CARDINAL SPLINE INTERPOLATION FROM $H^1(\mathbb{Z})$ TO $L_1(\mathbb{R})$

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Abstract. Let $H^1(\mathbb{Z})$ be the discrete Hardy space, consisting of those sequences $y = \{y_j\}_{j \in \mathbb{Z}} \in l_p(\mathbb{Z})$, such that $Hy = \{Hy_j\} \in l_1(\mathbb{Z})$, where $Hy_j = \sum_{k \neq j} (k - j)^{-1} y_j$, $j \in \mathbb{Z}$, is the discrete Hilbert transform of $y$. For a sequence $y = \{y_j\} \in l_1(\mathbb{Z})$, let $L_m y(x) \in L_p(\mathbb{R})$ be the unique cardinal spline of degree $m - 1$ interpolating to $y$ at the integers. The norm of this operator, $\|L_m\|_1 = \sup_m \|L_m y\|_{L_1(\mathbb{R})} / \|y\|_{l_1(\mathbb{Z})}$, is called a Lebesgue constant from $l_1(\mathbb{Z})$ to $L_1(\mathbb{R})$, and it was proved that $\sup_m \|L_m\|_1 = 1$.

It is proved in this paper that
\[ \sup_m \left\{ \|L_m y\|_{L_1(\mathbb{R})} / \|y\|_{l_1(\mathbb{Z})} + \|y\|_{l_1(\mathbb{Z})} \right\} \leq \left( 1 + \frac{\pi}{2} \right) \left( 1 + \frac{\pi}{3} \right). \]

1. Introduction

Denote the classical Lebesgue space on $\mathbb{R}$ by $L_p(\mathbb{R})$, $1 \leq p \leq \infty$, and let $\| \cdot \|_{L_p(\mathbb{R})}$ denote its norm.

For a natural number $m$, the space $S_{m,p}(\mathbb{R}) = \{s\}$ of cardinal splines of degree $m - 1$ is taken to consist of those functions satisfying:

(i) $s \in C^{m-2}(\mathbb{R})$,
(ii) $\|s\|_{L_p(\mathbb{R})} < \infty$, $1 \leq p \leq \infty$,
(iii) $s$ reduces to a polynomial of degree at most $m - 1$ on each of the intervals $[\nu + m/2, \nu + m/2 + 1]$, $\nu \in \mathbb{Z}$.

For a sequence $y = \{y_j\}_{j \in \mathbb{Z}} \in l_p(\mathbb{Z})$, we define the space of double infinite bounded sequences with the usual norm as follows:

\[
\|y\|_{l_p(\mathbb{Z})} = \left( \sum_{j \in \mathbb{Z}} |y_j|^p \right)^{1/p}, \quad 1 < p < \infty,
\]

\[
\|\{y_j\}\|_{l_1(\mathbb{Z})} := \|\{y_j\}\|_{l_1(\mathbb{Z})},
\]

\[
\|\{y_j\}\|_{l_\infty(\mathbb{Z})} = \sup_{j \in \mathbb{Z}} |y_j|.
\]

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Schoenberg [3] proved that there is a unique element $L_m y \in S_{m,p}(\mathbb{R})$ interpolating the given data at integers, i.e.,

$$L_m y(j) = y_j, \quad j \in \mathbb{Z}. \tag{1.1}$$

The operator $L_m : l_p(\mathbb{Z}) \to S_{m,p}(\mathbb{R})$ is called the cardinal spline interpolation operator of order $m$ from $l_p(\mathbb{Z})$ to $L_p(\mathbb{R})$ and its norm

$$\|L_m\|_p = \sup\{\|L_m y\|_{l_p(\mathbb{R})} : \|y\|_{l_p(\mathbb{Z})} \leq 1\} \tag{1.2}$$

is referred to as the $m$th Lebesgue constant for cardinal spline interpolation. These numbers were investigated previously by many authors (see [6]–[8]).

**Theorem A (6).** Let $1 < p < \infty$. Then

$$\|L_m\|_p \leq C_p, \tag{1.3}$$

where the constant $C_p$ is independent of $m$.

