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REGULARITY OF OPERATORS ON ESSENTIAL EXTENSIONS OF THE COMPACTS

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Abstract. A semiregular operator on a Hilbert C^*-module, or equivalently, on the C^*-algebra of 'compact' operators on it, is a closable densely defined operator whose adjoint is also densely defined. It is shown that for operators on extensions of compacts by unital or abelian C^*-algebras, semiregularity leads to regularity. Two examples coming from quantum groups are discussed.

1. Introduction

Hilbert C^*-modules arise in many different areas, for example, in the study of locally compact quantum groups and their representations, in KK-theory, in noncommutative geometry, and in the study of completely positive maps between C^*-algebras. A regular operator on a Hilbert C^*-module is an analogue of a closed operator on a Hilbert space that naturally arises in many of the above contexts. A closed and densely defined operator T on a Hilbert C^*-module E is called regular if its adjoint T^* is also densely defined and if the range of $(I + T^*T)$ is dense in E. Every regular operator on a Hilbert C^*-module E is uniquely determined by a (bounded) adjointable operator on E, called its z-transform. This fact is exploited when dealing with regular operators, as the adjointable operators, being bounded, are more easily manageable than unbounded operators. But given an unbounded operator, the first and the most basic problem is to decide whether or not it is regular. In [7], Woronowicz investigated this problem using graphs of operators, and proved a few results (see Proposition 2.2, Theorem 2.3 and Examples 1–3 in [7]). In particular, he was able to conclude the regularity of some very simple functions of a regular operator T, like $T + a$ where a is an adjointable operator, and Ta and aT where a is an invertible adjointable operator.

The problem was later attacked from a different angle in [5]. A somewhat larger class of operators, called the *semiregular* operators, were considered. A semiregular operator is a closable densely defined operator whose adjoint is also densely defined. Though regularity is quite difficult to ascertain, semiregularity is not. The problem then investigated in [5] was 'when is a semiregular operator regular?'. The first step was to reduce the problem to a problem on C^*-algebras by establishing that

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semiregular operators on a Hilbert C^*-module E correspond, in a canonical manner, to those on the C^*-algebra $\mathcal{K}(E)$ of 'compact' operators on E. The question to be answered next is then 'for what class of C^*-algebras is a closed semiregular operator regular (or admits regular extension)?' Among other things, it was established that for abelian C^*-algebras, as well as for subalgebras of $\mathcal{B}_0(\mathcal{H})$, closed semiregular operators are indeed regular. In the present paper, we will extend the results to a class of C^*-algebras that contain $\mathcal{B}_0(\mathcal{H})$ as an essential ideal. Most of the results, however, are valid in a more general situation where $\mathcal{B}_0(\mathcal{H})$ is replaced by any essential ideal K. Since it involves almost no extra work, the results are stated in this general set up. In section 2, we develop the necessary background for proving the main results which are presented in section 3. Finally in section 4, we discuss two examples that arise in the context of quantum groups and are covered by the results in section 3. We have assumed elements of C^*-algebra theory and Hilbert C^*-module theory as can be found, for example, in Pedersen ([6]) and Lance ([3]) respectively.

Now, why are essential extensions of the compacts important in the context of the problem? First, because they cover examples that arise naturally, like the quantum complex plane which has been discussed later in this paper. Second and perhaps more importantly, because they arise as irreducible representations of all type I C^*-algebras. For a large class of type I C^*-algebras, one would be able to conclude by the results here that all irreducible 'fibres' of a semiregular operator S are regular. This fact, along with some mild restrictions on S, should then lead to its regularity.

Notations. We will follow standard notations mostly. So, for example, $\mathcal{H}$ is a complex separable Hilbert space, $\mathcal{B}_0(\mathcal{H})$ is the algebra of compact operators on $\mathcal{H}$; $\mathcal{A}$ is a C^*-algebra, $M(\mathcal{A})$ and $LM(\mathcal{A})$ are the space of multipliers and left multipliers respectively of $\mathcal{A}$. For a topological space X, $C_0(X)$ will denote the C^*-algebra of continuous functions on X vanishing at infinity. The C^*-algebra $\mathcal{A}$ that we will primarily be interested in *will always be assumed to be separable* (this of course will not be true for all C^*-algebras that we deal with; for example the multiplier algebra of a nonunital C^*-algebra is never separable).

