DIAGONAL DOMINANCE
AND HARMLESS OFF-DIAGONAL DELAYS

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Abstract. Systems of linear differential equations with constant coefficients, as well as Lotka–Volterra equations, with delays in the off-diagonal terms are considered. Such systems are shown to be asymptotically stable for any choice of delays if and only if the matrix has a negative weakly dominant diagonal.

1. Introduction

Consider a system of retarded linear differential equations with constant coefficients of the form

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j (t - \tau_{ij}), \quad \text{for } i = 1, \ldots, n,$$

with

$$\tau_{ij} \geq 0 \quad \text{for } 1 \leq i \neq j \leq n \quad \text{and} \quad \tau_{ii} = 0 \quad \text{for } i = 1, \ldots, n.$$  

This paper deals with the following question:

(*) For which matrices $A = (a_{ij})$ is the trivial solution $x = 0$ of (1.1) asymptotically stable for any choice of delays satisfying (1.2)?

Such type of stability has been referred to as ‘absolute stability’ in the literature (cf. El’sgol’ts and Norkin [3, p. 175]). In the context of population modelling, the term ‘harmless’ delays has also been used (see Gopalsamy [5]). For $n = 2$, this question was answered by Lu and Wang [12]. They showed that (*) holds iff $a_{11}, a_{22} < 0$ and $a_{11} a_{22} > |a_{12} a_{21}|$ (with equality allowed in the latter if $a_{12} a_{21} < 0$). Actually, their result was for Lotka–Volterra equations but the proof applies to (1.1) with minor modifications. We extend this result to arbitrary $n$, both for the linear case (1.1) and the Lotka–Volterra case (3.1).

Definition. Let $\tilde{A} = (\tilde{a}_{ij})$ be the matrix with entries $\tilde{a}_{ii} = a_{ii}$ and $\tilde{a}_{ij} = |a_{ij}|$ for $i \neq j$. $A$ is said to be weakly diagonally dominant if all the principal minors of $-\tilde{A}$ are non-negative.

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Our main result is the following.

**Theorem 1.** (1.1) is asymptotically stable for all choices of delays of the form (1.2) if and only if $a_{ii} < 0$ for $i = 1, \ldots, n$, $\det A \neq 0$ and $A$ is weakly diagonally dominant.

For quasimonotone matrices $A$ (i.e., $a_{ij} \geq 0$ for $i \neq j$), a similar result was obtained by Győri (1992).

2. **Proof of Theorem 1**

The characteristic equation of the delay differential equation (1.1) is given by

$$
\det \begin{pmatrix}
  a_{11} - \lambda & a_{12}e^{-\lambda \tau_{12}} & \cdots \\
  a_{21}e^{-\lambda \tau_{21}} & a_{22} - \lambda & \cdots \\
  \vdots & \vdots & \ddots 
\end{pmatrix} = 0.
$$

Since $x(t) = ce^{\lambda t}$ is a solution of (1.1) for suitable $c \neq 0$ if $\lambda$ satisfies (2.1), by Corollary 6.1 of Hale and Verduyn Lunel [7, p. 215], the trivial solution $x = 0$ of (1.1) is asymptotically stable iff all the roots of (2.1) have negative real part.

2.1. **Sufficiency part.**

**Lemma 1.** If $A$ is weakly diagonally dominant, then all roots of (2.1) have negative real part, with the possible exception of $\lambda = 0$.

**Proof.** We first consider irreducible matrices $A$. By Theorem 5.9 of Fiedler [4, p. 124], for an irreducible weakly diagonally dominant matrix $A$, there is a $c > 0$ such that $\hat{A}c \leq 0$, i.e., there exist $c_i > 0$ such that

$$
a_{ii}c_i + \sum_{j \neq i} |a_{ij}|c_j \leq 0 \quad \text{for all } i = 1, \ldots, n. \quad (2.2)
$$

Suppose, for some set of delays $\tau_{ij}$ satisfying (1.2), there exists a root $\lambda$ of (2.1) with $\Re \lambda \geq 0$. Then $\lambda$ is an eigenvalue of the matrix $B = (b_{ij})$, where $b_{ij} = a_{ij}e^{-\lambda \tau_{ij}}$. Since $b_{ii} = a_{ii} \leq 0$ and $|b_{ij}| \leq |a_{ij}|$, (2.2) implies

$$
b_{ii}c_i + \sum_{j \neq i} |b_{ij}|c_j \leq 0 \quad \text{for } i = 1, \ldots, n.
$$

Applying Geršgorin’s theorem (cf. Lancaster and Tismenetsky [10, p. 371]) to the matrix $B = (c_i^{-1}b_{ij}c_j)$, which is similar to $B$, we know that the eigenvalue $\lambda$ of $B$ is contained in a circle with center $b_{ii} \leq 0$ and radius at most $|b_{ii}|$ (for some $i$). Hence either $\Re \lambda < 0$ or $\lambda = 0$.