**Theorem B (6).** The norms of the $m$th order cardinal spline interpolation operators from $l_1(\mathbb{Z})$ to $L_1(\mathbb{R})$ satisfy

$$\lim_{m \to \infty} (\|L_m\|_1 - (4/\pi^2) \log m) = (2A/\pi) + 4/\pi^2 [\log(4/\pi) + \gamma], \tag{1.4}$$

where $\gamma$ is the Euler–Mascheroni constant and

$$A = \int_0^\pi t^{-1} \left( \tan\left(\frac{t}{2}\right) - \frac{2}{\pi(\pi - t)} \right) \, dt. \tag{1.5}$$

From Theorem B, we know that $\sup_m \{\|L_m\|_1\} = \infty$.

Let $H^1(\mathbb{Z})$ be the discrete Hardy space, consisting of those double infinite bounded sequences $y = \{y_j\} \in l_1(\mathbb{Z})$, such that $H y = \{H y_j\} \in l_1(\mathbb{Z})$, where

$$H y_j = \sum_{k \neq j} y_j \frac{y_k}{k - j}, \quad j \in \mathbb{Z}, \tag{1.6}$$

is the discrete Hilbert transform of $y$. Thus $H^1(\mathbb{Z})$ is the subspace of $l_1(\mathbb{Z})$ consisting of those sequences $y = \{y_j\}$ for which the discrete Hilbert transform also belongs to $l_1(\mathbb{Z})$. Clearly

$$\|\{y_j\}\|_{H^1(\mathbb{Z})} := \|\{y_j\}\|_{l_1(\mathbb{Z})} + \|\{H y_j\}\|_{l_1(\mathbb{Z})} \tag{1.7}$$

is a norm of $H^1(\mathbb{Z})$.

$H^1(\mathbb{Z})$ was introduced by Coifman and Weiss [3, p. 622] as an important example of the Hardy space $H^1(\mathbb{X})$, associated with a space $\mathbb{X}$ of homogeneous type, in order to extend the atomic decomposition theory for the classical Hardy spaces to a more general setting. It is well known that the Hardy space $H^1(\mathbb{R})$ is a proper closed subspace of $L_1(\mathbb{R})$, and many results in harmonic analysis and approximation theory are valid on $H^1(\mathbb{R})$ but are not correct on $L_1(\mathbb{R})$. We have found the same situation exists with respect to the Lebesgue constant of the cardinal spline interpolation operator.

Our main result is the following:

**Theorem 1.** Let $\{(-1)^j y_j\} \in H^1(\mathbb{Z})$. Then for all $m \in \mathbb{N}$

$$\|L_m y\|_{L(\mathbb{R})} \leq \left(1 + \frac{\pi}{2}\right) \left(1 + \frac{\pi}{3}\right) \|\{(-1)^j y_j\}\|_{H^1(\mathbb{Z})}. \tag{1.8}$$
2. Interpolation operator of cardinal spline

Let \( j(x) \) be the unique integer satisfying \( j(x) - \frac{1}{2} < x < j(x) + \frac{1}{2} \), and let

\[
H_y(x) = \sum' y_j (x - j)^{-1},
\]

where the sum \( \sum' \) is taken over those \( j \in \mathbb{Z} \) for which \( j \neq j(x) \), and \( H_y \) is named the mixed Hilbert transform of the sequence \( y = \{ y_j \} \). Following some ideas of [6], we have

**Lemma 1.** Let \( y \in H^1(\mathbb{Z}) \). Then

\[
\| H_y \|_{1(\mathbb{R})} \leq \frac{\pi^2}{3} \| y \|_{H^1(\mathbb{Z})}.
\]

**Proof.** From the definition of \( j(x) \), we have \( |j(x) - x| \leq \frac{1}{2} \), and for \( j \neq j(x) \), we get

\[
|j(x) - j| \leq |j(x) - x| + |x - j| \leq \frac{1}{2} + |x - j|;
\]

therefore

\[
\left| \frac{j(x) - j}{x - j} \right| \leq 1 + \frac{1}{2} \left| \frac{x}{x - j} \right| \leq 2.
\]

For \( j \neq j(x) \),

\[
\frac{1}{x - j} = \frac{1}{j(x) - j} + \frac{(j(x) - x)(j(x) - j)}{x - j}(j(x) - j)^{-2}.
\]

