ON REFLEXIVITY OF DIRECT SUMS

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Abstract. Necessary and sufficient conditions are presented to insure that the direct sum of two reflexive representations of a finite dimensional algebra is reflexive, and it is shown that for each such algebra, there is an integer $k$ such that the direct sum of $k$ copies of each of its representations is reflexive. Given a ring $\Delta$, our results are actually presented in the more general setting of $\Delta$-representations of a ring $R$.

Over the last thirty years, a significant amount of attention has been given to the problem of determining when an algebra $A$ of operators on a vector space $V$ (often a Hilbert space) over a field $K$ is reflexive in the sense that no larger algebra of operators on that space has the same lattice of invariant subspaces. (See, for example, [2], [3], [6], [7], [12], [13], [14].) Of course, $A$ is an algebra of operators on a vector space $V$ if and only if $V$ is a faithful left $A$-module. Thus, to attack this problem from a more module theoretic point of view, based on notation of Halmos [14], the following notions were presented in [9]: For any $R-\Delta$-bimodule $M = RM_{\Delta}$ one defines

$$\text{alglat}(M) = \{ \alpha \in \text{End}(M_{\Delta}) \mid \alpha L \subseteq L \text{ for all } RL \subseteq RM \},$$

and, letting $\lambda : R \to \text{alglat}(M_{\Delta})$ denote the canonical ring homomorphism, $M$ is called a reflexive bimodule (or $\Delta$-representation of $R$) if $\lambda$ is surjective. Thus the $K$-algebra of operators $A$ on $V$ is reflexive if and only if $AV_K$ is a reflexive bimodule (and then we simply say that the $A$-module $V$ is reflexive).

The problems we shall deal with here have their roots in the papers [5] and [6] of Deddens and Fillmore and [2] of Azoff. Stated in module theoretic terms, in the first pair of papers the question was posed and answered (affirmatively) of whether a direct sum of two finitely generated reflexive modules over an algebra generated by a single complex matrix is again reflexive; and Azoff showed that for each finite dimensional module over a $\mathbb{C}$-algebra, there is a positive integer $k$, depending on its dimension, such that the direct sum of $k$ copies of that module is reflexive. Other results in this vein can be found in [3] where Brenner and Butler showed that the direct sum of two copies of the regular representation of a finite dimensional algebra is reflexive, and in Habibi and Gustafson’s [12] from which the same result follows for any faithful representation of a split serial algebra. (See [9], [10] and [11] for related and more general results.)

Our main objectives are to provide, in Theorem 1.2, a necessary and sufficient condition for a direct sum of two reflexive bimodules to be reflexive; and to show, in
Theorem\textsuperscript{3.2} that whenever $R$ is a left artinian ring with composition length $c(RR)$, there is a positive integer $k \leq c(RR) + 1$ such that the direct sum $M^{(k)}$ of $k$ copies of $M$ is reflexive for any bimodule $RM$. Along the way we show how Theorem\textsuperscript{1.2} can be employed to obtain simple proofs of some known results, and we examine a common generalization of generating and cogenerating called controlling that was introduced and employed in \cite{9} and \cite{10}, showing in particular that if $RM$ is faithful and reflexive, then $M \oplus N$ is reflexive if and only if $M$ controls $N$.

1. A characterization

Unless otherwise specified, all modules under consideration will be left-$R$, right-$\Delta$ bimodules for a fixed pair of rings $R$ and $\Delta$. If $R$ is an algebra over a field $K$, we shall assume that $\Delta = K$. If $\alpha \in \text{End}(M \Delta)$ and $\beta \in \text{End}(N \Delta)$, we shall write $(\alpha, \beta) \in \text{End}(M \oplus N)$ for the direct sum map $(\alpha, \beta) : (m, n) \mapsto (am, \beta n)$. Thus if $\gamma \in \text{alglat}(M \oplus N)$, then $\gamma = (\gamma|_M, \gamma|_N)$.

A common subquotient of a pair of left modules $RM$ and $RN$ is a module $RX$ that is isomorphic to a subquotient of both $RM$ and $RN$. This notion together with the following lemma allows us to determine just when the direct sum of a pair of reflexive modules is reflexive.