2. Restriction to an ideal

Let $\mathcal{A}$ be a nonunital C^*-algebra and let K be an essential ideal in $\mathcal{A}$. Since $\mathcal{A}$ is essential in $M(\mathcal{A})$, it follows that K is essential in $M(\mathcal{A})$. It is easy to see then that there is an injective homomorphism from $M(\mathcal{A})$ to $M(K)$ through which $M(\mathcal{A})$ can be thought of as sitting inside $M(K)$.

For the rest of this paper, we will always assume that $K \subseteq \mathcal{A} \subseteq M(\mathcal{A}) \subseteq M(K)$.

Before we proceed further, let us recall the definition of a semiregular operator.

Definition 2.1 ([5])**.** Let E and F be Hilbert $\mathcal{A}$-modules. An operator $T : E \to F$ is called *semiregular* if

(a) D_T is a dense submodule of E (i.e. $D_T\mathcal{A} \subseteq D_T$),
(b) T is closable,
(c) T^* is densely defined.

Next we list some elementary observations regarding the restriction of a semiregular operator to an essential ideal.

Proposition 2.2. *Let S be a closed semiregular operator on $\mathcal{A}$. Then*

1. $D_K := D(S) \cap K$ *is a dense right ideal in* K,
2. $S(D_K) \subseteq K$,
3. $S_0 := S|_K$ *is closed and semiregular,*
4. $D(S)K$ *is a core for* S_0,
5. $(S|_K)^* = S^*|_K$.

Proof. 1. That D_K is a right ideal is obvious. Let us show that it is dense. Choose any $a \in K$. Let $\{e_\alpha\}$ be an approximate identity in K. For any $\epsilon > 0$, there is an $\tilde{a} \in D(S)$ such that $\|a - \tilde{a}\| < \epsilon$. Hence for large enough α,

$$\begin{aligned} \|\tilde{a}e_\alpha - a\| &\leq \|\tilde{a} - a\|\|e_\alpha\| + \|ae_\alpha - a\| \\ &\leq 2\epsilon. \end{aligned}$$

Since $\tilde{a} \in D(S)$, $e_\alpha \in K$, $\tilde{a}e_\alpha \in D(S) \cap K$.

2. Take an $a \in D_K$. For any $b \in D(S^*)$, $b^*Sa = (S^*b)^*a \in K$. Since $D(S^*)$ is dense in $\mathcal{A}$, we have $b^*(Sa) \in K$ for all $b \in \mathcal{A}$. Put $b = Sa$ to get $(Sa)^*(Sa) \in K$. Hence $|Sa|^{1/2} \in K$. Now in $\mathcal{A}$, there exists an element u such that $Sa = u|Sa|^{1/2}$. Hence $Sa \in K$.

3. For $a \in D(S^*|_K)$ and $b \in D(S|_K)$, we have

$$\langle S^*|_K a, b\rangle = \langle S^*a, b\rangle = \langle a, Sb\rangle = \langle a, S|_K b\rangle.$$

Therefore $S^*|_K \subseteq (S|_K)^*$ and $(S|_K)^*$ is densely defined. Now suppose $a_n \in D(S|_K) = J_K$, and $a_n \to a$, $S|_K a_n \to b$. Since $S|_K a_n = Sa_n$ and S is closed, we conclude that $a \in D(S)$ and $Sa = b$. But $a \in K$ also. Hence $a \in J_K$, and $S|_K a = b$.

4. Take $a \in D(S|_K)$. If $\{e_\alpha\}$ is an approximate identity for K, then $ae_\alpha \to a$ and $S|_K(ae_\alpha) = (S|_K a)e_\alpha \to S|_K a$. Since $ae_\alpha \in D(S)K$, $D(S)K$ is a core for $S|_K$.