In the case of a reducible matrix $A$, we can (by suitably relabeling the indices) turn $A$ into an upper block triangular matrix with irreducible (or zero) blocks along the diagonal (cf. (3.6) of Berman and Plemmons [2, p. 39]). Since index relabeling is done via a permutation matrix and does not affect the principal minors of $A$, each diagonal block is itself weakly diagonally dominant. The result now follows by applying the previous argument to each irreducible diagonal block.

The solution $\lambda = 0$ of (2.1) is possible only if $\det A = 0$ which is excluded. This concludes the proof of the sufficiency part of Theorem 1.
2.2. Necessity part. We start with three lemmas.

**Lemma 2.** If \( a_{ii} < 0 \) for all \( i = 1, \ldots, n \) and \( \det(-\hat{A}) < 0 \), then there exist delays \( \tau_{ij} \) satisfying (1.2) such that (2.1) has a root \( \lambda \) with \( \Re \lambda > 0 \).

**Proof.** Consider the function

\[
F(z) = \det\begin{pmatrix}
a_{11} - z \varepsilon & a_{12}e^{-z\eta_{12}} & \cdots \\
a_{21}e^{-z\eta_{21}} & a_{22} - z \varepsilon & \cdots \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\]

where

\[
\eta_{ij} = \begin{cases} 
1, & \text{for } a_{ij} < 0, \\
\frac{1}{2}, & \text{for } a_{ij} \geq 0.
\end{cases}
\]

For \( z = x + 2\pi i \), where \( x \) is real, \( F_0(z) \) becomes

\[
D(x) = \det\begin{pmatrix}
a_{11} & |a_{12}|e^{-x\eta_{12}} & \cdots \\
|a_{21}|e^{-x\eta_{21}} & a_{22} & \cdots \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix}.
\]

By assumption, \((-1)^n D(0) = \det(-\hat{A}) < 0\), and \((-1)^n D(\infty) = (-1)^n a_{11} \cdots a_{nn} > 0\). Hence, by intermediate value theorem, there exists \( \hat{x} > 0 \) such that \( D(\hat{x}) = 0 \) and \( \hat{z} = \hat{x} + 2\pi i \) is a zero of \( F_0 \). It follows from Rouché's theorem (cf. Ahlfors [1, p. 153]) that the analytic function \( F \) has a zero \( \hat{z}(\varepsilon) \) near \( \hat{z} \) for small \( \varepsilon > 0 \). Clearly, \( \lambda = \hat{z}(\varepsilon) \varepsilon \) and \( \tau_{ij} = \frac{\eta_{ij}}{\varepsilon} \) satisfy (2.1), with \( \Re \lambda > 0 \). \(\square\)

**Lemma 3.** Let \( f(\lambda) \) be a real polynomial of degree \( n \) with \( n \) real zeros, including 0. Then the equation

\[ f(\lambda) = e^{-\lambda \tau} \]

has a solution \( \lambda \) with \( \Re \lambda > 0 \) for some \( \tau > 0 \).

**Proof.** Write \( f(\lambda) = \alpha \prod_{i=1}^n (\lambda - \lambda_i) \), where \( \alpha \neq 0 \in \mathbb{R} \) and \( \lambda_i \in \mathbb{R} \) with \( \lambda_1 = 0 \). We first show (2.3) has a purely imaginary root \( \lambda = i\omega \), where \( \omega > 0 \), for some \( \tau > 0 \). By suitably choosing \( \tau > 0 \), \( f(i\omega) = e^{-i\omega \tau} \) holds iff \( |f(i\omega)| = 1 \). Since \( |f(0)| = 0 \) and \( \lim_{\omega \to \infty} |f(i\omega)| = \infty \), the existence of \( \omega > 0 \) follows from the intermediate value theorem. Let \( \hat{f}(\lambda) = e^{-\lambda \hat{\tau}} \), where \( \hat{\lambda} = i\hat{\omega} \) and \( \hat{\tau} > 0 \).

