STABLE MAPS OF POLISH SPACES

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ABSTRACT. We define the notions of stable and transquotient maps and study the relation between these classes of maps. The class of stable maps contains all closed and open maps and their compositions. The transquotient maps preserve the property of being a Polish space, and every stable map between separable metric spaces is transquotient.

In particular, a composition of closed and open maps (the intermediary spaces may not be metric) preserves the property of being a Polish space. This generalizes the results of Sierpiński and Vainstein for open and closed maps.

In [5] the author introduced a concept of system \((\eta_y)_{y \in Y}\) associated to a map \(f : X \to Y\) with the help of which we give here new classes of stable and transquotient maps.

All the maps in this paper are continuous and onto. All the spaces are regular. Let \(f : X \to Y\) be a map from a separable, complete metrizable (= Polish) space \(X\) onto a metric space \(Y\). Then \(Y\) is also Polish in the following cases:

a) if \(f\) is an open or a closed map [7, 9];
b) if \(f\) is a compact-covering map or an \(s\)-covering map [2, 3, 5];
c) if \(f\) is a composition of open and perfect maps [3].

First, we will show that the class of stable maps contains the above-mentioned classes. Second, we obtain from Theorem 0 all the above-mentioned statements a)–c) and some new results. In particular, a composition of closed and open maps preserves the property of being a Polish space.

Notice, that in all the cases a)–c) there is a closed subset \(X_0 \subset X\) such that the restriction \(f|X_0\) is a perfect map onto \(Y\) (see [2, 3, 5]).

Theorem 0. Let \(f : X \to Y\) be a stable map from a Polish space \(X\) onto a metrizable space \(Y\). Then:

(1) There is a countable set \(Y_0 \subset Y\) and a Polish space \(X_0 \subset X\) such that the restriction \(f|X_0\) is a perfect map onto \(Y \setminus Y_0\).

(2) \(Y\) is a Polish space.

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1The system, which was studied in [5 Lemmas 1–3], was triquotient.

2A map \(f : X \to Y\) is compact-covering (resp. \(s\)-covering or countable-compact-covering) if every compact (resp. countable compact) \(K \subset Y\) is the image of some compact \(B \subset X\).
1. STABLE AND TRANQUOTIENT MAPS

Our approach will be to establish a characterization of stable, transquotient and harmonious maps in terms of transmittable systems and to give a relation between these classes of maps. We start first with some general notation:

A system associated to a map \( f : X \to Y \) is a family \( (\eta_y)_{y \in Y} \) where each \( \eta_y \) is a family of open subsets of \( X \) satisfying the condition \( U \cap f^{-1}(y) \neq \emptyset \) for all \( U \in \eta_y \).

The system is said to be transmittable if the following condition holds:

(a) for every \( U \in \eta_y \) there is a neighborhood \( O(y) \) of \( y \) such that \( U \in \eta_{y'} \) for every \( y' \in O(y) \).

It is clear that if \( U \in \eta_y \), then \( y \in \text{Int} f(U) \). A map may have many transmittable systems and a system is called trivial if \( \eta_y = \{X\} \) for every \( y \in Y \).

**Definition 1.** The map \( f : X \to Y \) is stable if it admits a transmittable system satisfying the following condition:

(b) If \( U \in \eta_y \) and \( V \) is open in \( X \) such that \( V \supset U \cap f^{-1}(y) \), then \( V \in \eta_y \).

**Definition 2.** A map \( f : X \to Y \) is transquotient if it admits the transmittable system satisfying the following condition:

(c) If \( U \in \eta_y \) and \( \gamma = \{U_\alpha\}_{\alpha \in A} \) is a family of open subsets of \( X \) such that \( \bigcup_{\alpha \in A} U_\alpha \supset f^{-1}(y) \cap U \), then there is a finite number \( U_{\alpha_1}, \ldots, U_{\alpha_n} \in \gamma \) and an open \( O(y) \) such that \( \bigcup_{i=1}^n U_{\alpha_i} \in \eta_{y'} \) for every \( y' \in O(y) \) 

**Remark 0.** Let \( \eta_y \) be a transquotient family for \( f \). Define a new family for \( f \):

\[ \tilde{\eta}_y = \{ V : V \text{ is open in } X \text{ and } 3U \in \eta_y \text{ such that } V \supset U \cap f^{-1}(y) \}. \]

In particular \( X \in \tilde{\eta}_y \). Notice that the transmittable property holds for \( \tilde{\eta}_y \) since for \( V \in \tilde{\eta}_y \) we can consider \( \gamma = \{V\} \) in (c) and then, obviously, \( V \in \eta_{y'} \) for every \( y' \in O(y) \). Since every cover of \( f^{-1}(y) \cap V \) is the cover of \( f^{-1}(y) \cap U \), we have by (c) that \( \tilde{\eta}_y \) is a transquotient family for \( f \).

If \( V \) is open in \( X \) and there is \( U \in \tilde{\eta}_y \) such that \( V \supset f^{-1}(y) \cap U \) (in particular, if \( V \supset f^{-1}(y) \)), then, obviously, \( V \) is also an element of \( \eta_y \). Hence, a transquotient map \( f \) is stable.