5. We have already seen that $S^*|_K \subseteq (S|_K)^*$. Let us prove the reverse inclusion here. For any $a \in D((S|_K)^*)$, $b \in D(S)$, $k \in K$, we have

$$\begin{aligned} \langle a, Sb\rangle k &= \langle a, S(bk)\rangle \\ &= \langle (S|_K)^*a, bk\rangle \\ &= \langle (S|_K)^*a, b\rangle k. \end{aligned}$$

Hence $\langle a, Sb\rangle = \langle (S|_K)^*a, b\rangle$, so that $a \in D(S^*)$. Thus $D((S|_K)^*) \subseteq D(S^*) \cap K = D(S^*|_K)$. □

Proposition 2.3. *Let S and T be semiregular operators on $\mathcal{A}$ such that $S|_K = T|_K$. Then*

1. $S = T$ *on* $D(S) \cap D(T)$,
2. $S^* = T^*$,
3. *if* $(S|_K)^{**} = S|_K$, *then there exists a maximal closed semiregular operator on* $\mathcal{A}$ *whose restriction to* K *equals* $S|_K$.

Proof. 1. Take $a \in D(S) \cap D(T)$. For any $k \in K$, $ak \in D(S|_K) = D(T|_K)$. Hence $(Sa)k = S(ak) = T(ak) = (Ta)k$. Therefore $Sa = Ta$.

2. Take any $a \in D(S^*)$, $b \in D(T)$. Then for any $k \in K$,

$$\begin{aligned} \langle a, Tb\rangle k &= \langle a, T(bk)\rangle \\ &= \langle a, S(bk)\rangle \\ &= \langle S^*a, bk\rangle \\ &= \langle S^*a, b\rangle k. \end{aligned}$$

Hence $\langle a, Tb\rangle = \langle S^*a, b\rangle$. Thus $S^* \subseteq T^*$. Similarly $T^* \subseteq S^*$.

3. S^{**} is the required operator. For, if T is any other semiregular operator whose restriction to K is $S|_K$, then $T^* = S^*$, thereby implying $S^{**} = T^{**}$, so that $T \subseteq S^{**}$. By part 5 of the foregoing proposition, $S^{**}|_K = (S^*|_K)^* = (S|_K)^{**} = S|_K$. □

Part 3 above tells us, in particular, that if $S|_K$ is regular, then S^{**} is the maximal semiregular operator on $\mathcal{A}$ whose restriction to K is same as that of S.

Lemma 2.4. *If T is regular on $\mathcal{A}$ with z-transform z, then $T(K) \subseteq K$, and $T|_K$ is a regular operator on K with the same z-transform z.*

Proof. Observe that $z \in M(\mathcal{A}) \subseteq M(K)$, and $(I-z^*z)^{1/2}K$ contains $(I-z^*z)^{1/2}\mathcal{A}K = D(T)K$ which is dense in K. Hence there exists a regular operator T_0 on K with z-transform z. Clearly $T_0 \subseteq T|_K$. By part 4 of Proposition 2.2, $T_0 = T|_K$. □

Proposition 2.5. *Let S be a closed semiregular operator on $\mathcal{A}$ such that $S|_K$ is regular with z-transform $z \in M(K)$. Then for any $a \in D(S)$, there is a $c \in M(K)$ such that*

$$\begin{aligned} a &= (I - z^*z)^{1/2}c, \\ Sa &= zc. \end{aligned}$$

Proof. Take an $a \in D(S)$. Let $\{e_\alpha\}$ be an approximate identity for K. For each α, one has $ae_\alpha \in D(S) \cap K = D(S|_K)$. Hence there is a $c_\alpha \in K$ such that

$$\begin{aligned} ae_\alpha &= (I - z^*z)^{1/2}c_\alpha, \\ S(ae_\alpha) &= zc_\alpha. \end{aligned} \tag{2.1}$$

From the above equations it follows that $c_\alpha = (I - z^*z)^{1/2}ae_\alpha + z^*(Sae_\alpha) = ce_\alpha$, where $c = (I - z^*z)^{1/2}a + z^*(Sa)$. Now using the fact that e_α is an approximate identity, we get

$$\begin{aligned} ak &= (I - z^*z)^{1/2}ck, \\ (Sa)k &= zck \end{aligned}$$

for all $k \in K$, which proves the result. □

The above proposition together with the one that follows will be the key ingredients in proving the regularity of certain semiregular operators later.

