CRITERIA FOR CONVEXITY IN BANACH SPACES

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Abstract. In this paper two convexity criteria are proven. The first one characterizes compact convex sets in a locally convex space and extends a previous result by G. Aumann, while the second one characterizes closed bounded convex sets with the Radon-Nikodým property in a Banach space.

INTRODUCTION

In the present paper the following two theorems are proven:

Theorem 1. Let $K$ be a closed subset of a locally convex space $X$ such that $L = \overline{\text{conv}}K$ is compact in $X$. The following are equivalent:

(i) $K = L$, that is, $K$ is convex.

(ii) For every closed hyperplane $H$ of $X$ which intersects $K$, $H \cap K$ is either contractible or has the fixed point property.

Theorem 2. Let $K$ be a weakly closed bounded subset of a Banach space $X$ such that $L = \overline{\text{conv}}K$ has the Radon-Nikodým property. The following are equivalent:

(i) $K = L$, that is, $K$ is convex.

(ii) For every closed hyperplane $H$ of $X$ which intersects $K$, $H \cap K$ is either contractible or has the fixed point property in the narrow sense.

(In the above theorems it is assumed that $\dim X \geq 2$.)

A set $A$ is assumed to have the fixed point property in the narrow sense if every continuous compact map of $A$ into itself has a fixed point. It is known that every closed convex subset of a Banach space has that property (Schauder’s Theorem, cf. [Du1]) and every compact convex subset of a locally convex space has the fixed point property in the usual sense (Tychonoff’s Theorem, cf. [Du1]). Also, as a direct consequence of the convexity we have that every convex set in a topological vector space is contractible. A closed bounded convex subset $A$ of a Banach space $X$ is assumed to have the Radon-Nikodým property (RNP) if every subset of $A$ has slices of arbitrarily small diameter (for a detailed study of this notion see [B]).

For the case $X = \mathbb{R}^n$ similar characterizations have been proven by G. Aumann ([A]) with methods from algebraic topology. Later I. Fáry ([F]) and A. Kosinski ([K]) gave other proofs for the same result.

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The approach of this paper is based on the notion of the $n$-extreme points for Theorem 1 and $n$-denting points of a convex set $K$ ($Ex_n(K)$, resp. $D_n(K)$) for Theorem 2. The precise definition of them is given in Section 1. These are subsets of $K$ strongly related with its extreme points and as is proven in the same section, the sets $\bigcup_{n=0}^{\infty}Ex_n(K)$ (for compact $K$), $\bigcup_{n=0}^{\infty}D_n(K)$ (for $K$ with the RNP) are weakly dense in the corresponding set $K$. In Section 2 the $n$-cycles are defined and a sufficient condition is considered for a set $K$ to have a non contractible and without the fixed point property hyperplane section. In Section 3 it is shown that under some assumptions $K$ satisfies the previous condition. Finally in Section 4 the proof of Theorems 1 and 2 is provided, through two more general theorems (Theorems 17 and 18) and some remarks are given.

Notation. The class of all closed subspaces of a locally convex space $X$ which have codimension $m$, $m \in \mathbb{N}$, is denoted by $\Sigma^m(X)$. The class of all affine closed subspaces of $X$ which have codimension $m$, $m \in \mathbb{N}$, is denoted by $A^m(X)$. We also set $\Sigma(X) = \bigcup_{m=0}^{\infty}\Sigma^m(X)$ and $A(X) = \bigcup_{m=0}^{\infty}A^m(X)$. The members of $A(X)$ will in general be called flats. If $A$ is a subset of a Banach space, then $\mathcal{P}(A)$ denotes the set of all points of continuity of the identity map $Id_{\mathcal{A}} : (A, \text{weak}) \to (A, \text{norm})$.

Some standard notation is also used. If $A \subset X$, then $\text{relint}A$ denotes the relative interior of $A$, $\text{relbd}(A)$ the relative boundary of $A$ and $\dim A$ the dimension of $A$, that is, the dimension of its affine hull.

Section 1

Let $X$ be a locally convex space, $L \subset X$ be a closed convex set and suppose that $x \in L$. Then $x$ is called an $n$-extreme point of $L$ ($n \in \mathbb{N}$) if $x$ is in the relative interior of no $(n+1)$-dimensional convex subset of $L$. The set of all $n$-extreme points of $L$ is denoted by $Ex_n(L)$. We also define $E(L) = \bigcup_{n=0}^{\infty}Ex_n(L)$.