It should be noted that the following lemma uses an idea of A.H. Stone’s proof [8, Lemma 1].

**Lemma 1.** Let \( f : X \to Y \) be a map. If \( Y \) is first-countable and if every fiber \( f^{-1}(y) \) is hereditarily Lindelöf, then the following conditions are equivalent:

(i) \( f \) is stable;

(ii) \( f \) is transquotient.

**Proof.** By Remark 0 (ii) \( \Rightarrow \) (i). We will show (i) \( \Rightarrow \) (ii). Suppose the family \( \eta_y \) satisfies Definition 1 and does not satisfy Definition 2. This means that (c) fails for some \( U \in \eta_y \) and family \( \gamma = \{U_\alpha\}_{\alpha \in \omega} \). It is obvious that there is an open decreasing base \( \{O_\alpha(y)\}_{\alpha \in \omega} \) at \( y \) such that for every \( \gamma_n = \bigcup_{\alpha=0}^n U_\alpha \) there is a \( y' \in O_\alpha(y) \setminus O_{\alpha+1}(y) \), for which \( \gamma_n \notin \eta_{y'} \), and, hence, \( M_n = (U \cap f^{-1}(y'_n)) \setminus \gamma_n \neq \emptyset \) (we assume that \( O_0(y) \) is a neighbourhood of \( y \) such that \( U \in \eta_{y'} \) for each \( y' \in O_0(y) \)).

Let \( M = \bigcup \{M_n : n \in \omega \} \). Now let us show that \( f^{-1}(y) \cap U \cup \text{cl}_X M \neq \emptyset \).

Suppose the contrary, then \( U' = U \setminus \text{cl}_X M \supset f^{-1}(y) \cap U \) and \( U' \notin \eta_y \). By the transmittable property, for some \( n \) we have \( U' \in \eta_{y'_n} \). By our construction, \( \gamma_n \supset \gamma_n \cap U \cap f^{-1}(y'_n) = U' \cap f^{-1}(y'_n) \) and, hence, \( \gamma_n \in \eta_{y'_n} \). That is impossible.
There is a perfect extension $f^{-1}(y) \cap U \cap \text{cl}_X \cap M$; then $x$ belongs to some $\gamma_n$. It is clear that $x \in T = f^{-1}(y) \cap U \cap \text{cl}_X (\bigcup \{M_k : k > n\})$. Taking into consideration that $M_k \subset U \setminus \gamma_n$ for $k > n$, and hence $T \cap \gamma_n = \emptyset$, we get a contradiction with $x \in \gamma_n$. \hfill \Box

Let us consider some examples of stable and transquotient maps.

**Proposition A.** Open maps and closed maps $f : X \to Y$ are stable.

For an open map: $\eta_y = \{U : U$ is open and $y \in f(U)\}$.

For a closed map: $\eta_y = \{U \subset X : U$ is open in $X$, and $U \supset f^{-1}(y)\}$. It is a well known fact that for every $y \in Y$ and every open $U \supset f^{-1}(y)$ there is an open $O(y)$ such that $f^{-1}(O(y)) \subset U$.

Every open map with $\eta_y = \{U : U$ is open and $y \in f(U)\}$ is transquotient but the space $Y = R^n \text{mod } Z$, obtained by identifying the integers $Z$ in the real line $R$ to one point $y$, gives a closed and non–transquotient map $f : R \to Y$. Indeed, let $\gamma_n$ be a cover of $f^{-1}(y)$ by open intervals of length $< 1/2$. Then for every finite subcover $\gamma_2 \setminus \gamma_1$ and every open $O(y)$ there is $y' \in O(y) \setminus \{y\}$ such that $\bigcup \gamma_2 \cap f^{-1}(y') = \emptyset$.

**Proposition B.** Suppose $f : X \to Y$ and $\varphi : Y \to Z$ are stable maps. Then the composition $g = \varphi \circ f : X \to Z$ is also a stable map.

Let $\eta_y(f)$ and $\eta_z(\varphi)$ be stable families for $f$ and $\varphi$, respectively. Let us define $\eta_z(g) = \{U : \exists T \in \eta_z(\varphi) \forall y \in T \cap \varphi^{-1}(z) (U \in \eta_y(f))\}$ and let us show that $g$ is a stable map. It is clear that $X \in \eta_z(g)$ for every $z \in Z$.

(a) Let us assume $U \in \eta_z(g)$ and $y \in T \cap \varphi^{-1}(z)$. By (a) for $f$ there is an open $O(y) \subset T$ such that $U \in \eta_y(f)$ for each $y' \in O(y)$. Let $T' = \bigcup \{O(y) : y \in T \cap \varphi^{-1}(z)\}$. Obviously, $T' \supset T \cap \varphi^{-1}(z)$. Hence, $T' \in \eta_z(g)$.

We choose (according to (a) for $\varphi$) an open $O(z)$ such that $T' \in \eta_z(\varphi)$ for every $z' \in O(z)$, and then we have $U \in \eta_z(g)$. (b) Suppose $V \supset U \cap g^{-1}(z)$ for an open $V$, where $U \in \eta_z(g)$. Then $V \in \eta_y(f)$ for every $y \in T \cap \varphi^{-1}(z)$ and, hence, $V \in \eta_z(g)$.