Hence

\[
\left| \sum' \frac{y_j}{x - j} \right| \leq \left| \sum' \frac{y_j}{j(x) - j} \right| + \sum' \left| \frac{y_j}{j(x) - j} \right|^2,
\]

from which we obtain

\[
\left\| \sum' \frac{y_j}{x - j} \right\|_{1(\mathbb{R})} \leq \int_{\mathbb{R}} \left( \left| \sum' \frac{y_j}{j(x) - j} \right| + \sum' \left| \frac{y_j}{j(x) - j} \right|^2 \right) dx
\]

\[
= \sum_{k \in \mathbb{Z}} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \left( \left| \sum_{j \neq k} y_j \right| + \sum_{j \neq k} \left| \frac{y_j}{k - j} \right|^2 \right) dx
\]

\[
= \sum_{k \in \mathbb{Z}} \left| \sum_{j \neq k} y_j \right| + \sum_{k \in \mathbb{Z}} \sum_{j \neq k} \left| \frac{y_j}{k - j} \right|^2
\]

\[
= \sum_{k \in \mathbb{Z}} \left| \sum_{j \neq k} y_j \right| + \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{k^2} \sum_{j \in \mathbb{Z}} |y_j|
\]

\[
\leq \| \{ H y_j \} \|_{l(\mathbb{Z})} + \frac{1}{3} \pi^2 \| \{ y_j \} \|_{H^1(\mathbb{Z})}
\]

\[
\leq \frac{1}{3} \pi^2 \| \{ y_j \} \|_{H^1(\mathbb{Z})},
\]

which completes the proof of Lemma 1. \( \square \)

Let \( W_\sigma y(x) = \sum_{k \in \mathbb{Z}} y_k \text{sinc} \sigma (x - k/\pi) \), and let

\[
\| W_\sigma \|_1 = \sup \left\{ \| W_\sigma y(x) \|_{1(\mathbb{R})} : \right\}
\]

\[
\left\{ \left\| (\pm 1)^j y_j \right\|_{H^1(\mathbb{Z})} \leq 1 \right\},
\]

where \( \text{sinc} x := x^{-1} \sin x \) for \( x \neq 0 \) and \( 1 \) for \( x = 0 \), \( W_\sigma \) is the well-known Whittaker operator and \( W_\sigma y \) is the Whittaker cardinal series. From Lemma 1, we have
Theorem 2. Let \( \sigma > 0 \). Then
\[
\|W_\sigma\|_1 \leq \left(1 + \frac{\pi}{\sigma}\right)^3.
\]

Proof. We first consider the case \( \sigma = \pi \):
\[
|W_\pi y(x)| = \left| \sum_{j \in \mathbb{Z}} y_j \text{sinc} \pi(x - j) \right|
\leq \frac{\sin \pi x}{\pi} \sum_{j \neq j(x)} (1 - 1)^j \left| \frac{y_j}{x - j} \right| + |y_j(x) \text{sinc} \pi(x - j(x))|
\leq \frac{1}{\pi} \sum_{j \neq j(x)} (1 - 1)^j \frac{y_j}{x - j} + |y_j(x)|.
\]

Therefore, it follows from Lemma 1 that we have
\[
\|W_\pi y(x)\|_{L^1(\mathbb{R})} \leq \frac{\pi}{3} \left\{ \|(-1)^j y_j\|_{H^1(\mathbb{Z})} + \|\{y_j\}\|_{L^1(\mathbb{Z})} \right\}
\leq \left(1 + \frac{\pi}{3}\right) \left\{ \|(-1)^j y_j\|_{H^1(\mathbb{Z})} \right\}.
\]

By changing scale, we obtain from (2.7) that
\[
\|W_\pi y\|_1 \leq \left(1 + \frac{\pi}{3}\right) \left(\frac{\pi}{\sigma}\right).
\]

\[\square\]

Denote by \( L^m_p(\mathbb{R}) \), \( 1 \leq p \leq \infty \), \( m \in \mathbb{N} \), the subspace of \( f \) in \( L_p(\mathbb{R}) \) for which the \((m - 1)\)th derivative of \( f \) exists and is locally absolutely continuous on \( \mathbb{R} \), and for which \( \|f^{(m)}\|_{p(\mathbb{R})} \) is finite. By \([4]\), if \( f \in L^m_p(\mathbb{R}) \), then
\[
\|\{f(j)\}\|_{p(\mathbb{Z})} \leq \|f\|_{p(\mathbb{R})} + \|f'\|_{p(\mathbb{R})} < \infty;
\]
therefore, it follows from Schoenberg \([9]\) that for every \( f \in L^m_p(\mathbb{R}) \), there is a unique \( \mathcal{L}_m f \in \mathcal{S}_{m,p}(\mathbb{R}) \), such that \( \mathcal{L}_m f(j) = f(j) \) for all \( j \in \mathbb{Z} \). Moreover, we have

