ON SINGULAR CRITICAL POINTS OF POSITIVE OPERATORS IN KREIN SPACES

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Abstract. We give an example of a positive operator $B$ in a Krein space with the following properties: the nonzero spectrum of $B$ consists of isolated simple eigenvalues, the norms of the orthogonal spectral projections in the Krein space onto the eigenspaces of $B$ are uniformly bounded and the point $\infty$ is a singular critical point of $B$.

An operator $A$ in the Krein space $(\mathcal{K}, [\cdot , \cdot ])$ is said to be positive if $[Ax, x] > 0$ for all nonzero $x$ in the domain of $A$. A bounded positive operator $A$ in the Krein space $(\mathcal{K}, [\cdot , \cdot ])$ has a projection valued spectral function $E$ with $0$ being its only possible critical point (see [1, Theorem IV.1.5] or [5, Section II.3.]). Recall that, by [5, Proposition 5.6], the condition

\[ \|E((-\infty, \alpha])\| \leq C_- < \infty \quad \text{for all } \alpha < 0 \]

is equivalent to the existence of the limit $\lim_{\alpha \to 0} E((-\infty, \alpha])$ in the strong operator topology. Similarly,

\[ \|E([\beta, \infty))\| \leq C_+ < \infty \quad \text{for all } \beta > 0 \]

is equivalent to the existence of the limit $\lim_{\beta \to 0} E([\beta, +\infty))$ in the strong operator topology. Since $0$ is not an eigenvalue of a positive operator $A$, [5, Proposition 3.2] implies that (1) and (2) are equivalent. Also, if $0$ is a critical point, it is said to be regular if one of the conditions (1) or (2) is fulfilled. If the critical point $0$ is not regular, it is called singular.

In the sequel the operator $A$ considered will have a discrete spectrum outside $0$. Examples of bounded positive operators in $\mathcal{K}$ having $0$ as a singular critical point can be constructed as follows (see also the examples in [2, Section 1], [3], [4]). Consider a sequence of two-dimensional Krein spaces $\mathcal{K}_n = \mathbb{C}^2$ with fundamental symmetry $J_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and positive operators $A_n$ in $\mathcal{K}_n$; denote by $\lambda_n^+$ ($\lambda_n^-$, respectively) its positive (negative, respectively) eigenvalues and by $P_n^+$ ($P_n^-$, respectively) the orthogonal (in $\mathcal{K}_n$) projection onto the corresponding eigenspace.

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If $A_n$ is chosen such that $\|A_n\| \leq C$ for all $n$, $\lambda_n^+ \downarrow 0$, $\lambda_n^- \uparrow 0$, $\|P_n\| \to \infty$ if $n \to \infty$, then $A = \bigoplus_{n=1}^{\infty} A_n$ is a bounded positive operator in $\mathcal{K} = \bigoplus_{n=1}^{\infty} \mathcal{K}_n$ having 0 as a singular critical point. Evidently,

$$\sigma(A) = \{\lambda_n^+, \lambda_n^- | n \in \mathbb{N}\} \cup \{0\},$$

and $\|E(\{\lambda_n^\pm\})\| \to \infty$ if $n \to \infty$, that is, the eigenvectors $f_n^+, f_n^-$ of $A$ corresponding to $\lambda_n^+$ and $\lambda_n^-$, respectively, become arbitrarily close if $n$ is large.

The question arises whether or not 0 can be a singular critical point of a positive operator $A$ in $\mathcal{K}$ with discrete spectrum $\{\lambda_n^+, \lambda_n^- | n \in \mathbb{N}\}$ in $\mathbb{C} \setminus \{0\}$ if the projections $E(\{\lambda_n^\pm\})$ are uniformly bounded. It is the aim of this note to show that the answer is yes: We will construct a bounded positive operator $A$ in a Krein space $\mathcal{K}$, such that the projections $E(\{\lambda_n^\pm\})$ corresponding to the single eigenvalues are uniformly bounded but, nevertheless,

$$\|E(\{\lambda_1^\pm, \ldots, \lambda_n^\pm\})\| \to \infty, \quad n \to \infty.$$

Our construction is based on the following two lemmas.

