HOLOMORPHIC SECTIONS
OF PRE-QUANTUM LINE BUNDLES ON G/(P,P)

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Abstract. Let \( G = K \mathcal{A} N \) be the Iwasawa decomposition of a complex connected semi-simple Lie group \( G \). Let \( P \subseteq G \) be a parabolic subgroup containing \( A \mathcal{N} \), and let \((P,P)\) be its commutator subgroup. In this paper, we characterize the \( K \)-invariant Kahler structures on \( G/(P,P) \), and study the holomorphic sections of their corresponding pre-quantum line bundles.

1. Introduction

Let \( K \) be a compact connected semi-simple Lie group, let \( G \) be its complexification, and let \( G = K \mathcal{A} N \) be an Iwasawa decomposition. Let \( T \) be the centralizer of \( A \) in \( K \), so that \( H = T A \) is a Cartan subgroup, and \( B = H N \) is a Borel subgroup of \( G \). Let \( P \) be a parabolic subgroup of \( G \) containing \( B \), and \((P,P)\) its commutator subgroup. Each \( P \) determines a subgroup \( A_P \subset A \) via Langlands decomposition \( P = M_P A_P N_P \) (2, p. 132). It also determines a subtorus \( T_P \subset T \), which makes \( H_P = T_P A_P \) complex. Since \( H_P \) normalizes \((P,P)\), it has right action on \( G/(P,P) \). In [3], we consider \( K \times T_P \)-invariant Kahler structures \( \omega \) on \( G/(P,P) \), and study the pre-quantum line bundle [8] corresponding to \( \omega \). We then observe that the holomorphic sections of the pre-quantum line bundle constitute a nice multiplicity-free \( K \)-representation. In this paper, we show that if the \( K \)-invariant \( \omega \) is not preserved by the right \( T_P \)-action, then the pre-quantum line bundle has no holomorphic section other than the zero section.

The Lie algebra of a Lie group shall always be denoted by its lowercase German letter. For instance, \( h \) and \( t_P \) are the Lie algebras of \( H \) and \( T_P \) respectively.

Consider the root system in \( h^* \). By declaring \( n \) to be the negative root spaces, we obtain a system of positive roots in \( h^* \). Let \( \Delta \) be the simple roots. There is a natural bijective correspondence between the parabolic subgroups \( P \) containing \( B \) and the subsets of \( \Delta \). Namely, \( P \) corresponds to \( \Delta_P \subset \Delta \) by

\[
\Delta_P = \{ \lambda \in \Delta : \langle \lambda, v \rangle \neq 0 \text{ for some } v \in t_P \}.
\]

Note that as \( P \) grows bigger, \( \Delta_P \) gets smaller. For example, \( \Delta_B = \Delta \).

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Fix one parabolic subgroup \( P \) containing \( B \), with corresponding simple roots \( \Delta_P = \{ \lambda_1, ..., \lambda_r \} \) via (1.1). Each \( \lambda_i \) is integral, in the sense that there is a multiplicative homomorphism \( \chi_i : H \longrightarrow \mathbb{C}^\times \) such that
\[
\chi_i(e^v) = \exp(\lambda_i, v)
\]
for all \( v \in \mathfrak{h} \). Let \( R_t \) denote the right action of \( t \in T_P \).

**Theorem 1.** Every \( K \)-invariant Kähler form on \( G/(P, P) \) can be uniquely expressed as
\[
\omega = \sqrt{-1} \partial \bar{\partial} F + \sum_{i=1}^{r} d\beta_i,
\]
where \( R_t^* \beta_i = \chi_i(t) \beta_i \) for all \( t \in T_P \). So \( \omega \) has a potential function if and only if it is right \( T_P \)-invariant, if and only if \( \sum_i d\beta_i \) vanishes.