Proposition 2.6. *For any $a \in D(S^*)$, there exists $c \in M(K)$ such that*

$$\begin{aligned} a &= (I - zz^*)^{1/2}c, \\ S^*a &= z^*c. \end{aligned}$$

Proof. Similar to the proof of the previous proposition. □

Let us denote by D the set $\{(I - z^*z)^{1/2}a + z^*(Sa) : a \in D(S)\}$ and by D_* the set $\{(I - zz^*)^{1/2}a + z(S^*a) : a \in D(S^*)\}$. Observe that for $c \in D$ and $d \in D_*$, zc and z^*d are in $\mathcal{A}$.

Lemma 2.7. *Let D be as above, and assume that $S = S^{**}$. Then*

1. *D is a Hilbert $\mathcal{A}$-module contained in $M(K)$,*
2. *$D = \Gamma(z) := (I - z^*z)^{-1/2}\mathcal{A} \cap z^{-1}\mathcal{A} \equiv \{c \in M(K) : (I - z^*z)^{1/2}c \in \mathcal{A}, zc \in \mathcal{A}\}$.*

Proof. Part 1 is straightforward. We will prove part 2 here. Define an operator $\tilde{S} : (I - z^*z)^{1/2}\Gamma(z) \to \mathcal{A}$ by

$$\tilde{S}((I - z^*z)^{1/2}c) = zc, \quad c \in \Gamma(z).$$

By Proposition 2.5, $D \subseteq \Gamma(z)$ and $S \subseteq \tilde{S}$. Hence $\tilde{S}$ is densely defined. From the injectivity of $(I - z^*z)^{1/2}$ it follows that $\tilde{S}$ is well-defined. It can easily be verified from the definition of $\tilde{S}$ that it is closed.

By Proposition 2.2, $S^*|_K = (S|_K)^*$ and hence has z-transform z^*. From Proposition 2.6, we conclude that $D_* \subseteq \Gamma(z^*)$. Now, for $d \in D_*$ and $c \in \Gamma(z)$,

$$\begin{aligned}\langle (I - zz^*)^{1/2}d, \tilde{S}((I - z^*z)^{1/2}c)\rangle &= \langle (I - zz^*)^{1/2}d, zc\rangle \\ &= \langle z^*(I - zz^*)^{1/2}d, c\rangle \\ &= \langle S^*((I - zz^*)^{1/2}d), (I - z^*z)^{1/2}c\rangle,\end{aligned}$$

so that $D(S^*) \subseteq D((\tilde{S})^*)$. Therefore $S^* \subseteq (\tilde{S})^*$. Thus $S \subseteq \tilde{S} \subseteq (\tilde{S})^{**} \subseteq S^{**} = S$. This implies $D(S) = D(\tilde{S}) \equiv (I - z^*z)^{1/2}\Gamma(z)$, i.e. $\Gamma(z) \subseteq D$. □

A similar statement about D_* also holds; except that in that case one need not assume $S^* = S^{***}$; it is automatic. The above proposition tells us that if $S|_K$ is regular, even though S may not be regular, it is uniquely determined by a bounded adjointable operator on K, as long as S is sufficiently nice (i.e. $S = S^{**}$).

Proposition 2.8. *Let S be a closed semiregular operator on $\mathcal{A}$ such that $S|_K$ is regular with z-transform z. Then one has the following inclusions:*

$$\begin{array}{rlrl}
\text{(i)} & z\mathcal{A} \subseteq \overline{(I - zz^*)^{1/2}\mathcal{A}}, & \text{(ii)} & z^*\mathcal{A} \subseteq \overline{(I - z^*z)^{1/2}\mathcal{A}}, \\
\text{(iii)} & \mathcal{A}z \subseteq \overline{\mathcal{A}(I - z^*z)^{1/2}}, & \text{(iv)} & \mathcal{A}z^* \subseteq \overline{\mathcal{A}(I - zz^*)^{1/2}}, \\
\text{(v)} & z^*z\mathcal{A} \subseteq \overline{(I - z^*z)\mathcal{A}}, & \text{(vi)} & zz^*\mathcal{A} \subseteq \overline{(I - zz^*)\mathcal{A}}, \\
\text{(vii)} & \mathcal{A} \subseteq \overline{(I - z^*z)\mathcal{A}}, & \text{(viii)} & \mathcal{A} \subseteq \overline{(I - zz^*)\mathcal{A}}.
\end{array}$$

(*Here the overline indicates closure in the norm topology.*)

Proof. We will prove (i) here. Proof of (ii) is similar. All the other inclusions follow from these two.