Lemma 1.1. If $r, s \in R$, the ordered pair $(r, s) \in \text{alglat}(M \oplus N)$ if and only if $(r - s)X = 0$, for every (equivalently, every cyclic) common subquotient of $RM$ and $RN$.

Proof. Note that $a \in R$ annihilates every common subquotient of $M$ and $N$ if and only if it annihilates each of their cyclic common subquotients. According to Gour-sats' Lemma, $W$ is an $R$-submodule of $M \oplus N$, if and only if there are submodules

$$M_2 \leq M_1 \leq M \quad \text{and} \quad N_2 \leq N_1 \leq N$$

and an $R$-isomorphism

$$f : M_1/M_2 \to N_1/N_2$$

such that

$$W = \{(m_1, n_1) \in M_1 \oplus N_1 \mid f(m_1 + M_2) = n_1 + N_2\}.$$ 

(Given $W$ and the orthogonal projections $\pi_M$ and $\pi_N$ for $M \oplus N$, one checks that $M_1 = \pi_M(W)$, $M_2 = M \cap W$, $N_1 = \pi_N(W)$, and $N_2 = N \cap W$.)

$(\Rightarrow)$ If $(m_1, n_1) \in W$ and $(rm_1, sn_1) \in W$, then

$$f(rm_1 + M_2) = sn_1 + N_2 = sf(m_1 + M_2)$$

so, since $f$ is an $R$-isomorphism, $(r - s)(M_1/M_2) = 0$.

$(\Leftarrow)$ Suppose $(r - s)(M_1/M_2) = 0$, and $f : M_1/M_2 \to N_1/N_2$ is an isomorphism and $(m_1, n_1) \in W$. Then

$$f(rm_1 + M_2) = rf(m_1 + M_2) = rm_1 + N_2 = sn_1 + N_2$$

so $(rm_1, sn_1) \in W$. \hfill $\square$

Now we are able to provide the promised characterization in terms of annihilators of subquotients of $M$ and $N$. The left annihilator of a module $M$ is $\ell_R(M) = \{r \in R \mid rM = 0\}$. 


Theorem 1.2. Let $M$ and $N$ be reflexive, and let $\{X_i \mid i \in I\}$ represent one copy of each of the (cyclic) common subquotients of $_RM$ and $_RN$. Then $M \oplus N$ is reflexive if and only if

$$\ell_R(M) + \ell_R(N) = \ell_R(\bigoplus_{i \in I} X_i).$$

Proof. $(\Rightarrow)$ Let $\alpha = (\beta, \gamma) \in \text{alglat}(M \oplus N)$. Then there are $r, s \in R$ with $\beta = r$ and $\gamma = s$, and by Lemma 1.1 $(r-s) \in \ell_R(\bigoplus_{i \in I} X_i)$. But then by hypothesis

$$r-s = p-q \text{ with } p \in \ell_R(M) \text{ and } q \in \ell_R(N)$$

so that, letting

$$t = r-p = s-q,$$

we have $\alpha = \lambda(t)$.

$(\Rightarrow)$ Let $r \in \ell_R(\bigoplus_{i \in I} X_i)$. Then by Lemma 1.1 $(r,0) \in \text{alglat}(M \oplus N)$. Thus, assuming that $M \oplus N$ is reflexive,

$$(r,0) = (s,s)$$

and

$$r = (r-s) + s \in \ell_R(M) + \ell_R(N).$$

Since always $\ell_R(M) + \ell_R(N) \subseteq \ell_R(\bigoplus_{i \in I} X_i)$, the proof is complete. \qed

We note that the proof $(\Rightarrow)$ above shows that if $M \oplus N$ is reflexive (regardless of reflexivity of $M$ and $N$), then $\ell_R(M) + \ell_R(N) = \ell_R(\bigoplus_{i \in I} X_i)$.