**2. Proof of Theorem 0**

A map $f : X \to Y$ from a metric space $X$ is uniformly sequentially complete if each $f^{-1}(S)$ is complete with the given metric on $X$ for every convergent, including its limit, sequence $S = \{y_i \} \cup \bigcup \{y_i : y_i \to y\}$ in $Y$.

**Proposition C.** If $f : X \to Y$ is transquotient with appropriate families $\eta_y, Y_0 \subset Y$, $Z = \bigcup_{\alpha \in A} U_\alpha$ and $U_\alpha \in \eta_y$ for each $y \in f(U_\alpha) \setminus Y_0$, then $f((Z \setminus f^{-1}(Y_0)) : Z \setminus f^{-1}(Y_0) \to f(Z) \setminus Y_0$ is transquotient.

The new transquotient families are defined as $\tilde{\eta}_y = \{U \setminus f^{-1}(Y_0) \subset Z \setminus f^{-1}(Y_0) : U \in \eta_y\}$.

**Lemma 2.** Let $f : X \to Y$ be a transquotient, uniformly sequentially complete map between separable metric spaces $X, Y$. Then there exists a $G_{\delta}$–set $X' \subset X$ and a countable set $Y_0 \subset Y$ such that the restriction $f|X'$ is a perfect map onto $Y \setminus Y_0$.

**Proof.** There is a perfect extension $f^* : X^* \to Y$ of $f$ and $Z \subset X^*$ such that $Z \supset X$, every $f^{-1}(S)$ is closed in $Z$ for every convergent, including its limit sequence $S \subset Y$ and $Z = \bigcap_{i \in \Omega} O_i$, where $O_i$ are open in $X^*$.

Indeed, suppose $B_Y \supset Y$ and $B_X$ are compact spaces containing $Y$ and the completion $\hat{X}$ of $X$, respectively. Then $\hat{X}$ is a $G_{\delta}$–set in $B_X$ and the first coordinate
projection $\pi$ on $BY \times B_{\tilde{X}}$ is a perfect map. Let $X^* = \pi^{-1}(Y)$, $Z = Y \times \tilde{X}$ and $f^* = \pi|X^*$. Then $f^* : X^* \to Y$ is also perfect. Finally, we can identify the graph $G \subset X^*$ of the map $f$ with $X$ and the projection of $G$ with $f$; then $f^*$ is the extension of $f$.

Notice that $f^{*-1}(S)$ is always compact for a convergent sequence $S$ including its limit because $f^{*-1}(S) = S \times B_{\tilde{X}}$ and this is a product of two compact sets.

Note also that by the construction of the transmittable system constructed in Remark 0 ($\tilde{\eta}_0$), we can assume that there exists a transmittable system $\eta_0$ that makes $f$ both transquotient and stable.

Let us choose for each $y \in Y$ some $U_0^y \in \eta_0$, $U_0^y \subset X$ and for each point $x \in U_0^y \cap f^{-1}(y)$ a neighborhood $U(x) \subset U_0^y$ such that $cl_X . U(x) \subset O_0$. According to Definition 2 there are finitely many $U(x_i)$, $i = 1, 2, ..., n$, and a neighborhood $O(y)$ such that $U_i^y = \bigcup_{i=1}^n U(x_i) \in \eta_k$ for every $y \in O(y) \setminus \{y\}$.

Hence, $\delta_1 = \{O(y) : y \in Y\}$ is an open cover of $Y$ and for every $y \in Y$ we have $f(U_i^y) \supset O(y)$.

Since every separable metric space is a Lindelöf space and every Lindelöf space is paracompact, we can consider the countable, locally finite open refinement $\gamma_1 = \{U_k\}_{k \in \omega}$ of $\delta_1$. We can choose for every $U_k \in \gamma_1$ some $y_k$ such that $O(y_k) \supset U_k$.

Denote $\delta_1(\gamma_1) = \{O(y_k) : k \in \omega\}$ and $Y_0 = \{y_k : k \in \omega\}$.

Choose for every $U_k \in \gamma_1$ the corresponding elements $O(y_k) \in \delta_1(\gamma_1)$, $U_k \in \eta_k$ and denote $W_{1,k} = U_k \cap f^{*-1}(U_k) \subset X$. By our construction

\[(*) \quad W_{1,k} \in \eta_k \quad \text{for every point} \quad y \in U_k \setminus Y_0.\]

In fact, for every $y' \in O(y_k) \setminus \{y_k\}$ (see the definition $\tilde{\eta}_0$) we obtain $(*)$. Let us denote

$$X_0 = \bigcup_{k \in \omega} W_{1,k} \setminus f^{-1}(Y_0), \quad f_0 = f|X_0 : X_0 \to Y \setminus Y_0.$$  

According to $(*)$ and Proposition C $f_0$ is transquotient. Let us denote

$$X^*_0 = (cl_X \cdot \bigcup_{k \in \omega} W_{1,k}) \setminus f^{*-1}(Y_0), \quad f^*_0 = f^*|X_0^* : X_0^* \to Y \setminus Y_0.$$  

It is clear that $f^*_0$ is a perfect map. Since $\{W_{1,k}\}_{k \in \omega}$ is a locally finite family, we have: $X_0 = (\bigcup_{k \in \omega} cl_X \cdot W_{1,k}) \setminus f^{*-1}(Y_0)$ and, hence, $X_0 \subset X^*_0 \subset O_0$.