Take any $a = (I - z^*z)^{1/2}d \in D(S)$. Then $za = z(I - z^*z)^{1/2}d = (I - zz^*)^{1/2}zd \in (I - zz^*)^{1/2}\mathcal{A}$. Thus $zD(S) \subseteq (I - zz^*)^{1/2}\mathcal{A}$. Since $D(S)$ is dense in $\mathcal{A}$, we have the required inclusion. □

Corollary 2.9. *With the notation as above, one has the following:*

$$\begin{aligned} D &\subseteq \overline{(I - z^*z)^{1/2}\mathcal{A}}, \\ D_* &\subseteq \overline{(I - zz^*)^{1/2}\mathcal{A}}. \end{aligned}$$

Proof. Any $d \in D$ is of the form $(I - z^*z)^{1/2}a + z^*Sa$ for some $a \in D(S)$. By part (ii) of the previous proposition, $z^*Sa \in \overline{(I - z^*z)^{1/2}\mathcal{A}}$. Hence we have the first inclusion. Proof of the other one is similar. □

Lemma 2.10. *Let S be as in Proposition 2.8. If $z \in M(\mathcal{A})$, then S^{**} is regular.*

Proof. From Corollary 2.9 and the given condition, it follows that $D \subseteq \mathcal{A}$. Therefore $(I - z^*z)^{1/2}\mathcal{A}$ contains $D(S)$ and is dense in $\mathcal{A}$. So z is indeed the z-transform of some regular operator T on $\mathcal{A}$. Clearly $S \subseteq T$, so that $T^* \subseteq S^*$. From Corollary 2.9 we also have $D_* \subseteq \mathcal{A}$. Therefore $D(S^*) = (I - zz^*)^{1/2}D_* \subseteq (I - zz^*)^{1/2}\mathcal{A} = D(T^*)$. It follows then that $S^* = T^*$. Hence $S^{**} = T^{**} = T$. Thus S^{**} is regular. □

Proposition 2.11. *Let S and z be as in the previous proposition. If $z^*z \in M(\mathcal{A})$, then S^{**} is regular.*

Proof. Let us first show that zz^* is also in $M(\mathcal{A})$. Take any a and b in $D(S^*)$. There are elements c, d in D_* such that $a = (I - zz^*)^{1/2}c$ and $b = (I - zz^*)^{1/2}d$. For any integer $n \geq 1$, we have $a^*(zz^*)^n b = c^*(I - zz^*)^{1/2}z(z^*z)^{n-1}z^*(I - zz^*)^{1/2}d = (z^*c)^*(I - z^*z)^{1/2}(z^*z)^{n-1}(I - z^*z)^{1/2}z^*d \in \mathcal{A}$. Since $D(S^*)$ is norm dense in $\mathcal{A}$, one has $a^*(zz^*)^n b \in \mathcal{A}$ for all $a, b \in \mathcal{A}$, which means in particular that zz^* and $(zz^*)^2$ both are in $QM(\mathcal{A})$, the space of quasi-multipliers of $\mathcal{A}$. By Proposition 5.3 in [8], $zz^* \in LM(\mathcal{A})$, and since zz^* is positive, it is actually in $M(\mathcal{A})$.

Now from parts (i) and (iii) of Proposition 2.8 and the foregoing lemma, it follows that S^{**} is regular. □

3. Regularity

We are now ready for the main results in this paper. Let π be the canonical projection of $M(K)$ onto $M(K)/K$. Restriction of π to $\mathcal{A}$ gives the canonical projection of $\mathcal{A}$ onto $\mathcal{A}/K$.

