ON REFLEXIVITY OF DIRECT SUMS

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Abstract. Necessary and sufficient conditions are presented to insure that the direct sum of two reflexive representations of a finite dimensional algebra is reflexive, and it is shown that for each such algebra, there is an integer $k$ such that the direct sum of $k$ copies of each of its representations is reflexive. Given a ring $\Delta$, our results are actually presented in the more general setting of $\Delta$-representations of a ring $R$.

Over the last thirty years, a significant amount of attention has been given to the problem of determining when an algebra $A$ of operators on a vector space $V$ (often a Hilbert space) over a field $K$ is reflexive in the sense that no larger algebra of operators on that space has the same lattice of invariant subspaces. (See, for example, [2], [3], [6], [7], [12], [13], [14].) Of course, $A$ is an algebra of operators on a vector space $V$ if and only if $V$ is a faithful left $A$-module. Thus, to attack this problem from a more module theoretic point of view, based on notation of Halmos [14], the following notions were presented in [9]: For any $R - \Delta$-bimodule $M = R\Delta$ one defines

$$ \text{alglat}(M) = \{ \alpha \in \text{End}(M\Delta) \mid \alpha L \subseteq L \text{ for all } RL \leq R\Delta \}, $$

and, letting $\lambda : R \to \text{alglat}(M\Delta)$ denote the canonical ring homomorphism, $M$ is called a reflexive bimodule (or $\Delta$-representation of $R$) if $\lambda$ is surjective. Thus the $K$-algebra of operators $A$ on $V$ is reflexive if and only if $AV_K$ is a reflexive bimodule (and then we simply say that the $A$-module $V$ is reflexive).

The problems we shall deal with here have their roots in the papers [5] and [6] of Deddens and Fillmore and [2] of Azoff. Stated in module theoretic terms, in the first pair of papers the question was posed and answered (affirmatively) of whether a direct sum of two finitely generated reflexive modules over an algebra generated by a single complex matrix is again reflexive; and Azoff showed that for each finite dimensional module over a $\mathbb{C}$-algebra, there is a positive integer $k$, depending on its dimension, such that the direct sum of $k$ copies of that module is reflexive. Other results in this vein can be found in [3] where Brenner and Butler showed that the direct sum of two copies of the regular representation of a finite dimensional algebra is reflexive, and in Habibi and Gustafson’s [12] from which the same result follows for any faithful representation of a split serial algebra. (See [9], [10] and [11] for related and more general results.)

Our main objectives are to provide, in Theorem 1.2, a necessary and sufficient condition for a direct sum of two reflexive bimodules to be reflexive; and to show, in

\[ \text{alglat}(M) = \{ \alpha \in \text{End}(M\Delta) \mid \alpha L \subseteq L \text{ for all } RL \leq R\Delta \}, \]

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Theorem 3.2 that whenever $R$ is a left artinian ring with composition length $c(R R)$, there is a positive integer $k \leq c(R R) + 1$ such that the direct sum $M^{(k)}$ of $k$ copies of $M$ is reflexive for any bimodule $R M$. Along the way we show how Theorem 1.2 can be employed to obtain simple proofs of some known results, and we examine a common generalization of generating and cogenerating called controlling that was introduced and employed in [9] and [10], showing in particular that if $R M$ is faithful and reflexive, then $M \oplus N$ is reflexive if and only if $M$ controls $N$.

1. A CHARACTERIZATION

Unless otherwise specified, all modules under consideration will be left-$R$, right-$\Delta$ bimodules for a fixed pair of rings $R$ and $\Delta$. If $R$ is an algebra over a field $K$, we shall assume that $\Delta = K$. If $\alpha \in \text{End}(M_\Delta)$ and $\beta \in \text{End}(N_\Delta)$, we shall write $(\alpha, \beta) \in \text{End}(M \oplus N)$ for the direct sum map $(\alpha, \beta): (m, n) \mapsto (\alpha m, \beta n)$. Thus if $\gamma \in \text{alglat}(M \oplus N)$, then $\gamma = (\gamma|_M, \gamma|_N)$.

