacency is induced by a collineation of  $\Pi$ . Observably, any collineation  $\varphi$  of  $\Pi$  with  $\delta\varphi = \varphi\delta$  preserves adjacency in both directions. From a different point of view, L.K. Hua [5], [6] proved the fundamental theorem in the geometry of symmetric matrices under further hypotheses. For a brief history of the development of this problem see Wan [9], [10]. We may consider the theorem of Chow as a Beckman-Quarles type theorem [8] on distance preserving mappings of the space  $I_r$ . Thus Chow's theorem may be seen as an early result in the discipline *characterizations of geometrical mappings under mild hypotheses* [3].

In the present paper we characterize the basic group under mild hypotheses:

**Theorem.** *Let  $r, n \in \mathbb{N}$ ,  $3 \leq n = 2r + 1$ . Let  $I_r$  be the space of all invariant  $r$ -dimensional subspaces of a null system  $\delta$  in an  $n$ -dimensional Pappian projective*

---

Received by the editors September 25, 1998.

1991 *Mathematics Subject Classification.* Primary 51A50; Secondary 51B25.

*Key words and phrases.* Null system, adjacency preserving mappings, symmetric matrices, Lie transformations.

space  $\Pi$ . Let  $\varphi : I_r \rightarrow I_r$  be a surjection satisfying

$$(1.1) \quad \text{if } a, b \text{ are adjacent, then } a^\varphi, b^\varphi \text{ are adjacent}$$

for all  $a, b \in I_r$ . Then  $\varphi$  is a transformation of the basic group of  $I_r$ .

## 2. PRELIMINARIES

In this paper, by dimension, intersection, and subspace we understand *projective* dimension, intersection, and subspace.

Let  $n, r$  be integers,  $3 \leq n = 2r + 1$ . Let  $\Pi$  be an arbitrary  $n$ -dimensional Pappian projective space, and let  $\delta$  be a null system on  $\Pi$ . For any subspaces  $a, b$  of  $\Pi$ , we have the following well-known properties:  $\dim a + \dim a^\delta = 2r$ ,  $(a + b)^\delta = a^\delta \cap b^\delta$ ,  $(a \cap b)^\delta = a^\delta + b^\delta$ , and  $a \subset b$  implies  $b^\delta \subset a^\delta$ . For  $-1 \leq k \leq n$  let  $[k]$  be the set of all  $k$ -dimensional subspaces of  $\Pi$ . For each  $a \in [k]$ , we call  $\bar{a} := a^\delta \in [2r - k]$  the *conjugate* of  $a$ . An element  $a \in [k]$  is called *invariant*, if  $a \subset \bar{a}$  or  $\bar{a} \subset a$ . Let

$$I_k := \{a \in [k] \mid a \text{ is invariant}\}.$$

Two elements  $a, b \in I_r$  are called *adjacent* if their intersection has dimension  $r - 1$ . Let  $a, b \in I_r$ . The *distance* between  $a$  and  $b$  is defined to be  $d(a, b) := r - \dim(a \cap b)$ . If  $a \neq b$ , then  $d = d(a, b)$  is the smallest positive integer with the property that there exists a sequence of  $d + 1$  invariant and consecutively adjacent subspaces  $a_1 = a, a_2, \dots, a_{d+1} = b \in I_r$  (see [4]). From this property we obtain the triangle inequality  $d(a, c) \leq d(a, b) + d(b, c)$  for all  $a, b, c \in I_r$ . Let  $P \in I_{r-1}$ ,  $a \in I_r$ . We define the distance between  $P$  and  $a$  by

$$d(P, a) := \min\{d(a, b) \mid b \in P^*\} = r - \dim(P \cap a) - 1$$

where  $P^* := \{l \in I_r \mid P \subset l\}$ .

A subset  $M$  of  $I_r$  is called a *maximal set* of adjacent elements in  $I_r$  if any two distinct elements of  $M$  are adjacent and if there are no other elements of  $I_r$  which are adjacent to each element of  $M$ .