Next, we show that by slightly increasing \( \tau \) past \( \hat{\tau} \), the solution \( \lambda \) crosses the imaginary axis. Indeed, let \( g(\lambda, \tau) = f(\lambda) - e^{-\lambda \tau} \). Then

\[ g(\lambda, \hat{\tau}) = e^{-\hat{\lambda} \hat{\tau}} \left[ \sum_{i=1}^n \frac{1}{\lambda - \lambda_i} + \hat{\tau} \right] \neq 0, \]

since

\[ \Im \left( \sum_{i=1}^n \frac{1}{\lambda - \lambda_i} \right) = -\hat{\omega} \sum_{i=1}^n \frac{1}{\lambda_i^2 + \hat{\omega}^2} < 0. \]

The implicit function theorem applied to (2.3) at \((\hat{\lambda}, \hat{\tau})\) yields the solution

\[ \lambda(\tau) = \hat{\lambda} + c(\tau - \hat{\tau}) + O((\tau - \hat{\tau})^2) \]

where \( c = -\frac{g_x(\hat{\lambda}, \hat{\tau})}{g_{xx}(\hat{\lambda}, \hat{\tau})} \). It is easy to see that \( \Re c > 0 \), and hence \( \Re \lambda(\tau) > 0 \) for \( \tau > \hat{\tau} \). \(\square\)
Lemma 4. If \( a_{ii} = 0 \) for some \( i \) and \( \det A \neq 0 \), then there exist delays \( \tau_{ij} \) satisfying (1.2) such that (2.1) has a root \( \lambda \) with \( \Re \lambda > 0 \).

Proof. Assume \( a_{11} = 0 \). Since \( \det A \neq 0 \), there exists a non-zero term in the expansion of \( \det A \), i.e. there is a permutation \( \sigma \) of \( \{1, \ldots, n\} \) such that \( \prod_{i=1}^{n} a_{i\sigma_i} \neq 0 \). The corresponding term in the expansion of (2.1) together with the product of the diagonal entries yields the ‘truncated’ equation

\[
(2.5) \quad \prod_{i=1}^{n} (a_{ii} - \lambda) + \text{sgn}(\sigma) \prod_{i=1}^{n} (a_{i\sigma_i} - \lambda \delta_{i\sigma_i}) \cdot e^{-\lambda \sum_{i=1}^{n} \tau_{i\sigma_i}} = 0.
\]

After cancelling the factor \( a_{ii} - \lambda \) (for each \( i \) with \( \sigma_i = i \)), there remains an equation of the form (2.3), where \( \tau = \sum_{i=1}^{n} \tau_{i\sigma_i} \). By Lemma 3, there is a solution \( \lambda \) of (2.5) with \( \Re \lambda > 0 \) for a suitable \( \tau > 0 \). By letting \( \tau_{ij} \to \infty \) whenever \( j \neq i \) and \( j \neq \sigma_i \), the remaining \( n! - 2 \) terms in the expansion of (2.1) left out in (2.5) can be made arbitrarily small (in a fixed neighbourhood of \( \lambda \)). Hence Rouche’s theorem again shows the existence of a root \( \lambda \) of (2.1) near \( \hat{\lambda} \). This shows the lemma.

Returning to the proof of the sufficiency part, the asymptotic stability assumption on \( x = 0 \) precludes \( \lambda = 0 \) from being a root of (2.1). Hence \( \det A \neq 0 \).

If \( a_{kk} > 0 \) for some \( k \), then Rouche’s theorem shows that (2.1) has a root near \( a_{kk} \) for large \( \tau_{ij} \). This contradiction together with Lemma 4 shows that \( a_{ii} < 0 \) for all \( i = 1, \ldots, n \).

Now suppose that some principal minor of \( -\tilde{A} \) is negative. Without loss of generality, we may assume \( \det(-\tilde{A}_k) < 0 \), where \( \tilde{A}_k = (a_{ij})_{1 \leq i,j \leq k} \). Applying Lemma 2 to this principal submatrix yields a root \( \hat{\lambda} \) with \( \Re \hat{\lambda} > 0 \) for the characteristic equation (2.1) corresponding to \( A_k \). By letting \( \tau_{ij} \to \infty \) whenever \( i > k \) or \( j > k \), a similar perturbation argument as above shows that the full characteristic equation (2.1) (corresponding to \( A \)) has a root near \( \hat{\lambda} \). In particular, (2.1) will have a root with positive real part, which contradicts the assumption of stability of \( x = 0 \). This completes the proof of Theorem 1.