A common subquotient of a pair of left modules $R M$ and $R N$ is a module $R X$ that is isomorphic to a subquotient of both $R M$ and $R N$. This notion together with the following lemma allows us to determine just when the direct sum of a pair of reflexive modules is reflexive.

**Lemma 1.1.** If $r, s \in R$, the ordered pair $(r, s) \in \text{alglat}(M \oplus N)$ if and only if $(r - s) X = 0$, for every (equivalently, every cyclic) common subquotient of $R M$ and $R N$.

**Proof.** Note that $a \in R$ annihilates every common subquotient of $M$ and $N$ if and only if it annihilates each of their cyclic common subquotients. According to Gour- sats' Lemma, $W$ is an $R$-submodule of $M \oplus N$, if and only if there are submodules 

$$M_2 \leq M_1 \leq M$$ and $N_2 \leq N_1 \leq N$

and an $R$-isomorphism

$$f : M_1/M_2 \to N_1/N_2$$

such that

$$W = \{(m_1, n_1) \in M_1 \oplus N_1 | f(m_1 + M_2) = n_1 + N_2\}.$$ 

(Given $W$ and the orthogonal projections $\pi_M$ and $\pi_N$ for $M \oplus N$, one checks that $M_1 = \pi_M(W)$, $M_2 = M \cap W$, $N_1 = \pi_N(W)$, and $N_2 = N \cap W$.)

($\Rightarrow$) If $(m_1, n_1) \in W$ and $(rm_1, sn_1) \in W$, then

$$f(rm_1 + M_2) = sn_1 + N_2 = sf(m_1 + M_2)$$

so, since $f$ is an $R$-isomorphism, $(r - s)(M_1/M_2) = 0$.

($\Leftarrow$) Suppose $(r - s)(M_1/M_2) = 0$, and $f : M_1/M_2 \to N_1/N_2$ is an isomorphism and $(m_1, n_1) \in W$. Then

$$f(rm_1 + M_2) = rf(m_1 + M_2) = rn_1 + N_2 = sn_1 + N_2$$

so $(rm_1, sn_1) \in W$. 

Now we are able to provide the promised characterization in terms of annihilators of subquotients of $M$ and $N$. The left annihilator of a module $M$ is $\ell_R(M) = \{r \in R | r M = 0\}$. 

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Theorem 1.2. Let $M$ and $N$ be reflexive, and let $\{X_i \mid i \in I\}$ represent one copy of each of the (cyclic) common subquotients of $RM$ and $RN$. Then $M \oplus N$ is reflexive if and only if

$$\ell_R(M) + \ell_R(N) = \ell_R(\bigoplus_{i \in I} X_i).$$

Proof. ($\Leftarrow$) Let $\alpha = (\beta, \gamma) \in \text{alglat}(M \oplus N)$. Then there are $r, s \in R$ with $\beta = r$ and $\gamma = s$, and by Lemma [1.1] $r - s \in \ell_R(\bigoplus_{i \in I} X_i)$. But then by hypothesis

$$r - s = p - q \quad \text{with} \quad p \in \ell_R(M) \quad \text{and} \quad q \in \ell_R(N)$$

so that, letting

$$t = r - p = s - q,$$

we have $\alpha = \lambda(t)$.

($\Rightarrow$) Let $r \in \ell_R(\bigoplus_{i \in I} X_i)$. Then by Lemma [1.1] $(r, 0) \in \text{alglat}(M \oplus N)$. Thus, assuming that $M \oplus N$ is reflexive,

$$(r, 0) = (s, s)$$

and

$$r = (r - s) + s \in \ell_R(M) + \ell_R(N).$$

Since always $\ell_R(M) + \ell_R(N) \subseteq \ell_R(\bigoplus_{i \in I} X_i)$, the proof is complete.