Theorem 3.1. *Let S be a closed semiregular operator on $\mathcal{A}$ such that its restriction to K is regular. If*

$$\Big(Z(\mathcal{A}/K) \cap \pi(D(S))\Big)\mathcal{A}/K \text{ is total in } \mathcal{A}/K, \tag{3.1}$$

where $Z(\mathcal{A}/K)$ is the centre of $\mathcal{A}/K$, then S^{**} is regular.

Proof. Let z be the z-transform of $S|_K$, and let $\{e_\alpha\}_\alpha$ be an approximate identity in $\mathcal{A}$. By part (iii) of Proposition 2.8, there exist elements f_α in $\mathcal{A}$ such that $\lim_\alpha \|e_\alpha z - f_\alpha(I - z^*z)^{1/2}\| = 0$. This implies that

$$\lim_\alpha \|z^*e_\alpha^2 z - (I - z^*z)^{1/2}{f_\alpha}^* f_\alpha(I - z^*z)^{1/2}\| = 0,$$

which, in turn, implies that

$$\lim_\alpha \|z^*zd - (I - z^*z)^{1/2}{f_\alpha}^* f_\alpha(I - z^*z)^{1/2}d\| = 0$$

for all $d \in D$. It follows then that

$$\lim_\alpha \|(I - z^*z)^{1/2}d - (I - z^*z)(I + {f_\alpha}^* f_\alpha)(I - z^*z)^{1/2}d\| = 0$$

for all $d \in D$, i.e.

$$\lim_\alpha \|a - (I - z^*z)(I + {f_\alpha}^* f_\alpha)a\| = 0 \quad \forall a \in D(S).$$

Applying π now, we get

$$\lim_\alpha \|\pi(a) - (I - \pi(z)^*\pi(z))(I + \pi(f_\alpha)^*\pi(f_\alpha))\pi(a)\| = 0 \quad \forall a \in D(S).$$

Now choose an $a \in D(S)$ such that $\pi(a) \in Z(\mathcal{A}/K)$; then $I + \pi(f_\alpha)^*\pi(f_\alpha)$ will commute with $\pi(a)$. Therefore using the facts that $\|(I + {f_\alpha}^* f_\alpha)^{-1}\| \leq 1$ and $(I + \pi(f_\alpha)^*\pi(f_\alpha))^{-1}$ also commutes with $\pi(a)$, we get

$$\lim_\alpha \|(I + \pi(f_\alpha)^*\pi(f_\alpha))^{-1}\pi(a) - (I - \pi(z)^*\pi(z))\pi(a)\| = 0 \tag{3.2}$$

for all $\pi(a) \in Z(\mathcal{A}/K) \cap \pi(D(S))$. From condition (3.1), it follows that (3.2) holds for all $\pi(a) \in \mathcal{A}/K$. That is, for any $a \in \mathcal{A}$, $\pi(z)^*\pi(z)\pi(a) \in \mathcal{A}/K$. Hence there are a $b \in \mathcal{A}$ and a $k \in K$ such that $z^*za = b + k$, which implies that $z^*za \in \mathcal{A}$. Thus $z^*z \in M(\mathcal{A})$. From Proposition 2.11, we conclude that S^{**} is regular. □

The following two corollaries are now immediate.

Corollary 3.2. *Let S be a closed semiregular operator on $\mathcal{A}$ such that its restriction to K is regular. If $\mathcal{A}/K$ is abelian, then S^{**} is regular.*

Proof. In this case, $Z(\mathcal{A}/K) \cap \pi(D(S)) = \pi(D(S))$. Therefore condition (3.1) holds. □

Corollary 3.3. *Let S be as in the earlier theorem. If $\mathcal{A}/K$ is unital, then S^{**} is regular.*

Proof. Since $\pi(D(S))$ is a dense right ideal in $\pi(\mathcal{A}) = \mathcal{A}/K$ which is unital, we have $\pi(D(S)) = \mathcal{A}/K$. Therefore $I \in Z(\mathcal{A}/K) \cap \pi(D(S))$. So (3.1) is satisfied. □

Remark 3.4. We will primarily be interested in the case $K = \mathcal{B}_0(\mathcal{H})$. By Proposition 5.1 of [5], the condition that the restriction of S to K is regular is automatic in this case.