From the inclusions

$$0 \subseteq \ell_R(M) + \ell_R(N) \subseteq \ell_R(\bigoplus_{i \in I} X_i) \subseteq \ell_R(X_i) \subseteq R$$

we easily obtain the following two corollaries:

Corollary 1.3. If $M$ and $N$ are reflexive and have a common faithful subquotient, then $M \oplus N$ is reflexive.

Corollary 1.4. If $R$ is semiperfect and $M$ and $N$ are reflexive and have no common composition factor, then $M \oplus N$ is reflexive.

Proof. Suppose that $e_1, \ldots, e_m, e_{m+1}, \ldots, e_n$ is a complete set of primitive idempotents such that $Re_i/J e_i$ is not a composition factor of $M$ for $i = 1, \ldots, m$, and $Re_i/J e_i$ is not a composition factor of $N$ for $i = m+1, \ldots, n$. Then $\ell_R(M) + \ell_R(N) = R$. (See [11] Section 32.) \qed

Regarding the Deddens-Fillmore result, the algebra generated by a matrix over a field $K$, being isomorphic to a proper factor of the polynomial ring $K[x]$, is an example of a split commutative uniserial algebra, i.e., a direct product of local uniserial rings. Since a module over a direct product of rings is reflexive precisely when its corresponding components are reflexive, to show that direct sums of reflexive modules are reflexive over such an algebra $R$, we may assume that $R$ is a local uniserial algebra, so that the ideals of $R$ are linearly ordered and every $R$-module is a direct sum of factors of $R$ (see [11] Section 32). In this case, $R/\ell_R(M)$ embeds in $M$ and every subquotient of $_RM$ is a factor of $R/\ell_R(M)$. In the presence of these facts Theorem 1.2 yields the following corollary almost at once. We note, however, that [11] Theorem 1] is more general.
Corollary 1.5. The direct sum of a finite number of reflexive modules over a split uniserial algebra is reflexive.

Proof. Since we may assume that the ideals of \( R \) are linearly ordered, given modules \( M \) and \( N \), we may also assume that \( \ell_R(M) \subseteq \ell_R(N) \subseteq \ell_R(X) \) for every common subquotient \( X \) of \( M \) and \( N \). So since \( R/\ell_R(N) \) embeds in \( N \), we see that both sides of the desired equality are equal to \( \ell_R(N) \).

Next, as further applications of this theorem, we shall see that it yields particularly nice proofs of some key results in [10].

In [10], in order to show that direct sums of reflexive modules may fail to be reflexive over a split \( K \)-algebra \( R \) with radical \( J \) whose quiver contains a triple arrow, a pair of reflexive modules was constructed with diagrams (as in [8])

\[
M: \begin{array}{ccc}
\bullet & \xrightarrow{u} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{v} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{w} & \bullet
\end{array}
\quad \text{and} \quad
N: \begin{array}{ccc}
\bullet & \xrightarrow{a} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{c} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{z} & \bullet
\end{array}
\]

with \( a, b, c \) linearly independent elements of \( J \setminus J^2 \). (The diagram indicates that \( au = x, aw = cw = z, \) etc.) Here

\[ \ell_R(M) + \ell_R(N) = \ell_R(M) + K(a - c) + Kb, \]

but the only common subquotients of \( M \) and \( N \) are simple modules, so

\[ \ell_R(\bigoplus_{i \in I} X_i) = \ell_R(M) + J \neq \ell_R(M) + \ell_R(N). \]

Thus the theorem shows that \( M \oplus N \) is not reflexive.

In the positive vein we shall employ the following lemma and Theorem 1.2 to obtain a simple proof of a key part of [10, Proposition 4].

Lemma 1.6. Let \( R \) be a split local \( K \)-algebra of \( \dim(KR) \leq 3 \). If \( M \) is a faithful left \( R \)-module, then every proper cyclic module is isomorphic to a subquotient of \( M \).