**Lemma 1.** Let $\mathcal{H}_n$ be an $n$-dimensional vector space with a positive definite scalar product $(\cdot, \cdot)$. Then there exist a basis $f_{n1}, \ldots, f_{nn}$ of $\mathcal{H}_n$ and a positive contraction $S_n$ in $\mathcal{H}_n$ such that

$$0 < 1 \leq \|f_{nk}\| \leq 2, \quad \|S_n^{-1}\| = n, \quad (S_n f_{nj}, f_{nk}) = \delta_{jk}, \quad j, k = 1, \ldots, n.$$

**Proof.** Let $e_{n1}, \ldots, e_{nn}$ be an orthonormal basis of $\mathcal{H}_n$, let $T_n$ be the selfadjoint transformation in $\mathcal{H}_n$ given by $T_n e_{n1} = \sqrt{n} e_{n1}$, $T_n e_{nj} = e_{nj}$, $j = 2, \ldots, n$, and put $S_n = T_n^{-2}$. Evidently, $S_n$ is a positive selfadjoint contraction in $\mathcal{H}_n$, and

$$\min \sigma(S_n) = 1/n.$$ 

Therefore $\|S_n^{-1}\| = n$. Let $(u_{k1} \ldots u_{kn})$, $k = 1, \ldots, n$, be an orthonormal basis of the $n$-dimensional space of row vectors with components in $\mathbb{C}$, such that $u_{1j} = 1/\sqrt{n}$, $j = 1, \ldots, n$. Then $U = (u_{kj})_{k,j=1}^n$ is a unitary matrix with $u_{1j} = 1/\sqrt{n}$, $j = 1, \ldots, n$. Put

$$\phi_{nj} = \sum_{k=1}^n u_{kj} e_{nk}, \quad j = 1, \ldots, n.$$ 

Then $\phi_{nj}$, $j = 1, \ldots, n$, is an orthonormal basis of $\mathcal{H}_n$ and

$$\|T_n \phi_{nj}\|^2 = n \frac{1}{n} + \sum_{k=2}^n |u_{kj}|^2 = 1 + 1 - \frac{1}{n}, \quad j = 1, \ldots, n.$$ 

Hence $1 \leq \|T_n \phi_{nj}\| \leq 2$. Let $f_{nj} = T_n \phi_{nj}$, $j = 1, \ldots, n$. Then $1 \leq \|f_{nj}\| \leq 2$ and $(S_n f_{nj}, f_{nk}) = (\phi_{nj}, \phi_{nk}) = \delta_{jk}$, $j, k = 1, \ldots, n$. The lemma is proved. \hfill $\Box$

**Lemma 2.** Let $(\mathcal{H}, (\cdot, \cdot))$ be a separable Hilbert space and let $P$ be a positive, bounded and boundedly invertible operator in $\mathcal{H}$. Let $\phi_j$, $j \in \mathbb{N}$, be a Riesz basis of $\mathcal{H}$ such that $(P \phi_j, \phi_k) = \delta_{jk}$, $j, k \in \mathbb{N}$, and let $\lambda_j \in \mathbb{C}$, $j \in \mathbb{N}$, be a bounded sequence. Define the operator $A$ in $\mathcal{H}$ by $A \phi_j = \lambda_j \phi_j$, $j \in \mathbb{N}$. Then, $A$ can be extended by continuity to a bounded linear operator in $\mathcal{H}$ such that $\|A\| \leq \sqrt{\|P\| \|P^{-1}\|} \sup \{\lambda_j, \quad j \in \mathbb{N}\}$.

**Proof.** For a bounded and boundedly invertible positive operator $P$ we have

$$\|P^{-1}||^{-1} (x, x) \leq (Px, x) \leq \|P\| (x, x), \quad x \in \mathcal{H}.$$
Since the vectors $\phi_j, j \in \mathbb{N}$, are orthonormal with respect to the inner product $(P \cdot, \cdot)$, it follows that

$$
(PAx, Ax) \leq (\sup\{ |\lambda_j|, j \in \mathbb{N} \})^2 (Px, x), \quad x \in \mathcal{H}.
$$

Combining (3) and (4) we get

$$
\|Ax\|^2 = (Ax, Ax) \leq \|P^{-1}\| (PAx, Ax) \leq \|P^{-1}\| (\sup\{ |\lambda_j|, j \in \mathbb{N} \})^2 (Px, x)
$$

$$
\leq \|P^{-1}\| \|P\| (\sup\{ |\lambda_j|, j \in \mathbb{N} \})^2 \|x\|^2
$$

and the lemma follows.

\[ \square \]

**Theorem.** There exist a Krein space $(\mathcal{K}, [\cdot, \cdot])$ and a bounded positive operator $A$ in $\mathcal{K}$ with the following properties:

(a) The nonzero spectrum of $A$ consists of isolated simple eigenvalues.