It is now natural to ask what happens in the general case, i.e. when $\mathcal{A}/K$ is neither unital nor abelian. We will give a counterexample to illustrate that the result may fail to hold in general. Before going to the example, let us observe that if S is a semiregular operator on $\mathcal{A}$, then the prescription

$$\begin{aligned} D(\pi(S)) &:= \pi(D(S)), \\ \pi(S)\pi(a) &:= \pi(Sa), \quad a \in D(S), \end{aligned}$$

defines a semiregular operator on $\pi(\mathcal{A})$. The example below, which appears in [2] as an example of a nonregular operator, will in fact show that even if $S|_K$ and $\pi(S)$ both are regular, S may fail to be so.

Let us first define an operator on the Hilbert C^*-module $E = C[0,1] \otimes \mathcal{H}$, where $\mathcal{H} = L_2(0,1)$. Let β be the following function on the interval $[0,1]$:

$$\beta(\pi) = \begin{cases} 1 & \text{if } \pi = 0, \\ \exp(i/\pi) & \text{if } 0 < \pi \leq 1. \end{cases}$$

Let

$$D_\pi = \{f \in L_2(0,1) : \ f \text{ absolutely continuous}, f' \in L_2(0,1), f(0) = \beta(\pi)f(1)\}.$$

For $f \in E$, denote by f_π the function $f(\pi, \cdot)$ in $\mathcal{H}$. Let T be the semiregular operator on E defined as follows:

$$D(T) = \{f \in E : f_\pi \in D_\pi \ \forall \pi, \pi \mapsto (f_\pi)' \text{ is continuous}\},$$

$$(Tf)_\pi := i(f_\pi)'.$$

It has been shown by Hilsum ([2]) that this is a self-adjoint nonregular operator. Also, from Proposition 2.9 in [2], it follows that the restriction of T to the submodule $F = C_0(0,1] \otimes \mathcal{H}$ is a self-adjoint regular operator.

Notice two things now. $\mathcal{A} = C[0,1] \otimes \mathcal{B}_0(\mathcal{H})$ is the C^*-algebra of 'compact' operators on E, and $K = C_0(0,1] \otimes \mathcal{B}_0(\mathcal{H})$ is the corresponding C^*-algebra for F. K can easily be seen to be an essential ideal in $\mathcal{A}$, and $\mathcal{A}/K \cong \mathcal{B}_0(\mathcal{H})$. Let ϕ_1 be the map introduced in section 3 of [5] for the Hilbert module E. Define S to be the operator $\overline{\phi_1(T)}$ on $\mathcal{A}$. Using Lemmas 3.1, 3.2 and 3.5 in [5], one can prove that for any semiregular operator $\boldsymbol{t}$ on E, $\overline{\phi_1(\boldsymbol{t}^*)} = \phi_1(\boldsymbol{t})^*$. Since in our case T is self-adjoint, it follows that S is self-adjoint too. Nonregularity of S is also clear by the discussion at the end of section 3 in [5]. Restriction of S to K is the ϕ_1-image of the restriction of T to F. Therefore $S|_K$ is regular. Since $\mathcal{A}/K \cong \mathcal{B}_0(\mathcal{H})$, the projection of S on $\mathcal{A}/K$ is also regular by Proposition 5.1 in [5].

Remark 3.5. If we write z for the z-transform of the restriction of S to K, then the above example tells us that the inclusions in Proposition 2.8 are not enough to guarantee that $z \in M(\mathcal{A})$, as in that case S would have been regular.