Evidently if $n = 0$, we get the usual notion of an extreme point of $L$. Also if $m < n$, then $Ex_m(L) \subset Ex_n(L)$. As can easily be proven, the basic property of the $n$-extreme points is the following: A point $x \in Ex_n(L)$ if and only if there exists a unique face $A_x$ of $L$ which contains $x$ in its relative interior and with $\dim A_x \leq n$. (As usually denoted a face of $L$ is a closed convex set $A \subset L$ such that each segment $[x_1, x_2] \subset L$ with $A \cap \text{relint} [x_1, x_2] \neq \emptyset$ is contained in $A$.) The reader can refer to [3] for a more extensive study of the notion of $n$-extreme points in finite dimensional spaces.

At this point let us give some examples for the $n$-extreme points:

(a) If $X = \ell^p$, $1 < p < \infty$, $L = B_{\ell^p}$ (the unit ball of $\ell^p$), then it can be shown that $Ex_0(L) = Ex_n(L) = S_{\ell^p}$ (the unit sphere of $\ell^p$), for every $n \in \mathbb{N}$.

(b) If $X = \ell^1$, $L = B_{\ell^1}$, then we have that:

$$Ex_{n-1}(L) = \{ \lambda_{m_1}e_{m_1} + \ldots + \lambda_{m_n}e_{m_n} : m_1 < \ldots < m_n, \sum_{i=1}^{n} |\lambda_{m_i}| = 1 \}$$

for every $n \geq 1$, where $\{e_1, e_2, \ldots\}$ is the usual basis of $\ell^1$.

(c) If $X = c_0$ and $L = B_{c_0}$, then $Ex_n(L) = \emptyset$ for every $n \geq 0$.

Lemma 3. Let $L$ be a closed convex subset of $X$ and suppose that a flat $F \in A^m(X)$ intersects $L$. Then $F \cap L$ is a closed convex subset of $X$ such that $Ex_0(F \cap L) \subset Ex_m(L)$.
Proof. If \( \text{Ex}_0(\mathcal{F} \cap L) = \emptyset \), then the lemma trivially holds. So let us assume that \( \text{Ex}_0(\mathcal{F} \cap L) \neq \emptyset \) and \( z \in \text{Ex}_0(\mathcal{F} \cap L) \).

Suppose that \( z \notin \text{Ex}_m(L) \). Then there exists an \((m + 1)\)-dimensional convex set \( K \) with \( z \in \text{relint}K \) such that \( K \subset L \). Since \( F \) is a flat of codimension \( m \) and \( z \in F \) is at the same time in the relative interior of an \((m + 1)\)-dimensional convex set, there exists a segment \( [x_1, x_2] \subset F \cap K \subset F \cap L \), such that \( z \) is the midpoint of \([x_1, x_2]\). Hence \( z \notin \text{Ex}_0(\mathcal{F} \cap L) \), which is a contradiction.

Proposition 4. If \( L \) is a compact convex subset of a locally convex space \( X \), then \( L = E(L) \).

Proof. If \( L \) is finite dimensional, then evidently \( L = E(L) = \text{Ex}_m(L) \) where \( m = \dim L \). If \( L \) is infinite dimensional, we can assume that \( L \) is weakly compact and so it is enough to prove that \( E(L) \) is weakly dense in \( L \).

Let \( V \) be a weakly open basic neighborhood of a point \( x_0 \in L \). Since \( E(L) = \bigcup_{n=0}^{\infty} \text{Ex}_n(L) \), it suffices to prove that there exists an \( n \in \mathbb{N} \) which depends on \( V \) such that \( V \cap \text{Ex}_n(L) \neq \emptyset \). Suppose that \( V \) is determined by the functionals \( f_1, \ldots, f_k \in X^* \). We set \( F = x_0 + \bigcap_{i=1}^{k} \ker f_i \). Then \( F \in A^m(X) \) for some \( m \leq k \), \( x_0 \in F \) and \( F \subset V \). Evidently \( F \cap L \) is a non empty compact convex subset of \( X \) and so \( \text{Ex}_0(F \cap L) \neq \emptyset \) by Krein-Milman’s Theorem. According to the previous lemma \( \text{Ex}_0(F \cap L) \subset \text{Ex}_m(L) \), hence \( \emptyset \neq \text{Ex}_0(F \cap L) \subset F \cap \text{Ex}_m(L) \subset V \cap \text{Ex}_m(L) \).

Remark 1. Using the same method as in the previous proposition, one can prove that if \( L \) is a closed convex subset of a locally convex space such that \( L \) has the Krein-Milman Property (that is, each closed convex subset of \( L \) has an extreme point), then \( L = \overline{E(L)}^{\sigma} \).