Let \( \omega \) be a \( K \)-invariant Kähler form on \( G/(P, P) \). By Theorem 1, \( \omega \) is exact. Therefore it is integral, and corresponds to a pre-quantum line bundle \( L \) in the sense of Kostant [3]. Namely the Chern class of \( L \) is \( [\omega] = 0 \), and \( L \) has a connection \( \nabla \) whose curvature is \( \omega \). A smooth section \( s \) of \( L \) is said to be holomorphic if \( \nabla_v s = 0 \) whenever \( v \) is an anti-holomorphic vector field [5]. Let \( H(L) \) denote the holomorphic sections of \( L \). The \( K \)-action on \( G/(P, P) \) lifts to a \( K \)-representation on \( H(L) \). In [3], we show that if \( \omega \) is right \( T_P \)-invariant, then every irreducible \( K \)-representation with highest weight in \( \mathfrak{t}_P \) occurs exactly once in \( H(L) \). The following theorem observes the opposite situation, when \( \omega \) is not right \( T_P \)-invariant.

**Theorem 2.** Suppose that \( \omega \) does not satisfy the equivalent conditions given in Theorem 1. Then \( H(L) = 0 \).

We remark that partial results of Theorems 1 and 2 appear in [1] and [4], for the special case where \( P \) is the Borel subgroup \( HN \). The present paper extends those results to general parabolic subgroups \( P \).

## 2. Proofs of theorems

In this section, we prove the two theorems mentioned in the introduction. We start with the following topological property of \( G/(P, P) \).

**Proposition 3.** \( H^2(G/(P, P), R) = 0 \).

**Proof.** Let \( K^P \) be the centralizer of \( T_P \) in \( K \), and \( K^P_{ss} = (K^P, K^P) \) be its commutator subgroup. As a manifold, \( G/(P, P) = (K/K^P_{ss}) \times A_P [3] \). Since \( A_P \) has the topology of an Euclidean space, it suffices to show that \( H^2(K/K^P_{ss}, R) = 0 \). But \( K \) is compact. So we only need to consider the DeRham subcomplex of \( K \)-invariant differential forms on \( K/K^P_{ss} \), and show that the \( H^2 \) of this subcomplex vanishes. This is done via relative Lie algebra cohomology as follows.

We restrict the coadjoint representation of \( K \) to \( K^P_{ss} \), and get
\[
Ad^* : K^P_{ss} \longrightarrow Aut(\mathfrak{t}^*).
\]
We extend this representation to exterior algebras, then differentiate to get the Lie algebra representation
\[
ad^* : \mathfrak{t}^P_{ss} \longrightarrow End(\wedge^q \mathfrak{t}^*).
\]
The relative Lie algebra cohomology is defined by the complex
\[
\wedge^q (\mathfrak{t}, \mathfrak{t}^P_{ss})^* = \{ \omega \in \wedge^q \mathfrak{t}^* : \iota(v)\omega = ad^*_v \omega = 0 \text{ for all } v \in \mathfrak{t}^P_{ss} \}.
\]
Here $\iota(v)\omega$ denotes the interior product. We write $H^0(\mathfrak{k}, \mathfrak{t}_{ss}^P)$ for the cohomology resulting from (2.1). The elements in (2.1) can be naturally identified with $K$-invariant differential forms on $K/K^P$. Hence to prove the proposition, it suffices to show that
\begin{equation}
H^2(\mathfrak{k}, \mathfrak{t}_{ss}^P) = 0.
\end{equation}

Let $\omega \in \wedge^2(\mathfrak{k}, \mathfrak{t}_{ss}^P)^*$, and suppose that $d\omega = 0$. Since $\mathfrak{k}$ is semi-simple, $H^2(\mathfrak{k}) = 0$ by the Whitehead lemma [6]. So since $\omega \in \wedge^2\mathfrak{k}^*$, there exists $\beta \in \wedge^1\mathfrak{k}^*$ such that $d\beta = \omega$. To prove (2.2), we need to show that $\beta \in \wedge^1(\mathfrak{k}, \mathfrak{t}_{ss}^P)^*$; namely
\begin{equation}
(\beta, v) = ad^*_v\beta = 0
\end{equation}
for all $v \in \mathfrak{t}_{ss}^P$.