3. Lotka–Volterra equations

In this section we prove a similar result for a class of nonlinear differential equations, widely used in population dynamics, the Lotka–Volterra systems

\[
\dot{y}_i = y_i \left( r_i + \sum_{j=1}^{n} a_{ij} y_j (t - \tau_{ij}) \right), \quad i = 1, \ldots, n.
\]

We assume in the following that there exists a positive vector \( \hat{y} \) with

\[
(3.2) \quad r + A\hat{y} = 0.
\]

This \( \hat{y} \) is then an equilibrium for (3.1), i.e., \( y(t) = \hat{y} \) is a (constant) solution of (3.1).

We are interested in the absolute stability of \( \hat{y} \); and we will show that this is the case under the same conditions on \( A \) as in the linear case (1.1). That the strict version of diagonal dominance (all principal minors of \( -A \) are positive) implies the global stability of the unique saturated equilibrium has been discussed in detail in Hofbauer and Sigmund \[8\] for ordinary Lotka–Volterra equations and in Kuang \[9\] for delayed versions like (3.1). That weak diagonal dominance is already sufficient
is more subtle, and its proof uses ideas from Lu and Takeuchi [11], Lu and Wang [12] and Redheffer [13].

**Theorem 2.** Suppose there exists a positive vector \( \hat{y} \) satisfying (3.2). Then \( \hat{y} \) is globally asymptotically stable for (3.1) (for initial conditions \( y_i(0) > 0 \)) for all delays \( \tau_{ij} \) satisfying (1.2) if and only if \( a_{ii} < 0 \) for \( i = 1, \ldots, n \), \( \det A \neq 0 \) and \( A \) is weakly diagonally dominant.

**Proof.** For the necessity part, first we observe that in the course of proving Theorem 1, we have shown that, under the assumption \( \det(A) \neq 0 \), the characteristic equation (2.1) has a root with positive real part (for a certain choice of \( \tau_{ij} \) satisfying (1.2)), provided \( a_{ii} > 0 \) for some \( i \) or \( A \) is not weakly diagonally dominant. Next, let \( B = (\hat{y}_i a_{ij}) \) be the linearization of (3.1) at \( \hat{y} \). Then \( \det(B) \neq 0 \), because otherwise \( \det(A) = 0 \) and (3.1) has many equilibrium solutions near \( \hat{y} \) (implying \( \hat{y} \) is not asymptotically stable). Now, since \( \hat{y} \) is asymptotically stable, none of the roots of (2.1), with \( A \) replaced by \( B \), can have positive real parts (cf. Theorem 2.1 of Hale and Verduyn Lunel [7, p. 314]). Hence, \( b_{ii} < 0 \) for all \( i \) and \( B \) is weakly diagonally dominant. This implies \( a_{ii} < 0 \) for all \( i \) and \( A \) is weakly diagonally dominant as well.

For the sufficiency part, the proof in section 2 only gives (local) asymptotic stability. When \( A \) is irreducible, it turns out that one can choose \( \alpha_i > 0 \) and \( \beta_{ij} > 0 \) appropriately (see later) so that

\[
V(y(.), t) = \sum_{i=1}^{n} \alpha_i \left( y_i(t) - \hat{y}_i \log y_i(t) \right) + \sum_{i,j=1}^{n} \beta_{ij} \int_{t-	au_{ij}}^{t} (y_j(s) - \hat{y}_j)^2 ds
\]

becomes a Lyapunov functional. For the case of a general \( A \), we will use induction to show that every solution \( y(t) \) of (3.1) with positive initial conditions converges to \( \hat{y} \).

