A REMARK ON THE BERGMAN STABILITY

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ABSTRACT. Let \( \{D_k\} \), \( k = 1, 2, \cdots \), be a sequence of bounded pseudoconvex domains that converges, in the sense of Boas, to a bounded domain \( D \). We show that if \( \partial D \) can be described locally as the graph of a continuous function in suitable coordinates for \( \mathbb{C}^n \), then the Bergman kernel of \( D_k \) converges to the Bergman kernel of \( D \) uniformly on compact subsets of \( D \times D \).

1. Introduction

Let \( D \) be a bounded domain in \( \mathbb{C}^n \). By \( K_D(z,w) \) we denote the Bergman kernel of \( D \). After the early paper of Ramadanov \[7\], there is a long list of papers concerning the stability problem of the Bergman kernels of a sequence of domains \( D_k \to D \) (cf. \[1\], \[2\], \[3\], \[4\], \[5\], \[8\]). The example \[8\] of decreasing concentric disks in the complex plane converging to a disk with a slit removed shows that it is not sufficient to require only that the \( D_k \) converge to \( D \) in the sense of Boas \[1\], i.e., the \( D_k \) eventually swallow every compact subset of \( D \) and are eventually swallowed by every open neighbourhood of \( \overline{D} \). Boas \[1\] proved stability when \( D \) has \( C^2 \) boundary and \( D_k \) are pseudoconvex. He also asked if the hypothesis could be reduced to \( C^1 \) boundary regularity. The answer is yes; in fact, we are going to prove the following

Main Theorem. Let \( \{D_k\} \) be a sequence of bounded pseudoconvex domains that converges, in the sense of Boas, to a bounded domain \( D \). Suppose that \( \partial D \) can be described locally as the graph of a continuous function in suitable coordinates for \( \mathbb{C}^n \). Then the sequence \( K_{D_k}(z,w) \) converges to \( K_D(z,w) \) uniformly on compact subsets of \( D \times D \).

Proof of the Main Theorem. As in \[1\], it is sufficient to show that if \( f \) is a square-integrable holomorphic function on \( D \), and if a positive \( \epsilon \) is prescribed, then for all sufficiently large \( k \) there exists a square-integrable holomorphic function \( f_k \) on \( D_k \) such that \( \| f_k - f \|_{L^2(D_k \cap D)} < \epsilon \) and \( \| f_k \|_{L^2(D_k \setminus D)} < \epsilon \).

For each \( \zeta^0 \in \partial D \), there is a neighbourhood \( U \) of \( \zeta^0 \) such that \( D \cap U = \{ z \in U | x_{2n} < \psi(x_1, \cdots, x_{2n-1}) \} \) in suitable coordinates \( z = (x_1+ix_2, \cdots, x_{2n-1}+ix_{2n}) \in \mathbb{C}^n \), where \( \psi \) is a continuous function. Then there exists a neighbourhood \( V \) of \( \zeta^0 \) with \( V \subseteq U \) such that for all sufficiently small \( \delta > 0 \) one has \( z = (0, \cdots, 0, i\delta) \in D \) for all \( z \in \overline{D} \cap V \) and \( z = (0, \cdots, 0, i\delta) \in \mathbb{C}^n \setminus \overline{D} \) for all \( z \in (\mathbb{C}^n \setminus D) \cap V \). Hence
we can assume that there exist finitely many points $\zeta_j \in \partial D$, $1 \leq j \leq l$, reals $\delta_0 > 0$, $r > 0$, and unit outward vectors $N_j$ at $\zeta_j$ such that
\[
(D \cap B(\zeta_j, r)) - \delta N_j \subset D,
\]
\[
((C^n \setminus D) \cap B(\zeta_j, r)) + \delta N_j \subset C^n \setminus \overline{D}
\]
for all $0 < \delta \leq \delta_0$, where $B(\zeta, r)$ is the ball in $C^n$ which is centered at $\zeta$ with radius $r$. Choose $U_0 \subset D$ such that $U_0 \cup (B(\zeta_j, r))_{1 \leq j \leq l}$ cover $D$. Let $(\varphi_j)_{0 \leq j \leq l}$ be a smooth partition of unity associated to the covering $U_0, B(\zeta_j, r)$. For each $0 < \delta \leq \delta_0$ we put
\[
h_\delta(z) = \sum_{j=1}^l f(z - \delta N_j) \varphi_j(z) + \varphi_0(z) f(z).
\]
Then $h_\delta$ is $C^\infty$ on an open neighbourhood of $\overline{D}$. For each $z \in D$, we have
\[
h_\delta(z) = \sum_{j=1}^l (f(z - \delta N_j) - f(z)) \varphi_j(z) + f(z),
\]
which gives
\[
\partial h_\delta = \sum_{j=1}^l (f(z - \delta N_j) - f(z)) \partial \varphi_j.
\]
Hence for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $\|\partial h_\delta\|_{L^2(D)} \leq \epsilon$ and $\|h_\delta - f\|_{L^2(D)} \leq \epsilon$. Now fix $\delta$. Then
\[
\|\partial h_\delta\|_{L^2(D_k)} = \|\partial h_\delta\|_{L^2(D_k \cap D)} + \|\partial h_\delta\|_{L^2(D_k \setminus D)} \leq \|\partial h_\delta\|_{L^2(D)} + \|\partial h_\delta\|_{L^2(D_k \setminus D)} \leq \epsilon + \|\partial h_\delta\|_{L^2(D_k \setminus D)} \leq 2\epsilon
\]
for all sufficiently large $k$. By a well-known theorem of Hörmander (cf. [8]), there exists a function $u_k$ on $D_k$ such that $\partial u_k = \partial h_\delta$ and
\[
\|u_k\|_{L^2(D_k)} \leq C\|\partial h_\delta\|_{L^2(D_k)} \leq 2C\epsilon,
\]
where $C$ is a positive constant depending only on the diameter of $D$. Put $f_k = h_\delta - u_k$. Then $f_k$ is a holomorphic function on $D_k$ satisfying
\[
\|f_k - f\|_{L^2(D_k \cap D)} \leq \|h_\delta - f\|_{L^2(D_k \cap D)} + \|u_k\|_{L^2(D_k)} \leq (1 + 2C)\epsilon
\]
and
\[
\|f_k\|_{L^2(D_k \setminus D)} \leq \|h_\delta\|_{L^2(D_k \setminus D)} + \|u_k\|_{L^2(D_k)} \leq (1 + 2C)\epsilon
\]
for all sufficiently large $k$. Q.E.D.

In the case of $n = 1$, a sufficient and necessary condition of Bergman stability given in [8] is the following.
Proposition. Let $\overline{D} = \bigcap_{k=1}^{\infty} D_k$, $D \subset D_{k+1} \subset D_k$, $D = \text{int} \overline{D}$ and $|\partial D| = 0$. Then $K_{D_k}(z, w) \to K_D(z, w)$ as $k \to \infty$ locally uniformly in $(z, w) \in D \times D$ if and only if the set of all $z \in \partial D$ which do not belong to the fine closure of the complement of $D$ has a zero logarithmic capacity.

Therefore, we can’t expect a similar phenomenon in the theorem to hold on a bounded domain which is only assumed to be the interior of its closure.

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References


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