Pick $v \in \mathfrak{t}_{ss}^P$. Up to linear combination, there exist $x, y \in \mathfrak{t}_{ss}^P$ such that $v = [x, y]$ because $\mathfrak{t}_{ss}^P$ is semi-simple. Then
\begin{align}
(\beta, v) &= (\beta, [x, y]) \\
&= d\beta(x, y) \\
&= \omega(x, y) \\
&= (\iota(x)\omega)(y).
\end{align}

Since $\omega \in \wedge^2(\mathfrak{k}, \mathfrak{t}_{ss}^P)^*$ and $x \in \mathfrak{t}_{ss}^P$, we get $\iota(x)\omega = 0$. Therefore, the last expression in (2.4) vanishes. This proves half of (2.3), and we next prove the other half of it.

Pick $x \in \mathfrak{t}_{ss}^P$ and $y \in \mathfrak{k}$. By following the arguments of (2.4), we get
\begin{equation}
(ad^*_x\beta, y) = (\beta, [x, y]) = (\iota(x)\omega)(y) = 0.
\end{equation}

Hence $ad^*_x\beta = 0$. This completes the proof of (2.3), which implies (2.2). Proposition follows.

Let $W$ be the Weyl group, acting on the roots in $\mathfrak{h}^*$. Given $\tau \in W$, we let $l(\tau)$ denote its length. Let $\rho$ denote half the sum of all positive roots.

**Proof of Theorem 7.** Let $\omega$ be a $K$-invariant Kähler form on $G/(P, P)$. By Proposition 3, $\omega = d\beta$ for some real 1-form $\beta$. We write
\begin{equation}
\beta = \alpha + \bar{\alpha},
\end{equation}
where $\alpha$ is a $(0, 1)$-form. Since $\omega$ is a $(1, 1)$-form, it follows from $d\beta = \omega$ that $\bar{\partial}\alpha = \partial\bar{\alpha} = 0$. In other words, we get a Dolbeault cohomology class $[\alpha] \in H^{0,1}(G/(P, P))$. We suppress $G/(P, P)$ and write $H^{0,1}$ for simplicity.

Consider $H^{0,1}$ as a $K \times T_P$-representation space. Let $H^{0,1}_K \subset H^{0,1}$ denote the $K$-invariant cohomology classes. Since $\omega$ is $K$-invariant, averaging by $K$ if necessary, we may assume that $\beta$ and $\alpha$ of (2.5) are also $K$-invariant. In other words, $[\alpha] \in H^{0,1}_K$. For an integral weight $\lambda \in \mathfrak{t}_P$, let $H^{0,1}_\lambda \subset H^{0,1}$ be the cohomology classes which transform by $\lambda$ under the right $T_P$-action. By Theorem 2 of [2], $H^{0,1}_K$ splits into 1-dimensional subrepresentations $H^{0,1}_\lambda$ for all $\lambda \in \mathfrak{t}_P$, in which we can find $\tau \in W$ satisfying
\begin{equation}
\tau(\lambda + \rho) - \rho = 0, \ l(\tau) = 1.
\end{equation}
But condition (2.6) simply means that $-\lambda$ is a simple root which lies in $\mathfrak{t}_P$. Equivalently $-\lambda \in \Delta_P$, where $\Delta_P = \{\lambda_1, ..., \lambda_r\}$ consists of the simple roots in (1.1).
Therefore, there exist $\bar{\partial}$-closed \((0,1)\)-forms $\alpha_1, \ldots, \alpha_r$ such that

\begin{equation}
[\alpha] = \left[ \sum_{i=1}^{r} \alpha_i \right] \in H^{0,1}_K
\end{equation}

and

\begin{equation}
[\alpha_i] \in H^{0,1}_{-\lambda_i} \subset H^{0,1}_K.
\end{equation}