We note that the proof ($\Rightarrow$) above shows that if $M \oplus N$ is reflexive (regardless of reflexivity of $M$ and $N$), then $\ell_R(M) + \ell_R(N) = \ell_R(\bigoplus_{i \in I} X_i)$.

From the inclusions

$$0 \subseteq \ell_R(M) + \ell_R(N) \subseteq \ell_R(\bigoplus_{i \in I} X_i) \subseteq \ell_R(X_i) \subseteq R$$

we easily obtain the following two corollaries:

Corollary 1.3. If $M$ and $N$ are reflexive and have a common faithful subquotient, then $M \oplus N$ is reflexive.

Corollary 1.4. If $R$ is semiperfect and $M$ and $N$ are reflexive and have no common composition factor, then $M \oplus N$ is reflexive.

Proof. Suppose that $e_1, \ldots, e_m, e_{m+1}, \ldots, e_n$ is a complete set of primitive idempotents such that $Re_i/Je_i$ is not a composition factor of $M$ for $i = 1, \ldots, m$, and $Re_i/Je_i$ is not a composition factor of $N$ for $i = m+1, \ldots, n$. Then $\ell_R(M) + \ell_R(N) = R$. (See [11] Section 27.)

Regarding the Deddens-Fillmore result, the algebra generated by a matrix over a field $K$, being isomorphic to a proper factor of the polynomial ring $K[x]$, is an example of a split commutative uniserial algebra, i.e., a direct product of local uniserial rings. Since a module over a direct product of rings is reflexive precisely when its corresponding components are reflexive, to show that direct sums of reflexive modules are reflexive over such an algebra $R$, we may assume that $R$ is a local uniserial algebra, so that the ideals of $R$ are linearly ordered and every $R$-module is a direct sum of factors of $R$ (see [11] Section 32)). In this case, $R/\ell_R(M)$ embeds in $M$ and every subquotient of $RM$ is a factor of $R/\ell_R(M)$. In the presence of these facts Theorem [1.2] yields the following corollary almost at once. We note, however, that [11] Theorem 1] is more general.
Corollary 1.5. The direct sum of a finite number of reflexive modules over a split uniserial algebra is reflexive.

Proof. Since we may assume that the ideals of $R$ are linearly ordered, given modules $M$ and $N$, we may also assume that $\ell_R(M) \subseteq \ell_R(N) \subseteq \ell_R(X)$ for every common subquotient $X$ of $M$ and $N$. So since $R/\ell_R(N)$ embeds in $N$, we see that both sides of the desired equality are equal to $\ell_R(N)$. 

Next, as further applications of this theorem, we shall see that it yields particularly nice proofs of some key results in [10].

In [10], in order to show that direct sums of reflexive modules may fail to be reflexive over a split $K$-algebra $R$ with radical $J$ whose quiver contains a triple arrow, a pair of reflexive modules was constructed with diagrams (as in [8])

\[
\begin{align*}
M : & \quad u \quad v \quad w \\
& a \downarrow \quad b \quad c \downarrow \quad x \quad y \quad z \\
N : & \quad a, c \downarrow \\
\end{align*}
\]

with $a, b, c$ linearly independent elements of $J \setminus J^2$. (The diagram indicates that $au = x, aw = cw = z$, etc.) Here

\[
\ell_R(M) + \ell_R(N) = \ell_R(M) + K(a-c) + Kb,
\]

but the only common subquotients of $M$ and $N$ are simple modules, so

\[
\ell_R(\bigoplus_{i \in I} X_i) = \ell_R(M) + J \neq \ell_R(M) + \ell_R(N).
\]

Thus the theorem shows that $M \oplus N$ is not reflexive.

In the positive vein we shall employ the following lemma and Theorem 1.2 to obtain a simple proof of a key part of [10, Proposition 4].

Lemma 1.6. Let $R$ be a split local $K$-algebra of $\dim(KR) \leq 3$. If $M$ is a faithful left $R$-module, then every proper cyclic module is isomorphic to a subquotient of $M$.