We may consider $O_1, X_0, X^*_0, f_0, f^*_0$ instead of $O_0, X, X^*, f, f^*$ and repeat the construction. In this way we obtain a countable subspace $Y_1 \subset (Y \setminus Y_0)$ and:

- a perfect map $f_1^* : X_1^* \to Y \setminus (Y_0 \cup Y_1)$, where $X_1^* \subset O_1$ and $X_1 \subset X_1^* \subset X^*_0$;
- a transquotient map $f_1|X_1$ onto $Y \setminus (Y_0 \cup Y_1)$, where $X_1 \subset X_0$.

Thus, we have the subspaces $O_i \supset X_i^* \supset X_{i+1}^*$, $i \in \omega$, where $X_i^* \subset X_i$ is closed in $X^* \setminus \bigcup_{k=0}^i f^{*-1}(Y_k)$ and every $f^*_i : X_i^* \to Y \setminus \bigcup_{k=0}^i Y_k$ is perfect. Denote $Y_\sigma = \bigcup_{i \in \omega} Y_i$. Evidently, $Y_\sigma$ is countable. It is obvious that $F = \bigcap_{i \in \omega} X_i^* \subset Z$ is closed in $X^* \setminus f^{*-1}(Y_\sigma)$ and, hence, $f^*|F$ is a perfect map onto $Y \setminus Y_\sigma$. Denote $X' = F \cap X$. We shall prove that $f' = f|X'$ is a perfect map onto $Y' = Y \setminus Y_\sigma$.

By inspection of the construction of the spaces $X_n^*$ one can see that they are of the form $X_n^* = T_n \setminus f^{*-1}(U_n)$, where the sets $T_n$ are closed in $X^*$, $T_0 \supseteq T_1 \supseteq \ldots$ and $T_n \subset O_n$. Let $T = \bigcap_{n} T_n$; then $T \subset Z$, because $T_n \subset O_n$. Let $S$ be a sequence contained in $Y'$ and assume that $S$ is convergent and contains its own limit. $f^{-1}(S)$ is closed in $Z$. Hence $f^{-1}(S \cap T)$ is closed in $T$, because $T \subset Z$. Hence $f'^{-1}(S) = f^{-1}(S) \cap F = f^{-1}(S) \cap T$ is closed in $f'^{-1}(S) \cap T$. But the set $f'^{-1}(S)$ is compact. Thus $f'^{-1}(S)$ is compact.
To prove that \( f'(X') = Y \setminus Y_\sigma \), note first that for every \( y \in Y \setminus Y_\sigma \) the set \( L_n = cl_X \cdot f^{-1}(y) \cap X_n^* \) is compact (\( n \in \omega \)). By our construction \( L_n \supset L_{n+1} \) and \( L_n \neq \emptyset \) since \( X_n^* \supset X_n \) and \( f^{-1}(y) \cap X_n \neq \emptyset \). Hence, there is \( x \in \bigcap_n L_n = cl_X \cdot f^{-1}(y) \cap F \). Since \( f^{-1}(y) \) is closed in \( Z \) and \( F \subset Z \), we have \( x \in f^{-1}(y) \cap F = f'^{-1}(y) \).

To verify that \( f'(A) \) is closed in \( Y' \) for every closed \( A \subset X \), note that if \( y_i = f'(x_i), x_i \in A, \) and \( y_i \rightarrow y \), then, as \( C = f'^{-1}(\{y_i : i \in \omega\} \cup \{y\}) \) is compact, there is a limit point \( x \) for the sequence \( \{x_i : i \in \omega\} \) of the points of \( C \). But then \( x \in A \) because \( x_i \) are in \( A \) and \( A \) is closed. Obviously, \( f'(x) = y \in f'(A) \).

Since \( f|X' \) is a perfect map onto \( Y \setminus Y_\sigma \), \( X' \) is a closed subset of \( X \setminus f^{-1}(Y_\sigma) \), hence \( X' \) is a \( G_\delta \)-set in \( X \).

Let \( X \) be a topological space. The strong Choquet game \( C \) is played as follows.

Players \( \alpha, \beta \) take turns, with \( \beta \) playing first, choosing nonempty open subsets of \( X \). Player \( \beta \) also plays a point in his open set at each move and \( \alpha \) must play an open set containing this point. Each open set played by any player is contained in the opponent's previous move

\[
\begin{array}{cccc}
\beta & U_0, x_0 & U_1, x_1 & \ldots \\
\alpha & V_0 & U_1 & \ldots \\
\end{array}
\]

where \( x_i \in U_i, x_i \in V_i \) and \( U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \ldots \). We say that \( \alpha \) wins a run of this game \( U_0, V_0, V_1, \ldots \) if \( \bigcap_n V_n = \bigcup U_n \neq \emptyset \). One can require without losing generality that at every step the first player plays a different point. We say that \( X \) is a strong Choquet space if \( \alpha \) has a winning strategy in the strong Choquet game \( C \) for \( X[1] \).