(b) The point 0 is a singular critical point of $A$.

(c) The norms of the orthogonal projections in the Krein space $(\mathcal{K}, [\cdot, \cdot])$ onto the eigenspaces of $A$ are uniformly bounded.

**Proof.** With the notation as in Lemma 1, choose $\mathcal{H}^+_n = \mathcal{H}^-_n = \mathcal{H}_n$. Let $\mathcal{K}_n = \mathcal{H}^+_n \oplus \mathcal{H}^-_n$ be the direct sum of the Hilbert spaces $(\mathcal{H}^+_n, \cdot, \cdot)$. The positive definite inner product on $\mathcal{K}_n$ is also denoted by $\cdot, \cdot$. All norms in $\mathcal{K}_n$ correspond to this inner product. Endow $\mathcal{K}_n = \mathcal{H}^+_n \oplus \mathcal{H}^-_n$ with the indefinite inner product $[\cdot, \cdot]$ given by the fundamental symmetry $J_n = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$. Consider the operator $K^+_n = (I_n - S_n)^{1/2}$ acting from $\mathcal{H}^+_n$ into $\mathcal{H}^-_n$ as an angular operator in $\mathcal{K}_n$. Here $S_n$ is the operator constructed in Lemma 1. Let $\mathcal{L}^+_n$ be the graph of $K^+_n$ in $\mathcal{K}_n = \mathcal{H}^+_n \oplus \mathcal{H}^-_n$. Then $\mathcal{L}^+_n$ is an $n$-dimensional maximal positive subspace in $\mathcal{K}_n$. It is spanned by the vectors $f^+_{nk} = \begin{pmatrix} f_{nk} \\ K^+_n f_{nk} \end{pmatrix}$, $k = 1, \ldots, n,$ and

$$
[f^+_{nk}, f^+_{nj}] = (f_{nk}, f_{nj}) - (K^+_n f_{nk}, K^+_n f_{nj}) = (S_n f_{nk}, f_{nj}) = \delta_{kj},
$$

$$
\|f^+_{nk}\|^2 = \|f_{nk}\|^2 + \|K^+_n f_{nk}\|^2 \leq 2 \|f_{nk}\|^2 \leq 8.
$$

Denote by $\mathcal{L}^-_n$ the orthogonal complement of $\mathcal{L}^+_n$ in the Krein space $\mathcal{K}_n$. Then $\mathcal{L}^-_n$ is a maximal negative subspace of $\mathcal{K}_n$. The operator $K^-_n = (I_n - S_n)^{1/2}$, acting from $\mathcal{H}^-_n$ into $\mathcal{H}^+_n$, is the angular operator of $\mathcal{L}^-_n$. The subspace $\mathcal{L}^-_n$ is spanned by the vectors $f^-_{nk} = \begin{pmatrix} K^-_n f_{nk} \\ I_n f_{nk} \end{pmatrix}$, $k = 1, \ldots, n$. This follows from the linear independence of $f_1, \ldots, f_n$ and the relation

$$
[f^-_{nj}, f^-_{nk}] = (f_{nj}, K^-_n f_{nk}) - (K^+_n f_{nj}, f_{nk})
$$

$$
= (f_{nj}, (I - S_n)^{1/2} f_{nk}) - ((I - S_n)^{1/2} f_{nj}, f_{nk}) = 0.
$$

The decomposition $\mathcal{K}_n = \mathcal{L}^+_n \oplus \mathcal{L}^-_n$ is a fundamental decomposition of $(\mathcal{K}_n, [\cdot, \cdot])$. Solving a corresponding system of vector equations we find that the orthogonal (fundamental) projections $Q^+_n$ of the Krein space $\mathcal{K}_n$ onto $\mathcal{L}^+_n$ are given by

$$
Q^+_n = \begin{pmatrix} I_n \\ K^+_n \end{pmatrix} S_n^{-1} (I_n - K^-_n), \quad Q^-_n = \begin{pmatrix} K^-_n \\ I_n \end{pmatrix} S_n^{-1} (I_n - K^+_n). \nonumber
$$

