A FINITENESS RESULT FOR ASSOCIATED PRIMES
OF LOCAL COHOMOLOGY MODULES

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Abstract. We show that the first non-finitely generated local cohomology module $H^i_a(M)$ of a finitely generated module $M$ over a noetherian ring $R$ with respect to an ideal $a \subseteq R$ has only finitely many associated primes.

1. Introduction

Apparently very little is known about the finiteness of the set $\text{Ass}_R(H^i_a(M))$ of associated primes of the local cohomology module $H^i_a(M)$ of a finitely generated module $M$ over a noetherian ring $R$ with respect to an ideal $a$ of $R$. So, in [M-S] the authors ask whether the sets $\text{Ass}_R(\text{Ext}_R^i(R/a^n, M))$ become stable for $n >> 0$. An affirmative answer to this would imply that the set $\text{Ass}_R(H^i_a(M))$ is always finite. On the other hand it seems not to be known whether the set $\text{Ass}_R(H^2_a(M))$ is finite in general even if $a$ is generated by two elements only. Moreover, if for two elements $a, b \in R$ the set $S_M(a, b) := \bigcup_{n \in \mathbb{N}} \text{Ass}_R(M/(a^n, b^n)M)$ is finite, then $\text{Ass}_R(H^2_{(a, b)}(M))$ is finite. But Katzman [K] has shown that $S_M(a, b)$ need not be finite. In [B-R-Sh] we have shown that $S_M(a, b)$ is finite under certain conditions. These conditions imply that $H^1_{(a, b)}(M)$ is finitely generated and so the resulting finiteness of $\text{Ass}_R(H^2_{(a, b)}(M))$ is not surprising, as $\text{Ass}_R(H^2_a(M))$ is finite whenever $H^1_a(M)$ is finitely generated (see [B-R-Sh] (2.4), (2.5)).

Finally, let us mention that the finiteness of the sets $\text{Ass}_R(H^i_a(M))$ is related to the local-global-principle for finiteness dimensions due to Faltings [F] (cf. [B-Sh] (9.6.2)) and also to the open problem of whether such a principle holds for the annihilation of local cohomology modules (see [R], [B-R-Sh]).

The aim of this note is to show that the set $\text{Ass}_R(H^i_a(M))$ is finite, whenever the modules $H^2_a(M), \ldots, H^{i-1}_a(M)$ are finitely generated. This generalizes the corresponding result which is shown in [B-R-Sh] for the special case $i \leq 2$ and which was already mentioned above.

Throughout this note, let $R$ be a noetherian ring, let $a \subseteq R$ be an ideal and let $M$ be a finitely generated $R$-module. If $i \in \mathbb{N}_0$, we write $H^i_a(M)$ for the $i$-th local cohomology module of $M$ with respect to the ideal $a$. For convenience we write...
\( H_a^i(M) = 0 \), whenever \( j \) is a negative integer. For the unexplained terminology we refer to [B-Sh].

2. The finiteness results

**Proposition 2.1.** Let \( i \in \mathbb{N}_0 \) be such that \( H_a^i(M) \) is finitely generated for all \( j < i \) and let \( N \subseteq H_a^i(M) \) be a finitely generated submodule. Then, the set \( \text{Ass}_R(H_a^i(M)/N) \) is finite.

**Proof.** We proceed by induction on \( i \). The case \( i = 0 \) is obvious as \( H_a^0(M) \) is finitely generated. So, let \( i > 0 \) and set \( \overline{M} := M/\Gamma_a(M) \). As \( H_a^0(\overline{M}) = 0 \) and in view of the natural isomorphisms \( H_a^k(\overline{M}) \cong H_a^k(M) \) for all \( k \in \mathbb{N} \), we may replace \( M \) by \( \overline{M} \) and hence assume that \( \Gamma_a(M) = 0 \). We thus find an \( M \)-regular element \( y \in \mathfrak{a} \).

By our choice of \( N \), there is some \( n \in \mathbb{N} \) with \( y^n N = 0 \).

We set \( x := y^n \) and apply cohomology to the exact sequence \( 0 \to M \to M/xM \to 0 \). It follows that \( H_a^i(M/xM) \) is finitely generated for all \( l < i - 1 \).

Moreover, we get the following commutative diagram with exact rows and columns in which \( \delta \) is the connecting homomorphism and in which \( \varepsilon \) and \( \varphi \) are the natural maps:

\[
\begin{array}{ccccccc}
H_a^{i-1}(M) & \xrightarrow{\varepsilon} & H_a^{i-1}(M/xM) & \xrightarrow{\delta} & H_a^i(M) & \xrightarrow{x} & H_a^i(M) \\
\downarrow & & \downarrow & & \varepsilon & & \downarrow \\
0 & \xrightarrow{\varphi} & H_a^{i-1}(M/xM)/\delta^{-1}(N) & \xrightarrow{\delta} & H_a^i(M)/N & \xrightarrow{\varphi} & H_a^i(M) \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & \\
\end{array}
\]

As \( \text{Ker}(\delta) = \varepsilon(H_a^{i-1}(M)) \) and \( N \) are both finitely generated, so is \( \delta^{-1}(N) \). Therefore, by induction

\( T := H_a^{i-1}(M/xM)/\delta^{-1}(N) \)

has only finitely many associated primes. It thus suffices to show the inclusion

\[
\text{Ass}_R(H_a^i(M)/N) \subseteq \text{Ass}_R(T) \cup \text{Ass}_R(N).
\]

So, let \( p \in \text{Ass}_R(H_a^i(M)/N) \setminus \text{Ass}_R(T) \). With an appropriate \( h \in H_a^i(M) \) we may write \( p = N : h \), hence \( p = 0 : R \varphi(h) \). As \( p \notin \text{Ass}_R(T) \), the last equality and the second row of (\*) show that \( p \in \text{Ass}_R(\varpi(\varphi(h))R) = \text{Ass}_R(xhR) \). This allows us to write \( p = 0 : xsh \) for some \( s \in R \).

As \( xsh \) is annihilated by some power of \( x \), we have \( x \in p \). By our choice of \( h \) this means that \( xsh \in N \). This implies that \( p \in \text{Ass}_R(N) \) and hence proves the inclusion (**) and thus our result.

Now, the announced result follows easily.

**Theorem 2.2.** Let \( i \in \mathbb{N}_0 \) be such that \( H_a^j(M) \) is finitely generated for all \( j < i \). Then the set \( \text{Ass}_R(H_a^i(M)) \) is finite.

**Proof.** Apply Proposition (2.1) with \( N = 0 \).
Next, let us introduce the \(a\)-finiteness dimension of \(M\) (see [B-Sh] (9.1.3)):

\[
f_a(M) := \min \{ j \in \mathbb{N}_0 \mid H^j_a(M) \text{ not finitely generated} \}.
\]

Using this notation we may write Theorem (2.2) in the form

**Corollary 2.3.** If \(i \leq f_a(M)\), then \(\mathrm{Ass}_R(H^i_a(M))\) is a finite set.

**Corollary 2.4** (see [B-R-Sh] (2.2)). The set \(\mathrm{Ass}_R(H^{\text{grade}_M(a)}_a(M))\) is finite.

**Proof.** Follows from Corollary (2.3) as \(\text{grade}_M(a) \leq f_a(M)\).

**References**


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