Proof. (This proof is essentially contained in the proof of [10, Proposition 4].) If \( R \) is uniserial, then \( _RR \) embeds in \( M \), so we may assume that \( J^2 = 0 \) and that \( \dim(KJ) = 2 \). Let \( u \in M \setminus \text{Soc} M \) and suppose that \( \ell_R(u) \neq 0 \). Then, since \( u \notin \text{Soc} M \), there is a \( b \in J \) with \( \ell_R(u) = Kb \). But then there is a \( v \in M \) with \( bv \neq 0 \), and there is an \( a \in J \) with \( Ka = \ell_R(v) \). Thus it follows that

\[ J = Ka \oplus Kb. \]

Now

\[ Ru \cong R/Rb \quad \text{and} \quad Rv \cong R/Ra \]

are non-isomorphic uniserial submodules of \( M \) of composition length 2. If \( Ru \cap Rv = 0 \), we see that \( \ell_R(u + v) = 0 \). Otherwise

\[ Ru \cap Rv = Ka \cong Kb = \text{Soc}(Ru + Rv), \]

so the annihilator of any non-zero \( K \)-linear combination of \( u \) and \( v \) is properly contained in \( J \), and we may assume that \( au = bv \). Then since

\[ (ka + lb)((k^{-1}u - \ell^{-1}v) = au - bv = 0, \]

we see that every proper cyclic \( R \)-module is a subquotient of \( M \). \( \square \)
Now we have the promised result of Fuller, Nicholson and Watters \cite{10}.

**Proposition 1.7.** Let \( R \) be a split local \( K \)-algebra of \( \dim(KR) \leq 3 \). If \( M \) and \( N \) are reflexive \( R \)-modules, then so is \( M \oplus N \).

**Proof.** We may assume that \( M \oplus N \) is faithful. If neither \( M \) nor \( N \) is faithful, then \( J = Ka \oplus Kb \) with \( aM = 0 \) and \( bN = 0 \). But then
\[
\ell_R(M) + \ell_R(N) = J = \ell_R(R/J)
\]
and the desired equality holds. If \( M \) is faithful and \( N \) is not, then by Lemma 1.6 every cyclic subquotient of \( N \) is a subquotient of \( M \), so
\[
\ell_R(M) + \ell_R(N) = \ell_R(N) = \ell_R(\bigoplus_{i \in I} X_i).
\]
Suppose both \( M \) and \( N \) are faithful. If \( R \) embeds in \( M \), then every cyclic subquotient of \( N \) is a common subquotient of \( M \) and \( N \), so
\[
\ell_R(M) + \ell_R(N) \subseteq \ell_R(\bigoplus_{i \in I} X_i) = \ell_R(N) = 0.
\]
Otherwise, the cyclic subquotients of \( M \) and \( N \) are all proper, so by Lemma 1.6 they are all common to both modules and again
\[
\ell_R(\bigoplus_{i \in I} X_i) = 0.
\]
\( \square \)

2. Controlling

The following the definition in \cite{9} or \cite{10}, given bimodules \( _RM_\Delta \) and \( _RN_\Delta \), we say that \( M \) controls \( N \) in case for each pair \((\alpha, n)\), with \( \alpha \in \End(N_\Delta) \) and \( n \in N \), there is a set
\[
C_{(\alpha, n)} = \{(m_i, n_i) \mid i \in I \} \subseteq M \times N,
\]
called a connection for \( \alpha \) and \( n \), such that, if there are \( r_i \in R \) with
\[
r_im_i = 0 \text{ and } \alpha n_i = r_in_i
\]
for all \( i \in I \), then \( \alpha n = 0 \).

Controlling is a particularly useful concept. In \cite{9} it was shown that if \( N \) is either generated or cogenerated by subquotients of \( M \), then \( M \) controls \( N \); and that if \( M \) is reflexive and controls \( N \), then \( M \oplus N \) is reflexive. Here we shall present an equivalent version of controlling that yields a partial converse to this last assertion.

**Proposition 2.1.** The bimodule \( _RM_\Delta \) controls \( _RN_\Delta \) if and only if, for each \( 0 \neq \alpha \in \End(N_\Delta) \), there is a pair
\[
P_\alpha = (m_\alpha, n_\alpha) \in M \times N
\]
with \( \alpha n_\alpha \notin \ell_R(m_\alpha)n_\alpha \) (i.e., for all \( r \in R \), if \( \alpha n_\alpha = rn_\alpha \), then \( rm_\alpha \neq 0 \)).