Denoting \( y - \hat{y} \) by \( x \), (3.1) can be written as

\[
\dot{y}_i(t) = y_i(t) \sum_{j=1}^{n} a_{ij} x_j(t - \tau_{ij}).
\]

The derivative of \( V \) along a solution \( y(t) \) of (3.1) is then given by

\[
\dot{V} = \sum_{i,j=1}^{n} \alpha_i x_i(t) a_{ij} x_j(t - \tau_{ij}) + \sum_{i,j=1}^{n} \beta_{ij} \left[ x_j^2(t) - x_j^2(t - \tau_{ij}) \right].
\]

First, we assume that \( A \) is an irreducible weakly diagonally dominant matrix. Then there exist \( c_i > 0 \) such that (2.2) holds. Moreover, there also exist \( d_i > 0 \) such that

\[
d_i a_{ii} + \sum_{j \neq i} d_j |a_{ji}| \leq 0 \quad \text{for} \quad i = 1, \ldots, n.
\]

(The vectors \( c \) and \( d \) may be chosen as right and left eigenvectors of \( \hat{A} \).)

By choosing \( \alpha_i = \frac{2 d_i}{c_i} \) and \( \beta_{ij} = \frac{d_i}{c_j} |a_{ij}| \), we have

\[
\dot{V} = \sum_{i \neq j} \frac{d_i}{c_j} |a_{ij}| x_j^2(t) + 2 \sum_{i,j=1}^{n} \frac{d_i}{c_i} a_{ij} x_i(t) x_j(t - \tau_{ij}) - \sum_{i \neq j} \frac{d_i}{c_j} |a_{ij}| x_j^2(t - \tau_{ij}).
\]
After collecting terms, the coefficient of \(x_i^2(t)\) (for \(i\) fixed) is given by
\[
\sum_{j=1, j \neq i}^{n} \frac{d}{c_i} |a_{ij}| + 2 \frac{d_i}{c_i} a_{ii}.
\]
By adding \(\frac{1}{c_i}\) times (3.5) to \(\frac{1}{c_i}\) times (2.2), we see that this coefficient is no larger than \(-\sum_{j=1, j \neq i}^{n} \frac{d_i c_j}{c_i^2} |a_{ij}|\). Hence (3.6) reduces to
\[
(3.7) \quad \dot{V} \leq - \sum_{i,j \neq i} \frac{d_i}{c_i} |a_{ij}| \left( \frac{c_j}{c_i} x_i(t) \text{sgn} a_{ij} - x_j(t - \tau_{ij}) \right)^2 \leq 0.
\]

The boundedness of \(V\) along a positive solution \(y(t)\) implies the existence of constants \(m, M > 0\) such that \(m \leq y_i(t) \leq M\) for all \(i\) and for all \(t \geq 0\). Hence forward orbits are precompact and the \(\omega\)-limit set exists and is nonempty, compact and invariant (cf. Lemma 1.3 of Hale and Verduyn Lunel [7, p. 103]). By LaSalle’s theorem, \(\omega(y(\cdot))\) is contained in the maximal compact invariant subset of \(\dot{V} = 0\).

Equality \(\dot{V} = 0\) is possible for \(y(\cdot) \neq \tilde{y}\) only if there is equality in (2.2) and (3.5) (i.e., \(\det A = 0\) and \(x_j(t - \tau_{ij}) = \frac{c_j}{c_i} x_i(t) \text{sgn} a_{ij}\) holds for all \(t\) and all \(i, j\) with \(i \neq j\) and \(a_{ij} \neq 0\). Inserting this into (3.1), we obtain
\[
\dot{y}_i = y_i \left[ a_{ii} x_i(t) + \sum_{j=1, j \neq i}^{n} a_{ij} x_j(t - \tau_{ij}) \right] = y_i \left[ a_{ii} x_i(t) + \sum_{j=1, j \neq i}^{n} |a_{ij}| \frac{c_j}{c_i} x_i(t) \right] = 0.
\]

Hence \(y(t)\) must be a constant solution of the differential equation (3.1). Since \(\det A \neq 0\) by assumption, \(x(t) \equiv 0\), i.e. \(y(t) \equiv \tilde{y}\) is the only positive constant solution of (3.1). Thus \(\{\tilde{y}\}\) is the global attractor and hence the maximal compact invariant subset in \(C([-\tau, 0], \mathbb{R}_+^n)\), the space of positive continuous functions on \([-\tau, 0]\), where \(\tau = \max_{i,j} \{\tau_{ij}\}\). In particular, every positive solution \(y(t)\) of (3.1) converges to \(\tilde{y}\) as \(t \to \infty\), when \(A\) is irreducible.