**Lemma 3.** Let \( f : X \rightarrow Y \) be a transquotient map of a strong Choquet space \( X \) onto \( Y \). Then \( Y \) is also a strong Choquet space.

**Proof.** Denote by \( \Sigma \) a winning strategy for \( \alpha \) in \( C \). Denote by \( C' \) the strong Choquet game on \( Y \) and by \( \alpha', \beta' \) its players. We will describe a winning strategy for \( \alpha' \) in \( C' \).

Step 0. Let \( \beta' \) start with \( (U'_0, y_0) \). For every \( x_0 \in f^{-1}(y_0) \) take an open \( U_0 \subset f^{-1}(U'_0), x_0 \in U_0 \). Let \( \beta \) play \( (U_0, x_0) \), then \( \alpha \) answers using \( \Sigma \), by \( f_0((x_0, U_0)) = V_0 \subseteq U_0, x_0 \in V_0 \).

By Remark 0 to Definition 2 we may suppose \( X \in \eta_{y_0} \). Since the elements \( V_0 \) cover \( X \cap f^{-1}(y_0) \), there is a finite family \( \gamma_0 \) of elements \( V_0 \) and an open \( V'_0 \subseteq V_0 \), \( y_0 \in V'_0 \) such that \( \bigcup \gamma_0 \in \eta_{y'} \) for every \( y' \in V'_0 \setminus \{y_0\} \). Then \( \alpha' \) answers by \( f_0((y_0, U'_0)) = V'_0 \subset U'_0, y_0 \in V'_0 \).

Step 1. Let \( \beta' \) play \( (U'_1, y_1) \), where \( y_1 \in U'_1 \subseteq V'_0 \).

Then \( \beta \) plays \( (U_1, x_1) \), where \( U_1 \subset V_0 \cap f^{-1}(U'_1) \) for every \( y_0 \in \gamma_0 \) and every \( x_1 \in V_0 \cap f^{-1}(y_1) \). Then \( \alpha \), using \( \Sigma \), answers by \( f_1((x_0, U_0), (x_1, U_1)) = V_1 \subset U_1, x_1 \in V_1 \).

By the definition of transquotient maps, there is a finite family \( \gamma_1 \) of elements \( V_1 \) and an open \( V'_1 \subseteq U'_1, y_1 \in V'_1 \) such that \( \bigcup \gamma_1 \in \eta_{y'} \) for every \( y' \in V'_1 \setminus \{y_1\} \). Then \( \alpha' \) answers by \( f'_1((y_0, U'_0), (y_1, U'_1)) = V'_1 \subseteq U'_1, y_1 \in V'_1 \). By construction \( U'_1 \supset f(\bigcup \gamma_1) \supset V'_1 \).

Step \( n + 1 \). Suppose we defined a map \( f'_n((y_0, U'_0), \ldots, (y_n, U'_n)) = V'_n \subset U'_n, y_n \in V'_n \) for \( (y_0, U'_0), \ldots, (y_n, U'_n) \) and finite families \( \gamma_n \) of open sets \( V_n \) such that \( \bigcup \gamma_n \in \eta_{y'} \) for every \( y' \in V'_n \) and \( U'_n \supset f(\bigcup \gamma_n) \supset V'_n \). Let \( \beta' \) play \( (U'_{n+1}, y_{n+1}) \).
Then $\beta$ plays $(U_{n+1}, x_{n+1})$ for every $V_n \in \gamma_n$ and every $x_{n+1} \in V_n \cap f^{-1}(y_{n+1})$ with $U_{n+1} \subseteq V_n \cap f^{-1}(U'_{n+1})$ and $\alpha$ answers, using $\Sigma$, $f_{n+1}(U_0, \ldots, (x_{n+1}, U_{n+1})) = V_{n+1} \subseteq U_{n+1}$, $x_{n+1} \in V_{n+1}$.

By the definition of transquotient maps there is a finite family $\gamma_{n+1}$ of elements $V_{n+1}$ and an open $V'_{n+1} \subseteq U'_{n+1}$, $y_{n+1} \in V'_{n+1}$ such that $\bigcup \gamma_{n+1} \in \eta_{y'}$ for every $y' \in V'_{n+1} \setminus \{y_{n+1}\}$. Then $\alpha'$ answers $f'_{n+1}(y_0, U'_0, \ldots, (y_{n+1}, U'_{n+1})) = V'_{n+1} \subseteq U'_{n+1}$, $y_{n+1} \in V'_{n+1}$.