Let us suppose for the rest of Section 1 that \( X \) is a Banach space. Let \( L \) be a closed bounded convex subset of \( X \). A point \( x \in L \) is an \( n \)-denting point of \( L \) \((n \in \mathbb{N})\) if: (a) \( x \) is an \( n \)-extreme point of \( L \) and (b) \( x \) is a point of continuity of the identity map \( \text{id}_L : (L, \text{weak}) \to (L, \text{norm}) \), that is, \( x \in \text{Ex}_n(L) \cap \mathcal{P}(L) \). The set of all \( n \)-denting points of \( L \) is denoted by \( D_n(L) \). We also define \( D(L) = \bigcup_{n=0}^{\infty} D_n(L) \).

Evidently if \( n = 0 \), we get the usual notion of a denting point of \( L \). Also if \( m < n \), then \( D_m(L) \subset D_n(L) \). Since every \( x \in D_n(L) \) is an \( n \)-extreme point of \( L \), there exists a unique face \( A_x \) of \( L \) which contains \( x \) in its relative interior and with \( \dim A_x \leq n \). The converse does not hold in general, but if \( L \) is finite-dimensional, then evidently \( \text{Ex}_n(L) = D_n(L) \) for every \( n \in \mathbb{N} \), since the weak and norm topologies coincide on \( L \). The examples (a) and (b) of \( n \)-denting points, given in the beginning of this section, are also examples for \( n \)-denting points.

The following result was suggested by Professor S. A. Argyros.

Proposition 5. Let \( L \) be a bounded convex subset of a Banach space \( X \) and suppose that \( V \) is a weakly open basic neighborhood of a point \( x_0 \in L \). Then \( \mathcal{P}(V \cap L) \subset \mathcal{P}(L) \).

Proof. Evidently if \( \mathcal{P}(V \cap L) = \emptyset \), the theorem trivially holds. So let us suppose that \( z_0 \in \mathcal{P}(V \cap L) \). Then for every \( \varepsilon > 0 \) there exists a weakly open basic neighborhood of \( z_0 \) which is denoted by \( W_z \) such that \( \text{diam}(W_z \cap V \cap L) < \varepsilon \). If \( z_0 \in V \), then obviously \( W_z \cap V \) is a weakly open neighborhood of \( z_0 \) such that \( \text{diam}(W_z \cap V \cap L) < \varepsilon \). Since \( \varepsilon \) is arbitrarily chosen, \( z_0 \in \mathcal{P}(L) \).
If \( z_0 \in \overline{V} \setminus V \), then without loss of generality we may suppose the following:

1. \( x_0 = 0 \) and \( V = \{ x \in X : |f_i (x)| < 1 \text{ for } i = 1, \ldots, k \} \) where \( f_1, \ldots, f_k \in X^* \) and so \( |f_i (z_0)| \leq 1 \text{ for } i = 1, \ldots, k \).

2. \( 0 \notin W_\varepsilon \) and \( W_\varepsilon = \{ x \in X : |g_j (x - z_0)| < 1 \text{ for } j = 1, \ldots, m \} \) where \( g_1, \ldots, g_m \in X^* \). Hence \( |g_j (z_0)| \geq 1 \), for some \( j \in \{ 1, \ldots, m \} \).

3. \( 0 < \varepsilon < \frac{M'}{2} \) where \( M = \sup \{ \| x \| : x \in L \} \) and \( C = \max \{ |g_j (z_0) : 1 \leq j \leq m \} \).

According to (2) it is clear that \( C \geq 1 \) and \( M > 0 \).

Let us define:
\[
V' = \{ x \in X : |f_i (x)| < 1 + \frac{\varepsilon}{M'} \text{ for } i = 1, \ldots, k \}
\]
and
\[
W'_\varepsilon = \{ x \in X : |g_j (x - z_0)| < 1 - \frac{\varepsilon}{M} \text{ for } j = 1, \ldots, m \}.
\]

Claim. (a) Let \( r \in (0, 1) \) such that \( \frac{1}{1 + \frac{\varepsilon}{M'}} < r < \frac{1}{1 + \frac{\varepsilon}{M}} \). Then \( r (V' \cap W'_\varepsilon \cap L) \subset V \cap W_\varepsilon \cap L \).

(b) \( \text{diam} (V' \cap W'_\varepsilon \cap L) < 4 \varepsilon \).

Proof of the Claim. (a) Let \( x \in V' \cap W'_\varepsilon \cap L \). Then

1. For \( i = 1, \ldots, k \), \( r |f_i (x)| < r \left( 1 + \frac{\varepsilon}{M'} \right) < 1 \). Hence \( rx \in V \).