**Proof.** \((\Rightarrow)\) Let \( \alpha \in \End(N_\Delta) \) with \( \alpha n \neq 0 \) and suppose that \( C_{(\alpha, n)} = \{(m_i, n_i) \mid i \in I \} \) is a connection for \( \alpha \) and \( n \). If \( \alpha n_i \in \ell_R(m_i)n_i \) for all \( i \in I \), then there are \( r_i \) with \( r_im_i = 0 \) and \( \alpha n_i = r_in_i \) for all \( i \in I \). But \( \alpha n \neq 0 \), so there is a pair \((m_i, n_i)\) with \( \alpha n_i \notin \ell_R(m_i)n_i \).
If \( an = 0 \), then any \( C(\alpha, n) = \{ (m_i, n_i) | i \in I \} \) is a connection for \( \alpha \) and \( n \).

If \( an \neq 0 \), let

\[
C(\alpha, n) = \{ P \alpha \} = \{ (m_\alpha, n_\alpha) \}.
\]

Then it never occurs that \( r_\alpha m_\alpha = 0 \) and \( \alpha n_\alpha = r_\alpha n_\alpha \), so \( C(\alpha, n) \) is a connection.

It is worthy of note that when employing Proposition 2.1 to test for controlling, one only needs to consider those \( 0 \neq \alpha \in \text{alglat}(N) \). Indeed, if \( 0 \neq \alpha \in \text{End}(N_\Delta) \setminus \text{alglat}(N) \), then there is a \( n_\alpha \) such that \( \alpha n_\alpha \notin \text{im}(\alpha) \), so for any \( m_\alpha \), \( \alpha n_\alpha \notin \ell_R(m_\alpha)n_\alpha \). Also one easily checks that \( \ell_R(M) \subseteq \ell_R(N) \) whenever \( M \) controls \( N \).

The necessity part of the following corollary was established in [9], and has been employed in several subsequent papers. The second statement generalizes the fact that if \( M \) is reflexive and controls \( N \), then \( M \oplus N \) is reflexive.

**Corollary 2.2.** The bimodule \( R M_\Delta \) controls \( R N_\Delta \) if and only if the restriction mapping

\[
\text{res} : \text{alglat}(M \oplus N) \rightarrow \text{alglat}(M)
\]

is injective.

Moreover, if these conditions hold and \( \text{Im}(\text{res}) \subseteq \lambda(R) \), then \( M \oplus N \) is reflexive.

**Proof.** (\( \Leftarrow \)) Suppose \( M \) does not control \( N \). Then there is an \( \alpha \neq 0 \) in \( \text{End}(N_\Delta) \) such that for all \( m \in M \) and all \( n \in N \), there is an \( r \in \ell_R(m) \) with \( \alpha n = rm \). But then \( 0 \neq \beta = (0, \alpha) \in \text{End}(\text{alglat}(M \oplus N)) \) and \( \beta(m, n) = (0, \alpha n) = r(m, n), \) so \( \beta \in \text{alglat}(M \oplus N) \), and \( \beta|_M = 0. \)

(\( \Rightarrow \)) If \( \beta = (0, \alpha) \in \text{alglat}(M \oplus N) \) with \( \alpha \neq 0 \), then for each pair \( (m, n) \in M \times N \) there is an \( r \in R \) with \( \beta(m, n) = (rm, rn) = (0, rn) \), contrary to the condition of Proposition 2.1.

For the last statement, assume that \( \text{res} \) is injective, \( \delta \in \text{alglat}(M \oplus N) \) and \( \text{res}(\delta) = \lambda_2(r) \) in the commutative diagram:

\[
\begin{array}{ccc}
\text{alglat}(M \oplus N) & \xrightarrow{\text{res}} & \text{alglat}(M) \\
\downarrow{\lambda_1} & & \downarrow{\lambda_2} \\
R & & R
\end{array}
\]

Then \( \text{res}(\delta) = \lambda_2(r) = \text{res}(\lambda_1(r)) \), so \( \delta = \lambda_1(r) \).