By our construction
\[(*) U'_{n+1} \supseteq f(\bigcup \gamma_{n+1}) \supset V'_{n+1}.
\]

By König’s theorem an infinite, finitely branching tree must have an infinite branch and we obtain in this way a sequence $V_0 \supset V_1 \supset \ldots \supset V_i \supset \ldots$, obtained according to a winning strategy for $\alpha$, where $V_i \in \gamma_i$. Since $\alpha$ wins, there is $x \in \bigcap_n V_n$ and, hence, $x \in \bigcup \gamma_n$ for every $n \in \omega$. Since by definition of strong Choquet game $V_n \supset U'_{n+1}$, we have according to $(*)$: $f(x) \in \bigcap_n V'_n$ and $\alpha'$ wins.

**Proof of Theorem 0.** Lemma 2 implies part (1) of Theorem 0. Notice that $X$ is completely metrizable if and only if $X$ is a metrizable strong Choquet space [11] and we obtain from Lemmas 1 and 3 part (2) of Theorem 0 (see also [3] Corollary 7.3).

### 3. Remarks

We begin by recalling the following definition [3]:

**Definition 3.** A map $f: X \to Y$ is harmonious if one can assign to every compact $K \subseteq Y$ a family $\eta_K$ of open subsets of $X$ satisfying the following conditions:

(a') If $U \in \eta_K$, then there exists a compact $B \subseteq U$ such that $f(B) = K$ and for every open $V \supset B$ we have $V \in \eta_K$.

(b') If $U \in \eta_K$, then there exists an open $O(K)$ such that $U \in \eta_{K'}$ for every compact $K' \subset O(K)$.

We obtain the definition of a point–harmonious map by replacing $K$ and $K'$ in the above definition by points $y$ and $y'$.

By Theorems 1 and 2 from [3] in case of separable metrizable spaces $X, Y$ the class of harmonious (resp. point–harmonious) maps coincides with the class of compact–covering (resp. $s$–covering) maps.

Recall that triquotient maps may be defined as transquotient maps satisfying condition (c) from Definition 2 for every $y' \in O(y)$.

**Proposition D.** Transquotient maps, triquotient maps, point–harmonious maps and harmonious maps are stable.

Let $f: X \to Y$ be a map. Then
1) $f$ is harmonious.
2) $f$ is point–harmonious.
3) $f$ is triquotient.
4) $f$ is transquotent.
5) $f$ is stable.

1) $\Rightarrow$ 2) $\Rightarrow$ 3) follow from the definition of harmonious and point–harmonious maps.
3) $\Rightarrow$ 4) $\Rightarrow$ 5) follow from the definition of transquotient maps and Remark 0.
**Proposition E.** Compact-covering maps and $s$-covering maps onto a metrizable space are stable.

The corresponding stable family $\eta_y$ is constructed in Example 1.

**Proposition F.** If $f : X \to Y$ is a stable map with compact fibres, then $f$ is point-harmonious.

Let $\eta_y$ be a stable family for $f$ and let $\varepsilon_y = \{ B = \text{cl}_X U \cap f^{-1}(y) : U \in \eta_y \}$. Define a transmittable family for $f$ as follows:

$$\eta'_y = \{ U' : U' \text{ is open in } X \text{ and there is } B \in \varepsilon_y \text{ such that } U' \supset B \}. $$

It is clear that the condition $(a')$ is satisfied. Let us consider $U' \in \eta'_y$ and some compact $B \in \varepsilon_y, B \subset U'$. Since $X$ is regular, there is an open set $V \supset B$ such that $\text{cl}_X V \subset U'$. Since $B \supset U \cap f^{-1}(y)$ where $U \in \eta_y$, there is an open $O(y)$ such that $V \in \eta'_y$ for every $y' \in O(y)$. Since $U' \supset B' = \text{cl}_X V \cap f^{-1}(y')$ and $B' \in \varepsilon_y$, we have $U' \in \eta'_y$ for every $y' \in O(y)$ and $(b')$ is also satisfied.

A map $f : X \to Y$ is feebly open if for every (non-empty) open $V \subset X$ we have $\text{Int}(V) \not= \emptyset$.

Recall that a space $Y$ is an $F_{\infty}$-space if there is no closed subset of $Y$ which is of the first category in itself. The following theorem shows that part (2) of Theorem 0 can be extended onto a larger class of $F_{\infty}$-spaces.

**Theorem 1.** Let $f : X \to Y$ be a transquotient map. If $X$ is an $F_{\infty}$-space, then $Y$ is also an $F_{\infty}$-space.

We need the following lemma.

**Lemma 4.** Let $f : X \to Y$ be a transquotient map. Then there is a closed $Z \subset X$ such that the restriction $f|Z$ is a feebly open, transquotient map onto $Y$.

**Proof.** Suppose $f$ is not feebly open; then there is a non-empty open $V \subset X$ such that $\text{Int}(V) = \emptyset$. Denote $X_1 = X \setminus V, f_1 = f|X_1, \eta^0_y = \{ U^1 = U \cap X_1 : U \in \eta_y \}$.

First, $U^1 \cap f_1^{-1}(y) \not= \emptyset$. Suppose not; then for some $U_2 \in \eta_y$, $V \supset U \cap f^{-1}(y)$. According to Remark 0, we may suppose that $f$ is not only transquotient but also stable with respect to $\eta_y$, hence there is an open $O(y)$ such that $f(V) \supset O(y)$, which is impossible.