2. For \( j = 1, \ldots, m \),
\[
|g_j (rx - z_0)| = |g_j (rx - rz_0 + rz_0 - z_0)| \leq r |g_j (x - z_0)| + (1 - r) |g_j (z_0)|
\]
\[
< r \left( 1 + \frac{\varepsilon}{M} \right) + (1 - r) C = r - r \left( \frac{\varepsilon}{M} + 1 \right) C < r - C + C = r < 1.
\]

Hence for \( j = 1, \ldots, m \), \( |g_j (rx - z_0)| < 1 \) and so \( rx \in W_\varepsilon \).

3. Since \( L \) is convex and \( 0 \in L \), then \( (1 - r) 0 + r x = rx \in L \).

By (1), (2) and (3) \( rx \in V' \cap W_\varepsilon \cap L \) for every \( x \in V' \cap W'_\varepsilon \cap L \).

(b) Let \( x \in V' \cap W'_\varepsilon \cap L \) and \( r \in (0, 1) \) as in (a). Then \( ||rx - x|| = (1 - r) ||x|| < \left( 1 - \frac{\varepsilon}{1 + \frac{\varepsilon}{M'}} \right) M = \frac{\varepsilon}{1 + \frac{\varepsilon}{M'}} \) and \( ||rx - z_0|| < \varepsilon \) since \( z_0, r x \in (V' \cap W_\varepsilon \cap L) \) (by (a)) and \( \text{diam} (V' \cap W_\varepsilon \cap L) < \varepsilon \). Therefore \( ||x - z_0|| \leq ||rx - x|| + ||rx - z_0|| < \varepsilon + \frac{\varepsilon}{1 + \frac{\varepsilon}{M'}} \).

Hence \( \sup \{ ||x - y|| : x, y \in V' \cap W'_\varepsilon \cap L \} < 4 \varepsilon \). The proof of the claim is complete.

Since \( V' \cap W'_\varepsilon \) is evidently a weakly open neighborhood of \( z_0 \) and \( \varepsilon \) is arbitrarily chosen, we conclude by (b) of the claim that \( z_0 \in P(L) \) and thus the proof of Proposition 5 is complete.

Lemma 6. Let \( L \) be a closed bounded convex subset of a Banach space \( X \) and suppose that \( V \) is a weakly open basic neighborhood of a point \( x_0 \in L \). Then \( \overline{V} \cap L \) is a closed bounded convex subset of \( X \) with \( Ex_0 (\overline{V} \cap L) \subset E (L) \).

Proof. It is evident that the set \( \overline{V} \cap L \) is a closed bounded convex subset of \( X \). If \( Ex_0 (\overline{V} \cap L) = \emptyset \), then the lemma trivially holds. So let us suppose that \( z \in Ex_0 (\overline{V} \cap L) \). Following the proof of Proposition 4, there exists an \( F \in A^m (X) \) for some \( m \in \mathbb{N} \) such that \( z \in F \) and \( F \subset \overline{V} \). Obviously \( z \in Ex_0 (F \cap L) \) and by Lemma 3 \( z \in Ex_m (L) \). Therefore \( z \in E (L) \).

Proposition 7. Let \( L \) be a closed bounded convex subset of a Banach space \( X \) and suppose that \( L \) has the RNP. Then \( L = \overline{D(L)}^w \).

Proof. Since \( L \) is a norm closed convex subset of \( X \), \( L \) is weakly closed as well. So it is enough to prove that \( D (L) \) is weakly dense in \( L \). To this end, let us assume that \( V \) is a weakly open basic neighborhood of a point \( x_0 \in L \). It is clear that there exists a
weakly open basic neighborhood \( V' \) of the same point such that \( V' \subset V \). Evidently \( V' \cap L \) is a closed convex subset of \( L \) and since \( L \) has the RNP, \( D_0 (V' \cap L) \neq \emptyset \) by the Phelps-Bourgain Theorem (cf. [B]). By Proposition 5 and Lemma 6 we have that \( D_0 (V' \cap L) \subset E(L) \cap P(L) = D(L) \). Hence \( \emptyset \neq V' \cap D(L) \subset V \cap D(L) \).

**Section 2**

Let \( S, F \) be closed subspaces of a locally convex space \( X \) where \( F \) is finite dimensional and \( S \cap F = \{0\} \). We write \( S \oplus F \) for the linear span of \( S \cup F \). It is obvious that every \( x \in S \oplus F \) can be uniquely written as \( x = s + f \), where \( s \in S \) and \( f \in F \). Hence we can define the projection \( P_F : S \oplus F \to F \), where \( P_F(x) = f \), if \( x = s + f \). In the trivial case where \( S = \{0\} \), \( P_F \) is the identity map on \( F \).

Since \( X \) is a locally convex space and \( F \) is finite dimensional, it can be proven that \( S \oplus F \) is a closed subspace of \( X \) and the projection \( P_F \) is continuous.