**Lemma 2.1.** *A set  $M \subset I_r$  is maximal iff there is a  $P \in I_{r-1}$  with  $M = P^*$ .*

*Proof.* Assume there exists a “triangle”  $a, b, c \in I_r$ , such that  $a, b$ , resp.  $b, c$ , resp.  $c, a$ , are adjacent and  $P := a \cap b \neq b \cap c =: Q$ . Then  $H := a + b = b + c = c + a \in [r + 1]$ . Since  $P \subset a$ , we have  $a = a^\delta \subset P^\delta \in I_{r+1}$  and  $H = a + b = P^\delta$ . Similarly,  $H = b + c = Q^\delta$ , i.e.  $P^\delta = Q^\delta$  in contradiction to that  $\delta$  is a one-to-one correspondence. □

Let  $\varphi : I_r \rightarrow I_r$  be a mapping which satisfies (1.1). Then  $(P^*)^\varphi$  is contained in a unique maximal set  $Q^*$ . We define

$$(2.1) \quad P^\varphi := Q \quad \text{if} \quad (P^*)^\varphi \subset Q^*.$$

Furthermore, following the definition of distance, for any  $a, b \in I_r$  we have

$$(2.2) \quad d(a, b) \geq d(a^\varphi, b^\varphi).$$

**Lemma 2.2.** *For any invariant subspaces  $x \in I_s, y \in I_t, t, s \leq r$ , the subspace  $x + (\bar{x} \cap y)$  is invariant and has dimension  $\leq r$ . In the case  $r = t, x + (\bar{x} \cap y)$  has dimension  $r$ .*

**Lemma 2.3.** *For any  $a \in I_r$  there exists  $b \in I_r$  with  $a \cap b = \emptyset$ , i.e.  $d(a, b) = r + 1$ .*

For a proof of Lemma 2.2 and Lemma 2.3 see [4].

**Lemma 2.4.** *For any  $a, b \in I_r$ ,  $a \neq b$ , there exists  $c \in I_r$  with  $d(a, c) = r + 1$  and  $d(b, c) < r + 1$ .*

*Proof.* Let  $l \in I_r$  with  $a \cap l = \emptyset$ . If  $b \cap l \neq \emptyset$ , then take  $c := l$ . Suppose  $l$  satisfies  $l \cap b = \emptyset$ . Let  $k := \dim(a \cap b) < r$ . Let  $s$  be an  $(r - k - 1)$ -dimensional subspace of  $b$  with  $s + (a \cap b) = b$ . Following Lemma 2.2,  $c := s + (\overline{s} \cap l)$  is an invariant element with dimension  $r$ . It is clear that  $b \cap c = s$  and  $a \cap b \cap c = \emptyset$ . Suppose  $d(a, c) \neq r + 1$ . There is an  $x \in a \cap c$ . Hence  $x \in c \subset \overline{s}$ , and  $s \subset \overline{x}$ . On the other hand,  $x \in a \subset \overline{x}$  implies  $a \cap b \subset \overline{x}$ ,  $b = (a \cap b) + s \subset \overline{x}$  and  $x \in b$ , a contradiction to  $a \cap b \cap c = \emptyset$ .  $\square$

**Lemma 2.5.** *Let  $P \in I_{r-1}$ ,  $a \in I_r$ . Then  $d(P, a) = k$  iff there is a uniquely determined  $b \in P^*$  with  $d(a, b) = k$  and  $d(a, l) = k + 1$  for any  $l \in P^* \setminus \{b\}$ .*

*Proof.* “ $\Rightarrow$ ”: Let  $b := P + (a \cap \overline{P}) \in I_r$ . Then  $a \cap b = a \cap \overline{P} = \overline{a + P}$ , hence  $\dim(a \cap b) = 2r - \dim(a + P) = r - k$  and  $d(a, b) = k$ . For any  $l \in P^* \setminus \{b\}$  we have  $k \leq d(a, l) \leq d(a, b) + d(b, l) = k + 1$ . Let  $l \in P^*$  with  $d(a, l) = k$ , i.e.  $\dim(a \cap l) = r - k$ . Since  $a \cap l \subset a \cap \overline{P}$  and  $\dim(a \cap l) = \dim(a \cap \overline{P})$ ,  $a \cap \overline{P} = a \cap l \subset l$ . Hence  $l = P + (a \cap \overline{P}) = b$ . “ $\Leftarrow$ ” follows straightforwardly from the definition.  $\square$

**Lemma 2.6.** *Let  $a, b \in I_r$  with  $d(a, b) = k$ . Then*

$$\min\{d(P, a) \mid P \subset b, P \in I_{r-1}\} = k - 1.$$

*Furthermore, for all  $P \subset b$ ,  $P \in I_{r-1}$  we have  $d(P, a) = k - 1$  iff  $a \cap b \subset P$ .*