It is clear that the transmittable property (a) for $f_1$ is satisfied.

Let $U^1 \in \eta^1_y$ and $\gamma^1 = \{ U^1 \} \cup_U U^1$ be an open cover of $f_1^{-1}(y) \cap U^1$. Let us take an open $U \in \eta_y$ such that $U \cap X_1 = U^1$. Obviously, $\gamma = \{ \gamma^1 = \bigcup_{\alpha} \gamma^1 \} \cup (X \setminus X_1) \cup f^{-1}(y)$ is an open cover of $U \cap f^{-1}(y)$. Hence, there exist $\gamma^1, \gamma^2, \ldots, \gamma^n, O(y) \subset Y$ such that

$$\gamma = \bigcup_{\alpha} \gamma^1 \cup \gamma^2 \cup \ldots \cup \gamma^n \cup O(y) \subset Y$$

for every $y' \in O(y) \setminus \{ y \}$. It is clear that $\gamma \cap X_1 = \bigcup_{\alpha} U^1 \cup U \cup f^{-1}(y)$ and, hence, $(c)$ is also satisfied and $f_1$ is transquotient.

Suppose we obtained for some $\alpha$ a strictly decreasing sequence of closed subsets

$$X_\beta \subset X \ (\beta < \alpha)$$

and transquotient maps $f_\beta = f|X_\beta$ onto $Y$ with hereditary transquotient family $\eta^0_\beta = \{ U_\beta = U \cap X_\beta : U \in \eta_y \}$.

If $\alpha$ is not a limit ordinal, that is, if $\alpha = \beta + 1$, and $f|X_\beta$ is not feebly open, we may consider $X_\alpha, X_{\beta}$ instead of $X_1, X$ and repeat the construction above.

For the limit ordinal $\alpha$ define: $X_\alpha = \bigcap_{\beta < \alpha} X_\beta, f_\alpha = f|X_\alpha, \eta^0_\alpha = \{ U_\alpha = U \cap X_\alpha : U \in \eta_y \}$ and prove that for every $y \in Y$ and every $U^\alpha \in \eta_y$ we have $U^\alpha \cap f_\alpha^{-1}(y) \not= \emptyset$.

Suppose the contrary. Then for some $U \in \eta_y$ and $U^\alpha = U \cap X_\alpha$, we have $U \cap X_\alpha \cap f^{-1}(y) = \emptyset$. It is clear that $\{ O_\beta = X \setminus X_\beta \} \beta < \alpha$ is an open cover of
$f^{-1}(y) \cap U$. Since $X_\beta \supset X_{\beta+1}$ and $f$ is transquotient, there is $\beta < \alpha$ and $y' \in O(y)$ such that $O_\beta \in \eta_{y'}$ and, hence, $O_\beta \cap X_\beta \in \eta_{y'}$. This contradicts the condition $\emptyset \notin \eta_{y'}$ because $O_\beta \cap X_\beta = \emptyset$.

Just like for $X_1$ we can prove that $f|X_\alpha$ is a transquotient map with hereditary transquotient family. We may repeat the above construction if $f|X_\alpha$ is not feebly open, etc. Since $\{X_\alpha\}$ is a strictly decreasing sequence of closed sets, there is $\alpha_0$ such that $Z = X_{\alpha_0} = X_{\alpha_0+1} = \ldots$. Let us denote $f_{\alpha_0} = f|Z$. Then $f_{\alpha_0}$ is a feebly open, transquotient map onto $Y$.

To prove Theorem 1 suppose the contrary and let $Y_0 \subset Y$ be a closed subspace of the first category and, hence, $Y_0$ can be represented as a countable union of nowhere dense and closed in $Y$ subsets $F_i$. It is not hard to prove (see Proposition C) that $f_0 = f|X_0$, where $X_0 = f^{-1}(Y_0)$, is a transquotient map. By Lemma 4 we can suppose that $f_0$ is feebly open and, hence, every $f^{-1}(F_i)$ is a closed nowhere dense subset. Then $X_0$ is a subspace of the first category in itself, which is impossible by definition of $F_{\Pi}$-space.

Let $X$ be a (non–separable) complete metric space, and let $Y$ be a metric space. I don’t know whether the map $f$ in the following cases 1 and 2 must be transquotient (or $Y$ completely metrizable):

1. $f: X \to Y$ is a countable–inductively perfect map (i.e., for every countable closed $Y_0 \subset Y$ there is $X_0 \subset X$ such that $f|X_0$ is a perfect map onto $Y_0$).

2. $f: X \to Y$ is a composition of closed and open maps (the domain and the range of all maps are regular).

4. Examples

The assumption that $X$ is separable cannot be omitted in Theorem 0:

**Example 1.** A stable, non-transquotient map $f: X \to Y$ from a completely metrizable space $X$ onto metric space $Y$ which does not satisfy conditions (1) and (2) of Theorem 0.