**Notation.** The relative boundary of an \((n+1)\)-dimensional compact convex set will be called an \( n \)-cycle. We denote by \( \mathcal{K}^{n}_0 \) the class of all \( n \)-dimensional compact convex subsets of \( X \) which contain \( 0 \) in their relative interior. The relative boundary of a set \( L \in \mathcal{K}^{n+1}_0 \) is called an \( n \)-cycle around \( 0 \).

**Lemma 8.** Let \( K \) be a subset of a locally space \( X \) and \( H \in \Sigma^1(X) \) with the following properties:

(i) \( H = F \oplus F_n \), where \( F \in \Sigma^{n+1}(X) \), \( F_n \) is an \( n \)-dimensional subspace of \( X \) and \( n \geq 1 \).

(ii) \( F \cap K = \emptyset \).

(iii) \( F_n \cap K \) contains an \((n-1)\)-cycle \( C \) around \( 0 \).

Then \( C \) is a retract of \( H \cap K \).

**Proof.** For every \( x \in F_n \setminus \{0\} \) we define \( P_C(x) = C \cap \{ \lambda x : \lambda > 0 \} \). Since \( C \) is an \((n-1)\)-cycle, \( P_C \) is well defined and continuous. Let \( P_{F_n} : H \to F_n \) be the projection onto \( F_n \). We note that \( P_{F_n}(H \cap K) \subset F_n \setminus \{0\} \), since \( P_{F_n}(x) = 0 \) if and only if \( x \in F \), whereas \( F \cap K = \emptyset \) by assumption (ii).

So the map \( S : H \cap K \to C \) defined by \( S(x) = P_C(P_{F_n}(x)) \), where \( x \in H \cap K \), is a well defined continuous map onto \( C \) leaving every point of \( C \) fixed. That is, \( C \) is a retract of \( H \cap K \).

**Remark 2.** Suppose that a set \( A \subset X \) is contractible or has the fixed point property in the usual or in the narrow sense. Then there is no \( n \)-cycle which is a retract of \( A \). Indeed, it is easy to see that every retract of \( A \) has the above mentioned properties, whereas an \( n \)-cycle as an homeomorphic image of the unit sphere \( S^n \) of \( \mathbb{R}^{n+1} \), neither is contractible by Brouwer’s Theorem (cf. [Du]) nor has the fixed point property in the usual or in the narrow sense.

**Section 3**

In the following and up to Proposition 11 let us denote by \( X \) a locally convex space \( X \) and by \( K \) a closed subset of \( X \) such that the set \( L = \text{cont} K \) is compact.

Let us define \( \text{Ex}^{m}_{m}(L) \) = \( \text{Ex}_{m}(L) \setminus \text{Ex}_{m-1}(L) \), \( m \geq 1 \), and suppose that \( 0 \in \text{Ex}^{m}_{m}(L) \). According to the basic property of the \( n \)-extreme points (Section 1), there exists a unique face \( A_0 \) of \( L \) which contains \( 0 \) in its relative interior. Let us denote by \( F_{m} \) the linear span of \( A_0 \). It is obvious that \( A_0 \in \mathcal{K}^{m}_0 \) and so \( F_{m} \) is an
m-dimensional (and thus closed) subspace of \( X \). Since \( F_m \) is finite dimensional, there exists a \( Y \in \Sigma^m (X) \) such that \( Y \oplus F_m = X \).

In the following Lemma 9 we shall need Choquet’s Lemma (cf. [HHZ]), that is, in every compact convex subset \( L \) of \( X \) the slices of \( L \) consist of a neighborhood base for the extreme points of \( L \). As is known, a slice of \( L \) is the intersection of \( L \) with an open halfspace of \( X \). Formally, if \( f \in X^* \) and \( a \in \mathbb{R} \), then a slice of \( L \) can be written as \( S(f,L,a) = \{ y \in L : f(y) > a \} \).

**Lemma 9.** Let \( K, L, A_0, F_m, Y \) be as above and \( 0 \in \text{Ex}^*_m (L) \setminus K, 1 \leq m < \text{dim} X \). Then:

(a) \( 0 \in \text{Ex}_0 (Y \cap L) \).

(b) There exist an \( H \in \Sigma^1 (X) \) and an \( F \in \Sigma^{m+1} (X) \), \( F \subset Y \) such that:

(i) \( H = F \oplus F_m \)

(ii) \( F \cap K = \emptyset \).