From Lemma 1 it follows that $\|S_n^{-1}\| = n$. This and the above matrix representations of $Q^+_n$ imply that

$$
n \leq \|Q^+_n\| \leq 2n.
$$
Consequently, for any $f \in \mathcal{K}_n$ we have

$$\|Q_n^+ f\| \leq 2n\|f\|.$$ 

It follows from (5) that the vectors $f_{n1}^+, \ldots, f_{nn}^+$ form an orthonormal basis in the Hilbert space $(\mathcal{L}_n^+, [\cdot, \cdot])$. Denote by

$$P_{nk}^+ = \frac{[\cdot, f_{nk}^+] [f_{nk}^+, \cdot]}{[f_{nk}^+, f_{nk}^+]^+}, \quad k = 1, \ldots, n,$$

the orthogonal projection in the Krein space $\mathcal{K}_n$ onto the subspace spanned by the vector $f_{nk}^+, k = 1, \ldots, n$. Then, by (5) and (10),

$$1 \leq \|P_{nk}^+\| = \frac{\|f_{nk}^+\|^2}{[f_{nk}^+, f_{nk}^+]^+} \leq 8, \quad k = 1, \ldots, n. \tag{9}$$

Further, the operator

$$J_{n1} := Q_n^+ - Q_n^-$$

is a fundamental symmetry in $(\mathcal{K}_n, [\cdot, \cdot])$. In particular, the inner product

$$(x, y)_1 := [J_{n1} x, y], \quad x, y \in \mathcal{K}_n,$$

is positive definite. Therefore, the operator $J_n J_{n1}$ is positive and invertible in the Hilbert space $(\mathcal{K}_n, (\cdot, \cdot))$. Note also that $J_{n1} = J_{n1}^{-1}$. It follows from (8) that

$$\|J_{n1}\| = \|J_{n1}^{-1}\| \leq \|Q_n^+\| + \|Q_n^-\| \leq 4n. \quad \text{Consequently,}$$

$$\|J_n J_{n1}\| = \|(J_n J_{n1})^{-1}\| \leq 4n. \tag{10}$$

The vectors $f_{nj}^+, f_{nk}^-$, $j, k = 1, \ldots, n$, are orthonormal in $(\mathcal{K}_n, (\cdot, \cdot))$. This follows from (5), (7) and the relation

$$\langle f_{nj}^+, f_{nk}^- \rangle_1 = \langle (Q_n^+ - Q_n^-) f_{nj}^+, f_{nk}^- \rangle = \langle Q_n^+ f_{nj}^+, f_{nk}^- \rangle = \langle f_{nj}^+, f_{nk}^- \rangle = 0.$$ 

Now we can apply Lemma 2 to the vectors $f_{nj}^+, f_{nk}^-$, $j, k = 1, \ldots, n$, and the positive operator $J_n J_{n1}$: For given $\lambda_1^+, \ldots, \lambda_n^+ \in \mathbb{C}$ define an operator $A_n$ by

$$A_n f_{nj}^\pm = \lambda_{nj}^\pm f_{nj}^\pm, \quad j = 1, \ldots, n,$$

and then extend it by linearity to $\mathcal{K}_n$. It follows from Lemma 2 and (10) that

$$\|A_n\| \leq 4n \max\{|\lambda_j^\pm|, \quad j = 1, \ldots, n\} \leq 4C. \tag{11}$$

Let $\mathcal{K}$ be the Krein space which is the direct orthogonal sum of the Krein spaces $\mathcal{K}_n$, $n \in \mathbb{N}$,

$$\mathcal{K} := \bigoplus_{n=1}^{\infty} \mathcal{K}_n.$$

The vectors $f_{nj}^\pm$, $j = 1, \ldots, n, n \in \mathbb{N}$, constructed above are considered as elements of $\mathcal{K}$ and the Krein spaces $\mathcal{K}_n$, $n \in \mathbb{N}$, are considered as mutually orthogonal subspaces of $\mathcal{K}$. The vectors $f_{nj}^\pm$, $j = 1, \ldots, n$, form a basis for $\mathcal{K}_n$. Let $\lambda_{nj}^\pm$, $j = 1, \ldots, n$, be distinct real numbers such that $\pm \lambda_{nj}^+ > 0$, $j = 1, \ldots, n$, and such that there exists a constant $C$ with

$$n \max\{|\lambda_{nj}^\pm|, \quad j = 1, \ldots, n\} \leq C \tag{12}$$

for all $n \in \mathbb{N}$.
Put

\[ A := \bigoplus_{n=1}^{\infty} A_n. \]