In fact, let $Y$ not be a union of a countable set and an absolutely $G_\delta$–set. For example, $Y = \mathbb{Q}^2$ (where $\mathbb{Q}$ is the space of rational numbers) and let $X$ be a topological sum of all compact subsets of $Y$ with the obvious map $f: X \to Y$. It is clear that $X$ is completely metrizable and, by Lemma 3, $f$ is not transquotient. Since a perfect image of a completely metrizable space is completely metrizable, $f$ does not satisfy the conditions of Theorem 0.

We’ll prove that $f$ is stable. In fact, let us define:

$\eta_y = \{U: U$ is open in $X$ and for every compact $K \subset Y$ containing $y$, there is a compact $B \subset X$ such that $f^{-1}(y) \cap B \subset U$ and $f(B) = K\}$.

Obviously, the condition (b) of Definition 1 is fulfilled. We will verify the transmittable property (a). Let us assume the contrary: for some $U \in \eta_y$ there is a base $\{O_i\}$ at the point $y$, points $y_i \in O_i$ and compacts $K_i \subset O_i$, which contain $y_i$ such that there is no compact subset $B_i$ of $X$ satisfying the conditions: $f^{-1}(y_i) \cap B_i \subset U$ and $f(B_i) = K_i$. It is clear that $S = \{y\} \cup \{K_i : i \in \omega\}$ is a compact set and $y \in S$. By definition of $\eta_y$ there is a compact $B \subset X$ for which $f(B) = S$ and $f^{-1}(y) \cap B \subset U$. Then compact $B \setminus U$ contains a closed countable discrete family of non-empty sets $(f^{-1}(y_i) \cap B) \setminus U$. This is impossible.
Example 2. A transquotient (and stable) map \( f : X \to Y \) from a countable, Polish space \( X \) onto a compact, metric space \( Y \) and the following conditions are satisfied:

1) \( f : X \to Y \) is not \( s \)-covering (= not point–harmonious) map.

2) \( f = f_2 \circ f_1 \), where \( f_1 \) is a closed map and \( f_2 \) is an open map\(^3\).

Let \( p_0 < p_1 < \ldots < p_n < \ldots \) be a sequence of prime numbers. Denote

\[
Y_n = \{1/p_n^{i+1} : i \in \omega\}, \quad Y'_n = \{0\} \cup Y_n \subset \mathbb{I} = [0, 1].
\]

For the prime numbers \( p_n, p_k \) we have \( p_n^{i+1} = p_k^{i+1} \) iff \( p_k = p_n \). Hence \( Y'_n \cap Y'_k = \{0\} \) (\( n \neq k \)). Denote also:

\[
X_1 = Y'_n \times \{n + 1\}, \quad X_2 = Y_n \times \{1/(n + 1)\},
\]

\[
X = \{0, 0\} \cup \bigcup \{X_1 \cup X_2 : n \in \omega\} \subset \mathbb{I} \times \mathbb{R}, \quad Y = \bigcup \{Y'_n : n \in \omega\}.
\]

Let \( \pi : \mathbb{I} \times \mathbb{R} \to Y \) be the projection onto the first coordinate and \( f = \pi|X \).

1. It is easy to see that \( f \) is stable (define \( \eta_n \) as a family of all open in \( X \) subsets \( U \) such that \( y \in f(U) \), if \( y \neq 0 \) and \( U \supset f^{-1}(0) \), if \( y = 0 \)).

2. If \( f \) is an \( s \)-covering map, then there is a compact \( X_0 \subset X \) for which \( f(X_0) = Y \) and \( X_0 \cap X_1 \neq \emptyset \) (since \( X_1 \) are open and closed in \( X \)) for a finite number of \( n \) only. Then there is an \( n \) for which the set \( X_2 \cap X_0 \) has cardinality \( \aleph_0 \). Since \( X_2 \) is closed in \( X \), \( X_2 \cap X_0 \) has an accumulation point in \( X_2 \). This is impossible.

3. Define \( f_1 \) by factorising \( f^{-1}(0) \) to one point \( z_0 \). Then \( f_1 \) is a closed map. The map \( f_2 \) from \( Z \) onto \( Y \) is defined as follows: \( f_2(z) = 0 \) if \( z = z_0 \), and \( f_2(z) = f(f^{-1}(z)) \) if \( z \neq z_0 \). Then \( f_2 \) is an open map and \( f = f_2 \circ f_1 \).

The product of two stable (even two transquotient) maps cannot be stable. Indeed, let us consider the product \( t = f \times id : X \times I \to Y \times I \) of the stable, transquotient map \( f \) from Example 2 and identity map \( id \). If \( t \) is stable, then by Theorem 0 there is a countable set \( Y_\sigma \subset Y \times I \) and \( X_0 \subset X \times I \) such that \( t|X_0 \) is a perfect map onto \( (Y \times I) \setminus Y_\sigma \). Hence, there is \( x_0 \in I \) such that \( (Y \times \{x_0\}) \cap Y_\sigma = \emptyset \) and \( t|t^{-1}(Y \times \{x_0\}) = f : X \to Y \) is \( s \)-covering. This is impossible.

REFERENCES


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\(^3\)2) was remarked by V. Popov. Note that \( f \) is not a triquotient map.