**Proof.** (a) Obviously \( Y \cap L \) is a compact convex subset of \( L \) containing 0. If \( 0 \notin \text{Ex}_0 (Y \cap L) \), then there exists a closed segment \([x_1, x_2] \subset Y \cap L\) with midpoint at 0 and \( x_1 \neq x_2 \). Since \( A_0 \) is a face of \( L \) and \( 0 \in \text{relint} [x_1, x_2] \cap A_0 \), we have that \([x_1, x_2] \subset A_0 \). Hence \([x_1, x_2] \subset Y \cap L \cap A_0 \subset Y \cap L \cap F_m = \{0\} \) which is a contradiction.

(b) We notice that \( \text{dim } Y \geq 1 \) since \( m < \text{dim } X \). Two cases are distinguished. Either \( Y \cap K = \emptyset \) or \( Y \cap K \neq \emptyset \).

If \( Y \cap K = \emptyset \), then for every \( F \in \Sigma^1 (Y) \) we have \( F \in \Sigma^{m+1} (X) \), \( F \oplus F_m = H \in \Sigma^1 (X) \) and \( F \cap K \subset Y \cap K = \emptyset \). (If \( \text{dim } Y = 1 \), then \( F \) is necessarily the trivial space \( \{0\} \).)

If \( Y \cap K \neq \emptyset \), then since \( 0 \notin A_0 \) and \( K \) is closed, there exists an open neighborhood \( V \) of 0 in \( X \) such that \( V \cap K = \emptyset \). But then \( V \cap (Y \cap L) \) is an open neighborhood of 0 in the compact convex set \( Y \cap L \). Since (by (a)) \( 0 \in \text{Ex}_0 (Y \cap L) \), Choquet’s lemma can be applied. Therefore there exists a slice \( S = S(f, Y \cap L, a) \) such that \( 0 \in S \) and \( S \subset V \cap (Y \cap L) \). Hence, \( S \cap K = \emptyset \).

Let \( G = K \cap f \). Then \( G \in \Sigma^1 (X) \) and since \( 0 \in S, G \cap (Y \cap L) \subset S \). We note that \( Y \) is not contained in \( G \), since otherwise \( Y \cap K = (G \cap Y) \cap (L \cap K) = G \cap (Y \cap L) \cap K \subset S \cap K = \emptyset \), which is a contradiction to our assumption where \( Y \cap K \neq \emptyset \).

Let \( F = G \cap Y \). Since \( Y \in \Sigma^m (X), G \in \Sigma^1 (X) \) and \( Y \notin G \) we conclude that \( F \in \Sigma^{m+1} (X) \). We also have that \( F \cap F_m = \{0\} \), since \( F \subset Y \), and so the subspace \( H = F \oplus F_m \) is a well defined closed hyperplane of \( X \). Finally \( F \cap K = (G \cap Y) \cap (L \cap K) = G \cap (Y \cap L) \cap K \subset S \cap K = \emptyset \). \( \square \)

**Lemma 10.** Let \( 0 \in \text{Ex}^*_m (L), m \geq 1 \) and \( \text{Ex}^*_{m-1} (L) \subset K \). Then \( \text{relbd} A_0 \subset K \) and therefore \( F_m \cap K \) contains an \((m-1)\)-cycle around 0.

**Proof.** Since \( A_0 \in K_0^m \), \( \text{relbd} A_0 \) is an \((m-1)\)-cycle around 0 and \( \text{relbd} A_0 = \text{Ex}^*_{m-1} (A_0) \). On the other hand \( A_0 \) is a face of \( L \). Hence every face of \( A_0 \) is a face of \( L \) as well. By the basic property of \( m \)-extreme points (as it is stated in Section 1), this means that \( \text{Ex}^*_{m-1} (A_0) \subset \text{Ex}^*_{m-1} (L) \). Since \( \text{Ex}^*_{m-1} (L) \subset K \), \( \text{relbd} A_0 \subset K \) as well. \( \square \)

**Proposition 11.** Let \( K \) be a closed subset of a locally convex space \( X \) with the following properties:

(i) \( L = \text{conv } K \) is a compact subset of \( X \).
We define that: (i) \( H \) and (ii) \( F \) trivially hold: (i) \( H = F \oplus F_m \) and \( F_m \) is an \( m \)-dimensional subspace of \( X \); (ii) \( F \cap K = \emptyset \); (iii) \( F_m \cap K \) contains a \((m-1)\)-cycle \( C \) around \( 0 \). By Lemma 8, \( C \) is a retract of \( H \cap K \).