Lemma 2 \([5]\). Let \( f \in L^m_1(\mathbb{R}) \), \( m \in \mathbb{N} \), and let \( \mathcal{L}_m f \) be the unique cardinal spline of degree \( m - 1 \) interpolating to \( \{f(j)\}_{j \in \mathbb{Z}} \) at the integers. Then
\[
\|f - \mathcal{L}_m f\|_{L^1(\mathbb{R})} \leq \frac{K_m}{\pi^m} \|f^{(m)}\|_{L^1(\mathbb{R})},
\]
where \( K_m \) is the Favard constant,
\[
K_m := \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k(m+1)}}{(2k+1)^{m+1}},
\]
and
\[
1 = K_0 < K_1 < \cdots < K_m < \cdots < K_3 < K_4 = \frac{\pi}{2}.
\]

Remark 1. de Boor and Schoenberg \([2]\) proved that equation (2.8) is also valid for \( m \) even and \( p = \infty \).

Let \( \mathcal{E}_\sigma(\mathbb{R}), \sigma > 0 \), be the restriction on \( \mathbb{R} \) of entire functions of exponential type \( \sigma \), and let
\[
B_{\sigma,p} = \mathcal{E}_\sigma(\mathbb{R}) \cap L_p(\mathbb{R}), \quad 1 \leq p \leq \infty, \quad B_\sigma := B_{\sigma,\infty}.
\]
It is well known that \( B_{\sigma,p} \subseteq B_{\sigma,q}, 1 \leq p < q \leq \infty \).
Lemma 3 (p. 211 Inequality of Bernstein’s type). Let $f \in B_{\sigma,p}$, $1 \leq p \leq \infty$, $\sigma > 0$. Then

$$
\|f'\|_{p(\mathbb{R})} \leq \sigma \|f\|_{p(\mathbb{R})}.
$$

Lemma 4 (10). Let $y = \{y_j\} \in l_2$. Then there is a unique $f \in B_{\sigma,2}$, interpolating the given data $y = \{y_j\}_{j \in \mathbb{Z}}$ at the integers, and $f$ is represented by

$$
f(x) = \sum_{j \in \mathbb{Z}} y_j \text{sinc} \pi(x - j), \quad \text{for all } x \in \mathbb{R},
$$

and the series $\sum_{j \in \mathbb{Z}} y_j \text{sinc} \pi(x - j)$ converges uniformly on $\mathbb{R}$.

Proof of Theorem 1. Let $\{(-1)^j y_j\} \in H^1(\mathbb{Z})$. Then $\{y_j\} \in l_2(\mathbb{Z})$. By Lemma 4, there exists a function $f \in B_{\sigma,2}$ such that $f(j) = y_j$ for all $j \in \mathbb{Z}$, hence $f \in B_{\sigma}$. It follows from Theorem 2 that $f \in L_1(\mathbb{R})$; therefore $f \in B_{\sigma,1}$. Using Lemma 2 and Bernstein’s inequality we get

$$
\|f - \mathcal{L}_m f\|_{1(\mathbb{R})} \leq \frac{K_m}{\pi^m} \|f^{(m)}\|_{1(\mathbb{R})}
$$

(2.10)

which together with (2.9) and Theorem 2 gives

$$
\mathcal{L}_m y\|_{1(\mathbb{R})} = \|\mathcal{L}_m f\|_{1(\mathbb{R})} \leq (1 + K_m) \|f\|_{1(\mathbb{R})}
\leq (1 + \frac{\pi}{2}) \left(1 + \frac{\pi}{3}\right) \|\{(-1)^j y_j\}\|_{H^1(\mathbb{Z})},
$$

which completes the proof of Theorem 1.

\[\square\]

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\textbf{References}

5. G. G. Magr"{i}l–Il'yaev, Average dimension, widths, and optimal recovery of Sobolev classes of functions on the line, Math. USSR Sbornik 74(1993), 381–403. [MR 92k:41034]

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