Proof. (This proof is essentially contained in the proof of [10, Proposition 4].) If $R$ is uniserial, then $R$ embeds in $M$, so we may assume that $J^2 = 0$ and that $\dim(KJ) = 2$. Let $u \in M \setminus \text{Soc} M$ and suppose that $\ell_R(u) \neq 0$. Then, since $u \notin \text{Soc} M$, there is a $b \in J$ with $\ell_R(u) = Kb$. But then there is a $v \in M$ with $bv \neq 0$, and there is an $a \in J$ with $Ka = \ell_R(v)$. Thus it follows that

\[
J = Ka \oplus Kb.
\]

Now

\[
Ru \cong R/Rb \quad \text{and} \quad Rv \cong R/Ra
\]

are non-isomorphic uniserial submodules of $M$ of composition length 2. If $Ru \cap Rv = 0$, we see that $\ell_R(u + v) = 0$. Otherwise

\[
Ru \cap Rv = Ka = Kb = \text{Soc}(Ru + Rv),
\]

so the annihilator of any non-zero $K$-linear combination of $u$ and $v$ is properly contained in $J$, and we may assume that $au = bv$. Then since

\[
(ka + lb)(k^{-1}u - \ell^{-1}v) = au - bv = 0,
\]

we see that every proper cyclic $R$-module is a subquotient of $M$. 

\[
\square
\]
Now we have the promised result of Fuller, Nicholson and Watters [10].

**Proposition 1.7.** Let $R$ be a split local $K$-algebra of $\dim_K R \leq 3$. If $M$ and $N$ are reflexive $R$-modules, then so is $M \oplus N$.

**Proof.** We may assume that $M \oplus N$ is faithful. If neither $M$ nor $N$ is faithful, then $J = Ka \oplus Kb$ with $aM = 0$ and $bN = 0$. But then

$$\ell_R(M) + \ell_R(N) = J = \ell_R(R/J)$$

and the desired equality holds. If $M$ is faithful and $N$ is not, then by Lemma 1.6 every cyclic subquotient of $N$ is a subquotient of $M$, so

$$\ell_R(M) + \ell_R(N) = \ell_R(N) = \ell_R(\bigoplus_{i \in I} X_i).$$

Suppose both $M$ and $N$ are faithful. If $R$ embeds in $M$, then every cyclic subquotient of $N$ is a common subquotient of $M$ and $N$, so

$$\ell_R(M) + \ell_R(N) \subseteq \ell_R(\bigoplus_{i \in I} X_i) = \ell_R(N) = 0.$$ 

Otherwise, the cyclic subquotients of $M$ and $N$ are all proper, so by Lemma 1.6 they are all common to both modules and again

$$\ell_R(\bigoplus_{i \in I} X_i) = 0.$$ 

\[ \square \]

2. CONTROLLING

The following the definition in [9] or [10], given bimodules $RM_\Delta$ and $RN_\Delta$, we say that $M$ controls $N$ in case for each pair $(\alpha, n)$, with $\alpha \in \text{End}(N_\Delta)$ and $n \in N$, there is a set

$$C_{(\alpha, n)} = \{(m_i, n_i) | i \in I\} \subseteq M \times N,$$

called a connection for $\alpha$ and $n$, such that, if there are $r_i \in R$ with

$$r_im_i = 0 \text{ and } \alpha n_i = r_in_i$$

for all $i \in I$, then $\alpha n = 0$.

Controlling is a particularly useful concept. In [9] it was shown that if $N$ is either generated or cogenerated by subquotients of $M$, then $M$ controls $N$; and that if $M$ is reflexive and controls $N$, then $M \oplus N$ is reflexive. Here we shall present an equivalent version of controlling that yields a partial converse to this last assertion.