4. Examples

We will restrict ourselves to two examples in this section that occur naturally in the study of quantum groups. The first one is the C^*-algebra corresponding to the quantum complex plane and the other one is the crossed product algebra $C_0(q^{\mathbb{Z}} \cup \{0\}) \ltimes_\alpha \mathbb{Z}$, where q is a fixed real number in the interval (0,1), $q^{\mathbb{Z}}$ stands for the set $\{q^k : k \in \mathbb{Z}\}$, and the action α of $\mathbb{Z}$ on $C_0(q^{\mathbb{Z}} \cup \{0\})$ is given by

$$\begin{aligned} \alpha_k f(q^r) &= f(q^{r-k}), \quad r,k \in \mathbb{Z}, \\ \alpha_k f(0) &= f(0). \end{aligned}$$

Let us start with the quantum complex plane. Let $\mathcal{H} = L_2(\mathbb{Z})$, with canonical orthonormal basis $\{e_n\}_n$. Let ℓ^* and q^N denote the following operators:

$$\begin{aligned} \ell^* e_k &= e_{k+1}, \quad k \in \mathbb{Z}, \\ q^N e_k &= q^k e_k, \quad k \in \mathbb{Z}. \end{aligned}$$

Let D denote the linear span of $\{(\ell^*)^k f_k(q^N) : k \in \mathbb{Z}, f_k \in C_0(q^{\mathbb{Z}} \cup \{0\}), f_k(0) = 0 \text{ for } k \neq 0\}$. The C^*-algebra of 'continuous vanishing-at-infinity functions' on the quantum plane, which we denote by $C_0(\mathbb{C}_q)$, is the norm closure of D. The quantum complex plane can be looked upon as the homogeneous space $E_q(2)/S^1$ (S^1 being the one dimensional torus) for the quantum $E(2)$ group ([4], [7]). $C_0(\mathbb{C}_q)$ was introduced in a slightly different form in [7] (for a proof of the fact that the C^*-algebra described above is isomorphic to the one in [7], see [4]).

Lemma 4.1. $C_0(\mathbb{C}_q)/\mathcal{B}_0(\mathcal{H}) \cong \mathbb{C}$.

Proof. Define a map $\phi : C_0(\mathbb{C}_q) \to \mathbb{C}$ by the prescription

$$\phi\Big(\sum_k (\ell^*)^k f_k(q^N)\Big) = f_0(0), \quad \sum_k (\ell^*)^k f_k(q^N) \in D.$$

It extends to a complex homomorphism of $C_0(\mathbb{C}_q)$. It is easy to see that $\ker\phi$ is the closure of $\{(\ell^*)^k f_k(q^N) : k \in \mathbb{Z}, f_k \in C_0(q^{\mathbb{Z}} \cup \{0\}), f_k(0) = 0 \text{ for all } k\}$, i.e. is isomorphic to $C_0(\mathbb{Z}) \ltimes \mathbb{Z}$, which in turn is isomorphic to $\mathcal{B}_0(\mathcal{H})$. $\square$

We can now apply Corollary 3.3 to conclude that for any closed semiregular operator S on $C_0(\mathbb{C}_q)$, S^{**} is regular. Indeed, since the restriction of S to $\mathcal{B}_0(\mathcal{H})$ is regular, by Proposition 2.3, S^{**} is an operator satisfying the assumptions of Corollary 3.3.

Our second example, the crossed product algebra $\mathcal{A} = C_0(q^{\mathbb{Z}} \cup \{0\}) \ltimes \mathbb{Z}$, is actually very similar to the previous one. Its relevance in quantum groups stems from the fact that for any infinite dimensional irreducible representation π of the C^*-algebra $C_0(E_q(2))$ corresponding to the quantum $E(2)$ group, $\pi(C_0(E_q(2)))$ is isomorphic to $\mathcal{A}$. From the definition of a crossed product algebra, it can be shown quite easily that $\mathcal{A}$ is the norm closure of the linear span of $\{(\ell^*)^k f_k(q^N) : k \in \mathbb{Z}, f_k \in C_0(q^{\mathbb{Z}} \cup \{0\})\}$. One then shows that $\mathcal{A}/\mathcal{B}_0(\mathcal{H}) \cong C(S^1)$. The proof is similar to the proof of Lemma 4.1, except that the map ϕ in this case maps $\mathcal{A}$ onto $C(S^1)$ and is defined by $\phi(\sum_k (\ell^*)^k f_k(q^N)) = \sum_k f_k(0)\zeta^k$, where ζ stands for the function $z \mapsto z$ on S^1.

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