Then \( F_m = X \), \( X \) is isomorphic to \( \mathbb{R}^m \) and \( A_0 = L \). Let us denote by \( BdL \) the boundary of \( L \). By Lemma 10, \( BdL \subset K \). For every \( H \in \Sigma^1(X) \) the following trivially hold: (i) \( H = \{0\} \oplus H \) and \( H \) is an \((m-1)\)-dimensional subspace of \( X \); (ii) \( \{0\} \cap K = \emptyset \); (iii) \( H \cap K \) contains an \((m-2)\)-cycle \( C \) around \( 0 \), where \( C = BdL \cap H \). By Lemma 8 for each \( H \in \Sigma^1(X) \) there exists an \((m-2)\)-cycle around \( 0 \) which is a retract of \( H \cap K \).

Hereafter let us denote by \( x \) a Banach space.

**Lemma 12.** Let \( L \) be a convex subset of a Banach space \( X \) and suppose that \( z \in \mathcal{P}(L) \). If \( z = rz_1 + (1-r)z_2 \), where \( r \in (0,1) \), and \( z_1, z_2 \in L \), then \( z_1, z_2 \in \mathcal{P}(L) \) as well.

**Proof.** We prove that \( z_1 \in \mathcal{P}(L) \) (the proof of \( z_2 \) is similar). Since \( z \in \mathcal{P}(L) \), for every \( \varepsilon > 0 \) there exists a weakly open neighborhood \( V_\varepsilon \) of \( z \) such that \( \text{diam}(V_\varepsilon \cap L) < \varepsilon \). We define \( V_\varepsilon^z = \frac{1}{2}(V_\varepsilon - (1-r)z_2) \). Evidently, \( V_\varepsilon^z \) is a weakly open neighborhood of \( z_1 \). It is easy to see that \( \text{diam}(V_\varepsilon^z \cap L) \leq \frac{1}{4}\varepsilon \). As \( \varepsilon \) is arbitrarily chosen, it follows that \( z_1 \in \mathcal{P}(L) \).

The lemma corresponding to Choquet’s Lemma in the case of 0-denting points, due to Troyanski, Lin and Lin (cf. [4], Chapter 5), is the following:

**Lemma 13.** Let \( L \) be a closed bounded convex subset of a Banach space \( X \) and suppose that \( z \in D_0(L) \). Then the slices of \( L \) consist a neighborhood base for the norm topology at \( z \).

Let us suppose in the sequel that \( K \) is a closed bounded subset of \( X \) and \( L = \text{conv}K \). Let us also define \( D_m^* \equiv D_m(L) \setminus D_{m-1}(L) \) for \( m \geq 1 \) and suppose that \( 0 \in D_m^* \). Hence \( 0 \in Ex_m^*(L) \) as well and so we can define \( A_0, F_m \) and \( Y \) just as we have defined them in the beginning of Section 3.

The proof of the following lemma is similar to that of Lemma 6 except that \( Y \cap L \) is a closed bounded subset of a Banach space \( X \), \( 0 \in D_0(Y \cap L) \) and we apply Lemma 14 instead of Choquet’s Lemma.

**Lemma 14.** Let \( K, L, A_0, F_m, Y \) be as above and \( 0 \in D_m^*(L) \setminus K, 1 \leq m < \text{dim} X \). Then:

(a) \( 0 \in D_0(Y \cap L) \).

(b) There exist an \( H \in \Sigma^1(X) \) and an \( F \in \Sigma^{m+1}(X) \), \( F \subset Y \) such that:

(i) \( H = F \oplus F_m \).

(ii) \( F \cap K = \emptyset \).
Lemma 15. Let \( 0 \in D_m (L) \), \( m \geq 1 \), and \( D_{m-1} (L) \subset K \). Then \( relbdA_0 \subset K \) and therefore \( F_m \cap K \) contains an \((m-1)\)-cycle around 0.

Proof. Using the arguments of Lemma 10 it can be shown that
\[
relbdA_0 = Ex_{m-1} (A_0) \subset Ex_{m-1} (L).
\]
Since \( 0 \in relint A_0 \), for every point \( z_1 \in relbd A_0 \), there exists another point \( z_2 \in relbd A_0 \) such that \( 0 = rz_1 + (1-r)z_2 \) for some \( r \in (0,1) \). By Lemma 12 since \( 0 \in P(L) \) we have that \( z_1, z_2 \in P(L) \). Hence \( relbd A_0 \subset Ex_{m-1} (L) \cap P(L) = D_{m-1} (L) \). Since \( D_{m-1} (L) \subset K \), \( relbd A_0 \subset K \) as well.

Proposition 16. Let \( K \) be a closed bounded subset of a Banach space \( X \) and \( L = \overline{conv} K \), with the following properties:
(i) \( D_{m-1} (L) \subset K \), for some \( m \geq 1 \).
(ii) \( 0 \in D_m (L) \setminus K \).