*Proof.* From Lemma 2.5, for all  $P \subset b$ ,  $P \in I_{r-1}$ ,

$$k = d(a, b) \in \{d(P, a), d(P, a) + 1\} = \{r - 1 - \dim(P \cap a), r - \dim(P \cap a)\}.$$

Thus  $a \cap b \subset P$  is equivalent to  $\dim(P \cap a) = \dim(a \cap b) = r - k$  and to  $d(P, a) = k - 1$ .  $\square$

**Lemma 2.7.** *For any  $a \in I_r$  and any projective point  $x \in a$ , there exists  $b \in I_r$  with  $d(a, b) = r$  and  $a \cap b = \{x\}$ .*

*Proof.* Let  $l \in I_r$ ,  $l \cap a = \emptyset$ . Define  $b := x + (\overline{x} \cap l) \in I_r$ .  $\square$

**Lemma 2.8.** *Let  $P \in I_{r-1}$ ,  $a \in I_r$ . Then for any mapping  $\varphi : I_r \rightarrow I_r$  satisfying (1.1) we have  $d(P, a) \geq d(P^\varphi, a^\varphi)$  where  $P^\varphi$  is defined in (2.1).*

*Proof.* Let  $b \in P^*$  with  $d(a, b) = d(P, a)$ . Then  $b^\varphi \in (P^\varphi)^*$  and

$$(2.3) \quad d(P, a) = d(a, b) \geq d(a^\varphi, b^\varphi) \geq d(P^\varphi, a^\varphi).$$

$\square$

### 3. PROOF OF THE THEOREM

**1.** If there are  $a, b \in I_r$  with  $d(a, b) = r + 1$  and  $d(a^\varphi, b^\varphi) = r$ , then for any  $P \in I_{r-1}$ ,  $P \subset a$  we have  $a^\varphi \cap b^\varphi \subset P^\varphi$ .

*Proof.* For any  $P \in I_{r-1}$ ,  $P \subset a$  we have  $P^\varphi \subset a^\varphi$ . Let  $c \in P^* \setminus \{a\}$  with  $d(b, c) = r$ . Then  $d(b^\varphi, c^\varphi) \leq r$  and  $c^\varphi \in (P^\varphi)^*$ . Since  $a, c$  are adjacent, also  $a^\varphi, c^\varphi$  are adjacent, and we have  $c^\varphi \neq a^\varphi$ . From Lemma 2.5,  $d(P^\varphi, b^\varphi) = r - 1$  and  $a^\varphi \cap b^\varphi \subset P^\varphi$ .  $\square$

**2.** For any  $a, b \in I_r$ ,  $d(a^\varphi, b^\varphi) = r$  implies  $d(a, b) = r$ .

*Proof.* Suppose not; then  $d(a, b) = r + 1$ . Choose  $c_1, \dots, c_r \in I_r$  with  $d(c_i^\varphi, a^\varphi) = r$  and such that the set  $\{a^\varphi \cap c_i^\varphi \mid i = 1, \dots, r\} \cup \{a^\varphi \cap b^\varphi\}$  is a projective basis of  $a^\varphi$ . Then  $r \leq d(c_i, a) \leq r + 1$  for all  $i = 1, \dots, r$ . Choose  $Q \subset a$ ,  $Q \in I_{r-1}$  with  $Q \cap c_i = a \cap c_i$  for all  $i = 1, \dots, r$ . In the case  $d(a, c_i) = r + 1$ , from 1.,  $a^\varphi \cap c_i^\varphi \subset Q^\varphi$ . In the other case  $d(a, c_i) = r$  we have  $d(Q^\varphi, c_i^\varphi) \leq d(Q, c_i) = r - 1$ , so  $a^\varphi \cap c_i^\varphi \subset Q^\varphi$  for all  $i = 1, \dots, r$ . Furthermore,  $a^\varphi \cap b^\varphi \subset Q^\varphi$ . This is a contradiction to that  $\{a^\varphi \cap c_i^\varphi \mid i = 1, \dots, r\} \cup \{a^\varphi \cap b^\varphi\}$  is a basis of  $a^\varphi$ .

**3.** For any  $a \in I_r$ ,  $Q \in I_{r-1}$ ,  $Q \subset a^\varphi$ , there exists  $P \in I_{r-1}$ ,  $P \subset a$  with  $P^\varphi = Q$ .