Here (2.7) says that

\begin{equation}
\alpha = \bar{\partial}f + \sum_{i=1}^{r} \alpha_i
\end{equation}

for some smooth function $f$. For the negative root $-\lambda_i$, the character corresponding to it via (1.2) is $\chi_i^{-1}$. Therefore, (2.8) says that for all right action of $R_t$ of $t \in T_P$,

\begin{equation}
R_t^* \alpha_i = \chi_i^{-1}(t^{-1}) \alpha_i = \chi_i(t) \alpha_i.
\end{equation}

Let $\beta_i = \alpha_i + \bar{\alpha}_i$ for all $i = 1, \ldots, r$. Then by (2.5) and (2.9),

\[
\beta = \alpha + \bar{\alpha} = \bar{\partial}f + \bar{\partial}\bar{f} + \sum_{i=1}^{r} (\alpha_i + \bar{\alpha}_i) = \bar{\partial}f + \bar{\partial}\bar{f} + \sum_{i=1}^{r} \beta_i.
\]

Therefore, by setting $F = \sqrt{-1} (\bar{f} - f)$,

\[
\omega = d\beta = \partial\bar{\partial}f + \bar{\partial}\partial\bar{f} + \sum_{i=1}^{r} d\beta_i = \sqrt{-1} \partial\bar{\partial}F + \sum_{i=1}^{r} d\beta_i.
\]

Since $\omega, \beta$ and $\alpha$ are $K$-invariant, we can take $\beta_i, \alpha_i$ and $f$ to be $K$-invariant too. Since $f$ is a $K$-invariant function on $G/(P, P) = (K/K_P^*) \times A_P$, it is automatically $K \times T_P$-invariant. Therefore, $F$ and $\sqrt{-1} \partial\bar{\partial}F$ are also $K \times T_P$-invariant.

Each $\beta_i$ behaves by $\chi_i$ in (2.10) under the right $T_P$-action. If $\sum_{i=1}^{r} d\beta_i$ does not vanish and has a potential function, then it is right $T_P$-invariant, which is impossible. Therefore, $\omega$ has a potential function if and only if $\sum_{i=1}^{r} d\beta_i$ vanishes. This can also be seen from the nontrivial Dolbeault cohomology classes $[\alpha_i] \neq 0$. Equivalently, the vanishing of $\sum_{i=1}^{r} d\beta_i$ leaves $\omega = \sqrt{-1} \partial\bar{\partial}F$ to be right $T_P$-invariant.

This proves Theorem 1. 

Let $\omega$ be a $K$-invariant Kähler form on $G/(P, P)$. By Theorem 1, $\omega$ is exact, so it is in particular integral. Let $L$ be the pre-quantum line bundle corresponding to $\omega$. We now prove Theorem 2 concerning the holomorphic sections on $L$.

**Proof of Theorem 2** Since $B \subset P$ and $N = (B, B) \subset (P, P)$, we have the natural fibration

\[
\pi : G/N \longrightarrow G/(P, P).
\]

Suppose that $\omega$ is not invariant under the right $T_P$-action. Since $\pi$ intertwines with the $K \times T_P$-action, $\pi^* \omega$ is $K$-invariant but not right $T_P$-invariant. Although
\( \pi^* \omega \) is not Kähler, it is a closed (1,1)-form on \( G/N \). Therefore, \( \pi^* \omega \) accepts most arguments in [4], including Theorem 1 there. Namely, the only holomorphic section on the pre-quantum line bundle of \( \pi^* \omega \) is the zero section.

Let \( L \) be the pre-quantum line bundle corresponding to \( \omega \). Then \( \pi^* L \) is the pre-quantum line bundle corresponding to \( \pi^* \omega \). If \( s \) is a holomorphic section on \( L \) and \( s \neq 0 \), then \( \pi^* s \) is a holomorphic section on \( \pi^* L \) and \( \pi^* s \neq 0 \). This is a contradiction, so the only holomorphic section on \( L \) is the zero section. Hence Theorem 2 is proved.

**References**


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