Next, we consider the case of a general \(A\). We will use induction on \(n\) to show that (i) for any positive solution \(y(t)\) (\(t \geq 0\)) of (3.1), there exist \(0 < m \leq M\) such that \(m \leq y_i(t) \leq M\) for all \(t \geq 0\) and \(i\), and that \(\lim_{t \to \infty} y(t) = \tilde{y}\) and (ii) if \(y(t)\) is a positive solution (3.1) defined for all \(t \in \mathbb{R}\) such that \(m \leq y_i(t) \leq M\) for all \(t\), where \(0 < m \leq M\), then \(y(t) \equiv \tilde{y}\).

If \(A\) is irreducible, then the assertions hold by the above Lyapunov functional argument. So we assume \(A\) is reducible. By a renumbering of indices, \(A\) can be written in block triangular form \(A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}\), with \(A_{11}\) irreducible of order \(k\). The delay equation (3.1) can then be written schematically as a \(2 \times 2\) system
\[
\begin{align*}
\dot{Y}_1 &= Y_1 (A_{11} X_1 + A_{12} X_2), \\
\dot{Y}_2 &= Y_2 A_{22} X_2.
\end{align*}
\]

Since the order \(n-k\) of \(A_{22}\) is strictly less than that of \(A\), by the induction hypothesis on the second half of (3.8), we can assume that \(y_i(t)\) remains bounded (both from 0 and \(\infty\)) for all \(i = k+1, \ldots, n\) and \(t \geq 0\) and \(Y_2(t) \to \bar{Y}_2 = (\bar{y}_{k+1}, \ldots, \bar{y}_n)\) as \(t \to \infty\). From the linear analysis of section 2, we can further infer that \(|X_2(t)| \leq C e^{-\varepsilon t}\) (\(t \geq 0\)) for suitable constants \(C, \varepsilon > 0\).
Now consider again the function $V$ from (3.3), with $n$ replaced by $k$, the size of the irreducible block $A_{11}$. Then a similar computation, taking into account (3.7) for the terms arising from $A_{11}$, leads to (with $|X_1|$ denoting the norm of the vector $X_1 = (x_1, \ldots, x_k)$)

\[
\dot{V} \leq \sum_{i=1}^{k} \alpha_i x_i(t) \left[ \sum_{j=k+1}^{n} a_{ij} x_j(t - \tau_{ij}) \right] \\
\leq C_1 |X_1(t)| e^{-\epsilon t} \leq C_2 (V(t) + C_3) e^{-\epsilon t}.
\]

This differential inequality shows that $V(t)$ stays bounded as $t \to \infty$; and hence the first $k$ components of $y(t)$ stay bounded (both from 0 and from $\infty$). This shows that there exist $0 < m \leq M$ such that $m \leq y_i(t) \leq M$ for all $i = 1, \ldots, n$ and $t \geq 0$.

Now consider a (full) solution $y(t) = (Y_1(t), Y_2(t))$ of (3.1) defined for all $t \in \mathbb{R}$ such that $m \leq y_i(t) \leq M$ for all $i$ and $t$, where $0 < m \leq M$. Applying the induction hypothesis to the second part of (3.8), we deduce that $Y_2(t) \equiv Y_2$. Thus $X_2(t) \equiv 0$. Hence $Y_1(t)$ satisfies $Y_1 = Y_1 A_{11} X_1$. Since $A_{11}$ is irreducible, we conclude $Y_1(t) \equiv Y_1$. Hence $y(t) \equiv \hat{y}$.

Lastly, let $y(t)$ ($t \geq 0$) be a positive solution of (3.1). Pick any $z$, a full orbit in $\omega(y_\infty)$, the $\omega$-limit set of the orbit $y_1$. Then $z(t)$ is a solution of (3.1) defined for all $t \in \mathbb{R}$ which is bounded (from 0 and $\infty$). As was shown in the previous paragraph, $z(t) \equiv \hat{y}$. Hence $\omega(y_\infty) = \{\hat{y}\}$ and $\lim_{t \to \infty} y(t) = \hat{y}$, which completes the induction proof.

\[\square\]

**Remark.** We note that the sufficiency part for the linear case (Theorem 1) can also be shown by basically the same argument as above, using the Lyapunov function $V(x(\cdot), t) = \sum_{i=1}^{n} \alpha_i x_i^2(t) + \sum_{i,j=1}^{n} \beta_{ij} \int_{t-\tau_{ij}}^{t} x_j^2(s) \, ds$ instead of (3.3).

**References**


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