Then there exists a hyperplane \( H \in \Sigma^1 (X) \) and an \( n \)-cycle \( C \) around 0 (where \( n = m-1 \) if \( m < \dim X \) or \( n = m-2 \) if \( \dim X = m \)), such that \( C \) is a retract of \( H \cap K \).

Proof. Similar to that of Proposition 11 except that now we use Lemmas 14 and 15 instead of Lemmas 9 and 10.

Section 4

(a) Proof of Theorem 2 (i) \( \Rightarrow \) (ii) If \( K \) is a compact convex subset of \( X \) and \( H \in \Sigma^1 (X) \) such that \( H \cap K \neq \emptyset \), then evidently \( H \cap K \) is a non empty compact convex subset of \( X \) and therefore it is contractible and has the fixed point property, as is noted in the Introduction.

(ii) \( \Rightarrow \) (i) According to Remark 2 it is enough to prove the following:

Theorem 17. Let \( K \) be a closed subset of a locally convex space \( X \) such that \( L = \overline{conv} K \) is compact in \( X \) and suppose that for every closed hyperplane \( H \) of \( X \) which intersects \( K \), there is no \( n \)-cycle, \( n \in \mathbb{N} \), which is a retract of \( H \cap K \). Then \( E(L) \subset K \) and \( K \) is convex.

Proof. We will prove inductively that \( Ex_m (L) \subset K \), for every \( m \in \mathbb{N} \). For \( m = 0 \) it is a consequence of Choquet’s Lemma. Let us suppose that for some \( m \geq 1 \), \( Ex_{m-1} (L) \subset K \) and \( Ex_m (L) \nsubseteq K \). Then there exists a point \( x_0 \in Ex_m (L) \setminus K \) which without loss of generality is the point 0. But then by Proposition 11 there exist an \( H \in \Sigma^1 (X) \) and an \( n \)-cycle \( C \) such that \( C \) is a retract of \( H \cap K \), a contradiction. Hence \( Ex_m (L) \subset K \) for every \( m \in \mathbb{N} \) and so \( E(L) \subset K \). Since by Proposition 14 \( L = E(L) \), we have that \( K = L \) and so \( K \) is convex.

The proof of Theorem 4 is complete.

(b) Proof of Theorem 4 (i) \( \Rightarrow \) (ii) It is similar to that of Theorem 11 except that we use Schauder’s fixed point theorem in place of Tychnoff’s.

(ii) \( \Rightarrow \) (i) Like in Theorem 11 it is enough to prove the following:

Theorem 18. Let \( K \) be a closed bounded subset of a Banach space \( X \) such that \( L = \overline{conv} K \) has the RNP and suppose that for every closed hyperplane \( H \) of \( X \) which intersects \( K \), there is no \( n \)-cycle which is a retract of \( H \cap K \). Then \( D(L) \subset K \) and \( K \) is weakly dense in \( L \).
Proof. Similar to that of the Theorem 17 except that we use Lemma 13, Proposition 16 and Proposition 7 instead of Choquet’s Lemma, Proposition 11 and Proposition 4, respectively.

The proof of Theorem 2 is complete.

Remark 3. In Theorem 18 the conclusion that $K$ is weakly dense in $L$ cannot be improved. For example let $X = \ell^2$ and $K = S_{\ell^2}$; then $L = \overline{\text{conv}}(K) = B_{\ell^2}$ and as is known $B_{\ell^2}$ has the RNP. If $H$ is a closed hyperplane of $\ell^2$ that intersects $S_{\ell^2}$, then $H \cap S_{\ell^2}$ is contractible and has the fixed point property in the narrow sense. Indeed $H \cap S_{\ell^2}$ is a single point or a retract of $H \cap B_{\ell^2}$, as follows by the related theorem of Dugundji ([Du2]). Of course $S_{\ell^2}$ is not convex but $S_{\ell^2}^w = B_{\ell^2}$.

Remark 4. Let us denote by $(X, \tau)$ a Banach space $X$ equipped with a topology $\tau$ where $\tau$ is the norm or the weak or the weak* topology (if $X$ is a dual space). Then $(X, \tau)$ is a locally convex space. Various theorems (such as Krein’s theorem if $\tau$ is the weak topology) establish that if $K$ is a compact subset of $(X, \tau)$, then $L = \overline{\text{conv}}(K)$ (where the closure is taken with respect to the topology $\tau$ ) is compact as well. So, as a direct consequence of Theorem 1, we have the following:

Theorem 19. Let $K$ be a compact subset of $(X, \tau)$. Then $K$ is convex, if and only if for every closed hyperplane $H$ of $X$ which intersects $K$, $H \cap K$ is either contractible or has the fixed point property.

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