*Proof.* Choose  $c_1, \dots, c_r \in I_r$  with  $d(a^\varphi, c_i^\varphi) = r$  such that  $\{a^\varphi \cap c_i^\varphi \mid i = 1, \dots, r\}$  is a basis of  $Q$ . Then for any  $P \in I_{r-1}$  with  $P \subset a$ ,  $a \cap c_i \subset P$  implies  $a^\varphi \cap c_i^\varphi \subset P^\varphi$  and  $P^\varphi = Q$ .

**4.** For any  $a, b \in I_r$  with  $d(a, b) = r + 1$ , we have  $d(a^\varphi, b^\varphi) = r + 1$ .

*Proof.* We prove 4. by induction. From 2.,  $d(a^\varphi, b^\varphi) \neq r$ . Let  $d(a^\varphi, b^\varphi) \neq r + 1 - k$  for some  $k \in \{1, \dots, r\}$ . Assume that  $d(a^\varphi, b^\varphi) = r - k$ . Let  $Q \subset a^\varphi$  with  $d(Q, b^\varphi) = r - k$ . Let  $P \subset a$  with  $P^\varphi = Q$ . Choose  $l \in P^* \setminus \{a\}$  with  $d(l, b) = r + 1$ . Then, by Lemma 2.5,  $l^\varphi \in Q^* \setminus \{a^\varphi\}$  implies  $d(l^\varphi, b^\varphi) = r + 1 - k$ , a contradiction. Hence  $d(a^\varphi, b^\varphi) \neq r - k$ .

**5.**  $\varphi$  is injective.

*Proof.* For any  $a \neq b \in I_r$ , from Lemma 2.4 there exists  $c \in I_r$  with  $d(a, c) = r + 1$  and  $d(b, c) < r + 1$ . Since  $d(a^\varphi, c^\varphi) = r + 1$  and  $d(b^\varphi, c^\varphi) \leq d(b, c) < r + 1$ , we have  $a^\varphi \neq b^\varphi$ .

**6.**  $a, b \in I_r$  are adjacent if  $a^\varphi, b^\varphi$  are adjacent.

*Proof.* Choose  $c \in I_r$  with  $d(a^\varphi, c^\varphi) = r + 1$  and  $d(b^\varphi, c^\varphi) = r$ . Denote  $Q := a^\varphi \cap b^\varphi$ . Let  $P \subset a$ ,  $P \in I_{r-1}$  with  $P^\varphi = Q$ . Let  $l \in P^*$  with  $d(l, c) = r$ ; then  $l \neq a$ . Since  $l^\varphi \in Q^*$ ,  $r = d(Q, c^\varphi) \leq d(l^\varphi, c^\varphi) \leq d(l, c) = r$ . Following Lemma 2.5, we have  $l^\varphi = b^\varphi$ .  $\varphi$  is injective, hence  $l = b$ . So  $a$  and  $b$  are adjacent.

**7.**  $\varphi$  is a transformation of the basic group of  $I_r$ .

*Proof.* From 5. and 6.,  $\varphi$  is a bijection of  $I_r$ , and  $\varphi$  and  $\varphi^{-1}$  preserve adjacency of pairs of elements of  $I_r$ . Chow's theorem completes the proof of the theorem.  $\square$

#### 4. APPLICATION TO LIE GEOMETRY

Let  $\mathbf{Q}$  denote the Lie quadric

$$x_1x_2 + x_3x_4 + x_5^2 = 0$$

in the four-dimensional projective space  $\Pi^4(K)$  over a commutative field  $K$ ,  $\text{ch } K \neq 2$ . We call two elements  $X = K(x_1, \dots, x_5)$ ,  $Y = K(y_1, \dots, y_5)$  of  $\mathbf{Q}$  conjugate if

$$X \sim Y \quad :\Leftrightarrow \quad x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3 + 2x_5y_5 = 0.$$

The Lie transformations of  $(\mathbf{Q}, \sim)$  are defined as bijections  $\alpha$  of  $\mathbf{Q}$  satisfying  $X \sim Y$  iff  $X^\alpha \sim Y^\alpha$ . Every Lie transformation is induced by a collineation of  $\Pi^4(K)$  (see e.g. [2]).