Then \( A \) is a positive operator in the Krein space \((\mathcal{K}, [\cdot, \cdot])\), and from (11) and (12) we get \( \|A\| \leq 4C \). Since the linear span of the vectors \( f_{n,j} \), \( j = 1, \ldots, n \), \( n \in \mathbb{N} \), is dense in \( \mathcal{K} \), it follows from the spectral theorem (see [1, Theorem IV.1.5] or [5, Theorem 3.1]) that the nonzero spectrum of \( A \) consists of the simple eigenvalues \( \lambda_{n,j}^\pm, j = 1, \ldots, n \), \( n \in \mathbb{N} \). Consequently, the left-hand side of the inequality (8) implies that 0 is a singular critical point of \( A \) and the right-hand side of the inequality (9) implies that the norms of the orthogonal projections in \((\mathcal{K}, [\cdot, \cdot])\) onto the eigenspaces of \( A \) are uniformly bounded by 8. The theorem is proved.

**Remark.** We can arrange the numbers \( \lambda_{n,j}^\pm, j = 1, \ldots, n \), \( n \in \mathbb{N} \), in an lower triangular table. Also, we can put the sequence \( \{\frac{1}{m}, m \in \mathbb{N}\} \) in a lower triangular table by ending each row with a triangular number \( \frac{n(n+1)}{2} \) in the denominator. A comparison of these two tables leads to

\[ \lambda_{n,j}^\pm := \pm \left( \frac{n(n-1)}{2} + j \right)^{-1}, \quad j = 1, \ldots, n, \quad n \in \mathbb{N}. \] (13)

In this way we get

\[ \{\lambda_{n,j}^\pm, j = 1, \ldots, n, \quad n \in \mathbb{N}\} = \left\{ \pm \frac{1}{m}, \quad m \in \mathbb{N} \right\}. \]

The numbers \( \lambda_{n,j}^\pm \) in (13) satisfy (12) with \( C = 2 \). The proof of the Theorem implies that the nonzero spectrum of the operator \( A \), which was constructed by means of the numbers \( \lambda_{n,j}^\pm \) from (13), consists of the simple eigenvalues \( \pm \frac{1}{m}, \quad m \in \mathbb{N} \).

If we consider the inverse \( B = A^{-1} \) of the operator \( A \) from the previous theorem and with the specific choice of numbers \( \lambda_{n,j}^\pm \) as in the Remark, we get:

**Corollary.** There exist a Krein space \((\mathcal{K}, [\cdot, \cdot])\) and an unbounded positive operator \( B \) in \( \mathcal{K} \) with the following properties:

(a) The nonzero spectrum of \( B \) consists of isolated simple eigenvalues.

(b) The point \( \infty \) is a singular critical point of \( B \).

(c) For each positive number \( \mu \) we have

\[ \|E([a,b])\| \leq 8|\mu| \quad \text{whenever} \quad b - a < \mu, \]

where \( E \) is the spectral function of \( B \) and \( |\mu| \) denotes the largest integer smaller than \( \mu \).

**Proof.** Let \( A \) be the operator defined in the proof of the Theorem with the specific choice of the numbers \( \pm \lambda_{n,j} \) as in the Remark. Then \( B = A^{-1} \) is a positive operator with a nonempty resolvent set (see e.g. [5, Proposition 3.1]), and \( \sigma(B) = \mathbb{Z} \setminus \{0\} \). Let \( \mu > 0 \) be arbitrary and let \( 0 < b - a < \mu \). Then the interval \([a,b])\) contains at most \( |\mu| \) eigenvalues of \( B \). Therefore, \( \|E([a,b])\| \leq 8|\mu| \). \( \square \)
REFERENCES


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