**Proposition 2.1.** The bimodule $RM_\Delta$ controls $RN_\Delta$ if and only if, for each $0 \neq \alpha \in \text{End}(N_\Delta)$, there is a pair

$$P_\alpha = (m_\alpha, n_\alpha) \in M \times N$$

with $\alpha n_\alpha \notin \ell_R (m_\alpha)n_\alpha$ (i.e., for all $r \in R$, if $\alpha n_\alpha = rm_\alpha$, then $rm_\alpha \neq 0$).

**Proof.** ($\Rightarrow$) Let $\alpha \in \text{End}(N_\Delta)$ with $\alpha n \neq 0$ and suppose that $C_{(\alpha, n)} = \{(m_i, n_i) | i \in I\}$ is a connection for $\alpha$ and $n$. If $\alpha n_i \in \ell_R (m_i)n_i$ for all $i \in I$, then there are $r_i$ with $r_im_i = 0$ and $\alpha n_i = r_in_i$ for all $i \in I$. But $\alpha n \neq 0$, so there is a pair $(m_i, n_i)$ with $\alpha n_i \notin \ell_R (m_i)n_i$. 

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(⇐) If αn = 0, then any C(α, n) = \{(m_i, n_i) | i ∈ I\} is a connection for α and n. If αn ≠ 0, let

\[ C(α, n) = \{P_α\} = \{(m_α, n_α)\}. \]

Then it never occurs that r_α m_α = 0 and αn_α = r_α n_α, so C(α, n) is a connection. □

It is worthy of note that when employing Proposition 2.1 to test for controlling, one only needs to consider those \(0 ≠ α ∈ \text{alglat}(N)\). Indeed, if \(0 ≠ α ∈ \text{End}(N_Δ) \setminus \text{alglat}(N)\), then there is a \(n_α\) such that \(αn_α ≠ Rn_α\), so for any \(m_α\), \(αn_α ≠ \ell_R(m_α)n_α\). Also one easily checks that \(\ell_R(M) \subseteq \ell_R(N)\) whenever \(M\) controls \(N\).

The necessity part of the following corollary was established in [9], and has been employed in several subsequent papers. The second statement generalizes the fact that if \(M\) is reflexive and controls \(N\), then \(M ⊕ N\) is reflexive.

**Corollary 2.2.** The bimodule \(RM_Δ\) controls \(RN_Δ\) if and only if the restriction mapping

\[ \text{res} : \text{alglat}(M ⊕ N) → \text{alglat}(M) \]

is injective.

Moreover, if these conditions hold and \(\text{Im}(\text{res}) ⊆ \lambda(R)\), then \(M ⊕ N\) is reflexive.

**Proof.** (⇐) Suppose \(M\) does not control \(N\). Then there is an \(α ≠ 0\) in \(\text{End}(N_Δ)\) such that for all \(m ∈ M\) and all \(n ∈ N\), there is an \(r ∈ \ell_R(m)\) with \(αn = rn\). But then \(0 ≠ β = (0, α) ∈ \text{End}((M ⊕ N)_Δ)\) and \(β(m, n) = (0, αn) = r(m, n)\), so \(β ∈ \text{alglat}(M ⊕ N)\), and \(β|M = 0\).

(⇒) If \(β = (0, α) ∈ \text{alglat}(M ⊕ N)\) with \(α ≠ 0\), then for each pair \((m, n) ∈ M × N\) there is an \(r ∈ R\) with \(β(m, n) = (rm, rn) = (0, rn)\), contrary to the condition of Proposition 2.1.

For the last statement, assume that \(\text{res}\) is injective, \(δ ∈ \text{alglat}(M ⊕ N)\) and \(\text{res}(δ) = λ_2(r)\) in the commutative diagram:

\[
\begin{array}{ccc}
\text{alglat}(M ⊕ N) & \xrightarrow{\text{res}} & \text{alglat}(M) \\
\lambda_1 \downarrow & & \lambda_2 \nearrow \\
R & & \end{array}
\]

Then \(\text{res}(δ) = λ_2(r) = \text{res}(λ_1(r))\), so \(δ = λ_1(r)\). □

Suppose that \(M\) controls \(N\). Then \(N\) is an \(R/ℓ_R(M)\)-module, and if \(M\) is reflexive, then so is \(M ⊕ N\). Conversely we have

**Corollary 2.3.** If \(RM\) is faithful and \(M ⊕ N\) is reflexive, then \(M\) controls \(N\). Thus if \(M\) is a faithful reflexive \(R\)-module, then \(M ⊕ N\) is reflexive if and only if \(M\) controls \(N\).

