A REMARK ON THE BERGMAN STABILITY

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(Communicated by Steven R. Bell)

ABSTRACT. Let \( \{D_k\}, k = 1, 2, \cdots \), be a sequence of bounded pseudoconvex domains that converges, in the sense of Boas, to a bounded domain \( D \). We show that if \( \partial D \) can be described locally as the graph of a continuous function in suitable coordinates for \( \mathbb{C}^n \), then the Bergman kernel of \( D_k \) converges to the Bergman kernel of \( D \) uniformly on compact subsets of \( D \times D \).

1. Introduction

Let \( D \) be a bounded domain in \( \mathbb{C}^n \). By \( K_D(z, w) \) we denote the Bergman kernel of \( D \). After the early paper of Ramadanov [7], there is a long list of papers concerning the stability problem of the Bergman kernels of a sequence of domains \( D_k \to D \) (cf. [1], [2], [3], [4], [5], [8]). The example [8] of decreasing concentric disks in the complex plane converging to a disk with a slit removed shows that it is not sufficient to require only that the \( D_k \) converge to \( D \) in the sense of Boas [1], i.e., the \( D_k \) eventually swallow every compact subset of \( D \) and are eventually swallowed by every open neighbourhood of \( \overline{D} \). Boas [1] proved stability when \( D \) has \( C^2 \) boundary and \( D_k \) are pseudoconvex. He also asked if the hypothesis could be reduced to \( C^1 \) boundary regularity. The answer is yes; in fact, we are going to prove the following

Main Theorem. Let \( \{D_k\} \) be a sequence of bounded pseudoconvex domains that converges, in the sense of Boas, to a bounded domain \( D \). Suppose that \( \partial D \) can be described locally as the graph of a continuous function in suitable coordinates for \( \mathbb{C}^n \). Then the sequence \( K_{D_k}(z, w) \) converges to \( K_D(z, w) \) uniformly on compact subsets of \( D \times D \).

Proof of the Main Theorem. As in [1], it is sufficient to show that if \( f \) is a square-integrable holomorphic function on \( D \), and if a positive \( \epsilon \) is prescribed, then for all sufficiently large \( k \) there exists a square-integrable holomorphic function \( f_k \) on \( D_k \) such that \( \|f_k - f\|_{L^2(D_k \cap D)} < \epsilon \) and \( \|f_k\|_{L^2(D_k \setminus D)} < \epsilon \).

For each \( \zeta^0 \in \partial D \), there is a neighbourhood \( U \) of \( \zeta^0 \) such that \( D \cap U = \{z \in U|x_2n < \psi(x_1, \cdots, x_{2n-1})\} \) in suitable coordinates \( z = (x_1 + ix_2, \cdots, x_{2n-1} + ix_{2n}) \in \mathbb{C}^n \), where \( \psi \) is a continuous function. Then there exists a neighbourhood \( V \) of \( \zeta^0 \) with \( V \subseteq U \) such that for all sufficiently small \( \delta > 0 \) one has \( z = (0, \cdots, 0, i\delta) \in D \) for all \( z \in \overline{D} \cap V \) and \( z = (0, \cdots, 0, i\delta) \in \mathbb{C}^n \setminus \overline{D} \) for all \( z \in (\mathbb{C}^n \setminus D) \cap V \). Hence

Received by the editors July 20, 1998 and, in revised form, October 30, 1998.
1991 Mathematics Subject Classification. Primary 32H10.
Key words and phrases. Bergman kernel.

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we can assume that there exist finitely many points $\zeta_j \in \partial D$, $1 \leq j \leq l$, reals $\delta_0 > 0$, $r > 0$, and unit outward vectors $N_j$ at $\zeta_j$ such that
\[
(\overline{D} \cap B(\zeta_j, r)) - \delta N_j \subset D,
((C^n \setminus D) \cap B(\zeta_j, r)) + \delta N_j \subset C^n \setminus \overline{D}
\]
for all $0 < \delta \leq \delta_0$, where $B(\zeta, r)$ is the ball in $C^n$ which is centered at $\zeta$ with radius $r$. Choose $U_0 \subset D$ such that $U_0 \cup (B(\zeta_j, r))_{1 \leq j \leq l}$ cover $\overline{D}$. Let $(\varphi_j)_{0 \leq j \leq l}$ be a smooth partition of unity associated to the covering $U_0, B(\zeta_j, r)$. For each $0 < \delta \leq \delta_0$ we put
\[
h_\delta(z) = \sum_{j=1}^l f(z - \delta N_j) \varphi_j(z) + \varphi_0(z) f(z).
\]
Then $h_\delta$ is $C^\infty$ on an open neighbourhood of $\overline{D}$. For each $z \in D$, we have
\[
h_\delta(z) = \sum_{j=1}^l (f(z - \delta N_j) - f(z)) \varphi_j(z) + f(z),
\]
which gives
\[
\partial h_\delta = \sum_{j=1}^l (f(z - \delta N_j) - f(z)) \partial \varphi_j.
\]
Hence for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $\|\partial h_\delta\|_{L^2(D)} \leq \epsilon$ and $\|h_\delta - f\|_{L^2(D)} \leq \epsilon$. Now fix $\delta$. Then
\[
\|\partial h_\delta\|_{L^2(D_k)} = \|\partial h_\delta\|_{L^2(D_k \cap D)} + \|\partial h_\delta\|_{L^2(D_k \setminus D)} \leq \epsilon + \|\partial h_\delta\|_{L^2(D_k \setminus D)} \\
\leq \epsilon + \epsilon + \|\partial h_\delta\|_{L^2(D_k \setminus D)} \\
\leq 2\epsilon
\]
for all sufficiently large $k$. By a well-known theorem of Hörmander (cf. [3]), there exists a function $u_k$ on $D_k$ such that $\partial u_k = \partial h_\delta$ and
\[
\|u_k\|_{L^2(D_k)} \leq C \|\partial h_\delta\|_{L^2(D_k)} \leq 2C\epsilon,
\]
where $C$ is a positive constant depending only on the diameter of $D$. Put $f_k = h_\delta - u_k$. Then $f_k$ is a holomorphic function on $D_k$ satisfying
\[
\|f_k - f\|_{L^2(D_k \cap D)} \leq \|h_\delta - f\|_{L^2(D_k \cap D)} + \|u_k\|_{L^2(D_k)} \\
\leq (1 + 2C)\epsilon
\]
and
\[
\|f_k\|_{L^2(D_k \setminus D)} \leq \|h_\delta\|_{L^2(D_k \setminus D)} + \|u_k\|_{L^2(D_k)} \\
\leq (1 + 2C)\epsilon
\]
for all sufficiently large $k$. Q.E.D.

In the case of $n = 1$, a sufficient and necessary condition of Bergman stability given in [5] is the following.
Proposition. Let $\overline{D} = \bigcap_{k=1}^{\infty} D_k$, $D \subset D_{k+1} \subset D_k$, $D = \inf \overline{D}$ and $|\partial D| = 0$. Then $K_{D_k}(z,w) \to K_D(z,w)$ as $k \to \infty$ locally uniformly in $(z,w) \in D \times D$ if and only if the set of all $z \in \partial D$ which do not belong to the fine closure of the complement of $\overline{D}$ has a zero logarithmic capacity.

Therefore, we can’t expect a similar phenomenon in the theorem to hold on a bounded domain which is only assumed to be the interior of its closure.

Acknowledgment

We would like to express our sincere thanks to the referee for the last version of this paper.

References


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