Suppose that \( M \) controls \( N \). Then \( N \) is an \( R/\ell_R(M) \)-module, and if \( M \) is reflexive, then so is \( M \oplus N \). Conversely we have

**Corollary 2.3.** If \( R M \) is faithful and \( M \oplus N \) is reflexive, then \( M \) controls \( N \). Thus if \( M \) is a faithful reflexive \( R \)-module, then \( M \oplus N \) is reflexive if and only if \( M \) controls \( N \).

**Proof.** As in the previous proof, we have \( \text{res} \circ \lambda_1 = \lambda_2 \). Thus if \( \lambda_1 \) is epic and \( \lambda_2 \) is monic, then \( \text{res} \) is monic, so Corollary 2.2 applies.

A QF-3 algebra \( R \) is a finite dimensional algebra with a (unique minimal faithful) module \( U \) that embeds as a direct summand in every faithful module. In [9] it was shown that over a QF-2 algebra (a special type of QF-3 algebra, see [11, Section 31]) every faithful module is reflexive if \( U \) is reflexive; and the problem was posed of determining whether this is the case for QF-3 algebras. Some recent progress has been made by Snashall in [15]. Perhaps this last corollary may help to shed more light on this problem.
3. UNIVERSAL $k$-REFLEXIVITY

In [2] Azoff showed, from an operator theory point of view, that for an integer $k \geq 3$, if $\dim(c(M)) \leq k$, then the direct sum $M^{(k-1)}$ of $k-1$ copies of $M$ is reflexive. This topic was treated later using algebraic methods in [11]. We conclude by showing that for any left artinian ring $R$ there is an integer $k$ such that the direct sum of $k$ copies of every bimodule $RM$ is reflexive. To do so we shall employ

**Lemma 3.1.** If $R/\ell_R(M)$ embeds in $N$ and $RM$ controls $RN$, then $M \oplus N$ is reflexive.

**Proof.** Since $R/\ell_R(M)$ embeds in $N$, there is an $n_0 \in N$ with $\ell_R(n_0) = \ell_R(M)$. Let $\delta = (\beta, \gamma) \in \text{alglat}(M \oplus N)$ and suppose that $\gamma(n_0) = s n_0$. Then for any $m \in M$, there are $r, t \in R$ with

$$r(m, n_0) = \delta(m, n_0) = (\beta m, \gamma n_0) = (tm, sn_0),$$

so $r - s \in \ell_R(n_0) = \ell_R(M)$ and

$$\beta m = rm = sm.$$

Thus $\text{res}(\delta) = \beta = \lambda(s)$, and Corollary 2.2 applies.

**Theorem 3.2.** For each left artinian ring $R$ there is a positive integer $k$ such that every $R$-bimodule is $k$-reflexive. Indeed if

$$k = \sup\{c(Soc(RR/I)) | I \leq RRR\} + 1,$$

then $M^{(k)}$ is reflexive for every bimodule $RM$.

**Proof.** Let $c = c(Soc(RR/\ell_R(M)))$ and let $N = M^{(c)}$. Then it is easy to see that $R/\ell_R(M)$ embeds in $N$ and, of course $M$ controls $N$. Thus $M \oplus N = M^{(c+1)}$ is reflexive by Lemma 3.1.

This value of $k$ cannot be improved since, over the uniserial $K$-algebra $R = K[x]/x^2$, the regular module $RR$ is not reflexive (see [2] or [9]).

According to [11, Corollary 3] the result of Azoff mentioned above can be extended to the $K$-algebra-bimodule case. Thus from this result and the proof of Theorem 3.2 we have

**Corollary 3.3.** Let $R$ and $\Delta$ be finite dimensional $K$-algebras and let $M$ be an $R-\Delta$-bimodule such that $RM$ is faithful. If $k \geq 2$ and

$$\min\{c(Soc(RR)) + 1, c(M) - 1\} \leq k,$$

then $M^{(k)}$ is reflexive.

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