On the three-dimensional projective space  $\Pi^3(K)$  we define a null system  $\delta$  by

$$P^\delta = \{Q \mid p_1q_3 - p_3q_1 + p_2q_4 - p_4q_2 = 0\}$$

where  $P = K(p_1, \dots, p_4)$ ,  $Q = K(q_1, \dots, q_4)$ . There is a one-to-one correspondence  $\gamma$  between the space of invariant lines  $I_1$  and the Lie quadric  $\mathbf{Q}$  which satisfies

$$a, b \text{ are adjacent} \quad \Leftrightarrow \quad a^\gamma, b^\gamma \text{ are distinct and conjugate}$$

for all  $a, b \in I_1$ . This transformation  $\gamma$  is defined as follows. For every line  $a$  of  $\Pi^3(K)$  consider the Plücker coordinates  $K(a_{12}, a_{13}, a_{14}, a_{34}, a_{42}, a_{23})$  of  $a$  where  $a_{ij} = p_i q_j - p_j q_i$  for any two distinct points  $P, Q \in a$ ,  $P = K(p_1, \dots, p_4)$ ,  $Q = K(q_1, \dots, q_4)$ . Then  $a_{13} = a_{42}$  if, and only if,  $a = a^\delta$ , i.e.  $a \in I_1$ . Define  $\gamma : I_1 \rightarrow \mathbf{Q}$  by  $a \mapsto K(a_{12}, a_{34}, a_{14}, a_{23}, a_{13})$ .  $\gamma$  is a bijection. Furthermore, any distinct  $a, b \in I_1$  with Plücker coordinates  $K(a_{12}, a_{34}, a_{14}, a_{23}, a_{13})$ ,  $K(b_{12}, b_{34}, b_{14}, b_{23}, b_{13})$  are adjacent if, and only if,

$$\begin{aligned} & a_{12}b_{34} + a_{34}b_{12} + a_{13}b_{42} + a_{42}b_{13} + a_{14}b_{23} + a_{23}b_{14} = 0 \\ \Leftrightarrow & a_{12}b_{34} + a_{34}b_{12} + a_{14}b_{23} + a_{23}b_{14} + 2a_{13}b_{13} = 0 \\ \Leftrightarrow & a^\gamma, b^\gamma \text{ are conjugate.} \end{aligned}$$

In the case  $(r, n) = (1, 3)$ , the theorem implies the following corollary.

**Corollary 4.1.** *Let  $\psi : \mathbf{Q} \rightarrow \mathbf{Q}$  be a surjective mapping which takes pairs of distinct conjugate points of  $\mathbf{Q}$  to pairs of distinct conjugate points. Then  $\psi$  is a Lie transformation.*

#### REFERENCES

- [1] R. Baer. *Linear Algebra and Projective Geometry*. Academic Press, New York, San Francisco, London, 1952. MR **14**:675j
- [2] W. Benz. *Geometrie der Algebren*. Springer-Verlag, Berlin Heidelberg New York, 1973. MR **50**:5623
- [3] W. Benz. *Geometrische Transformationen*. BI Wissenschaftsverlag, Mannheim; Leipzig; Wien; Zürich, 1992. MR **93i**:51002
- [4] W.-L. Chow. On the geometry of algebraic homogeneous spaces. *Ann. Math.*, 50(1):32–67, 1949. MR **10**:396d
- [5] L.K. Hua. Geometries of matrices III. Fundamental theorems in the geometries of symmetric matrices. *Trans. Amer. Math. Soc.*, 61:229–255, 1947. MR **9**:171e
- [6] L.K. Hua. Geometries of symmetric matrices over any field with characteristic other than two. *Ann. Math.*, 50:8–31, 1949. MR **10**:424h
- [7] W.-l. Huang. Adjacency preserving mappings of Grassmann spaces. *Abh. Math. Sem. Univ. Hamburg*, 68:65–77, 1998. CMP 99:05
- [8] J. A. Lester. Distance preserving transformations. In F. Buekenhout, editor, *Handbook of Incidence Geometry*, pages 921–944, Amsterdam, 1995. Elsevier. MR **96j**:51019
- [9] Z.-X. Wan. Geometry of matrices. *Adv. Stud. Pure Math.*, 24:443–453, 1996. MR **97h**:15017
- [10] Z.-X. Wan. *Geometry of matrices*. World Scientific, Singapore, 1996. MR **98a**:51001

MATHEMATISCHES SEMINAR, UNIVERSITÄT HAMBURG, BUNDESSTRASSE 55, 20146 HAMBURG, GERMANY

*E-mail address:* [huang@math.uni-hamburg.de](mailto:huang@math.uni-hamburg.de)