**Proof.** As in the previous proof, we have \(\text{res} ∘ λ_1 = λ_2\). Thus if \(λ_1\) is epic and \(λ_2\) is monic, then \(\text{res}\) is monic, so Corollary 2.2 applies. □

A QF-3 algebra \(R\) is a finite dimensional algebra with a (unique minimal faithful) module \(U\) that embeds as a direct summand in every faithful module. In [9] it was shown that over a QF-2 algebra (a special type of QF-3 algebra, see [11, Section 31]) every faithful module is reflexive if \(U\) is reflexive; and the problem was posed of determining whether this is the case for QF-3 algebras. Some recent progress has been made by Snashall in [17]. Perhaps this last corollary may help to shed more light on this problem.
3. Universal $k$-reflexivity

In [2] Azoff showed, from an operator theory point of view, that for an integer $k \geq 3$, if $\dim(c(M)) \leq k$, then the direct sum $M^{(k-1)}$ of $k-1$ copies of $M$ is reflexive. This topic was treated later using algebraic methods in [11]. We conclude by showing that for any left artinian ring $R$ there is an integer $k$ such that the direct sum of $k$ copies of every bimodule $R M$ is reflexive. To do so we shall employ

**Lemma 3.1.** If $R/\ell_R(M)$ embeds in $N$ and $R\Delta$ controls $R N\Delta$, then $M \oplus N$ is reflexive.

**Proof.** Since $R/\ell_R(M)$ embeds in $N$, there is an $n_0 \in N$ with $\ell_R(n_0) = \ell_R(M)$. Let $\delta = (\beta, \gamma) \in \text{alglat}(M \oplus N)$ and suppose that $\gamma(n_0) = s n_0$. Then for any $m \in M$, there are $r, t \in R$ with

$$r(m, n_0) = \delta(m, n_0) = (\beta m, \gamma n_0) = (tm, sn_0),$$

so $r - s \in \ell_R(n_0) = \ell_R(M)$ and

$$\beta m = rm = sm.$$

Thus $res(\delta) = \beta = \lambda(s)$, and Corollary 2.2 applies.

**Theorem 3.2.** For each left artinian ring $R$ there is a positive integer $k$ such that every $R$-bimodule is $k$-reflexive. Indeed if

$$k = \sup\{c(Soc(RR/I)) \mid I \leq RR_R \} + 1,$$

then $M^{(k)}$ is reflexive for every bimodule $R M\Delta$.

**Proof.** Let $c = c(Soc(RR/\ell_R(M)))$ and let $N = M^{(c)}$. Then it is easy to see that $R/\ell_R(M)$ embeds in $N$ and, of course $M$ controls $N$. Thus $M \oplus N = M^{(c+1)}$ is reflexive by Lemma 3.1.

This value of $k$ cannot be improved since, over the uniserial $K$-algebra $R = K[x]/x^2$, the regular module $R R$ is not reflexive (see [2] or [9]).

According to [11, Corollary 3] the result of Azoff mentioned above can be extended to the $K$-algebra-bimodule case. Thus from this result and the proof of Theorem 3.2 we have

**Corollary 3.3.** Let $R$ and $\Delta$ be finite dimensional $K$-algebras and let $M$ be an $R - \Delta$-bimodule such that $R M$ is faithful. If $k \geq 2$ and

$$\min\{c(Soc(RR)) + 1, c(M\Delta) - 1\} \leq k,$$

then $M^{(k)}$ is reflexive.

**References**


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