EXOTIC SMOOTH STRUCTURES ON $3\mathbb{CP}^2 \# n\mathbb{CP}^2$, PART II

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(Communicated by Ronald A. Fintushel)

Abstract. We construct exotic $3\mathbb{CP}^2 \# 11\mathbb{CP}^2$ and $3\mathbb{CP}^2 \# 13\mathbb{CP}^2$ using the surgery techniques of R. Fintushel and R.J. Stern. We show that these 4-manifolds are irreducible by computing their Seiberg-Witten invariants.

1. Introduction

This paper is a sequel to [P]. For some history and general remarks on distinguishing smooth structures on $3\mathbb{CP}^2 \# n\mathbb{CP}^2$, we refer to Section 1 of [P]. Our main result is the following

Theorem 1.1. There exists a smooth closed simply-connected irreducible symplectic 4-manifold $X_n$ that is homeomorphic but not diffeomorphic to $3\mathbb{CP}^2 \# n\mathbb{CP}^2$ for each integer $10 \leq n \leq 13$.

The even cases $n = 10, 12$ were dealt with in [P]. In this paper we prove the remaining $n = 11, 13$ cases and complete our picture. Together these constructions provide the “smallest” known examples of an exotic closed simply-connected oriented 4-manifold with $b_2^+ > 1$. It remains an open problem whether there is an exotic $3\mathbb{CP}^2 \# n\mathbb{CP}^2$ with $n < 10$.

2. Construction of $X_n$

We start out by recalling the following

Proposition 2.1 (See [T]). Let $X$ be a closed symplectic 4-manifold and suppose that $b_2^+(X) > 1$. Then $X$ does not admit any Riemannian metric of positive scalar curvature.

Our main building block will be a homotopy rational elliptic surface of Fintushel and Stern in [FS2]. First, let us recall some basic properties of the rational elliptic surface $E(1) = \mathbb{CP}^2 \# 9\mathbb{CP}^2$. Let $B_{2,1} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ denote the union of four vertical and two horizontal spheres in the direct product $\mathbb{CP}^1 \times \mathbb{CP}^1$. More precisely, we choose six distinct points $p_1, \ldots, p_4, q_1, q_2$ in $\mathbb{CP}^1$ and define the nodal curve

$$B_{2,1} := \bigcup_{i=1}^4 (p_i \times \mathbb{CP}^1) \cup \bigcup_{j=1}^2 (\mathbb{CP}^1 \times \{q_j\}) \subset \mathbb{CP}^1 \times \mathbb{CP}^1.$$
Let $D_{2,1}$ be the double branched cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along $B_{2,1}$. Then $E(1)$ is the desingularization of $D_{2,1}$

$$p : E(1) \to D_{2,1} \to \mathbb{CP}^1 \times \mathbb{CP}^1.$$ 

Let $pr_1 : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ denote the projection onto the first factor and define $\pi = pr_1 \circ p$. Then $\pi : E(1) \to \mathbb{CP}^1$ is a fibration with generic fiber $\mathbb{CP}^1$ since the generic fiber of $pr_1$ meets the branch locus $B_{2,1}$ at two points. On the other hand, note that the composition $pr_2 \circ p$ is the standard elliptic fibration, where $pr_2$ is the projection to the second factor. We will denote the generic torus fiber of $pr_2 \circ p$ by $F$. For more details on these two fibrations, we refer to [FS] (§7.3).

Next we perform a knot surgery on $E(1)$ as in [FS2]. Let $K$ be the trefoil knot in $S^3$ and $m$ a meridional circle to $K$. Perform the 0-framed surgery on $K$ and call the resulting 3-manifold $S^3(K)$. In $S^3(K) \times S^1$ we have the smoothly embedded torus $T = m \times S^1$ of self-intersection 0. Let $E(1)_K$ denote the fiber sum

$$E(1)\#_{F \sim T}(S^3(K) \times S^1) = \left[ E(1) - (F \times D^2) \right] \cup_{\psi} \left[ (S^3(K) \times S^1) - (T \times D^2) \right],$$

where the two pieces are glued together so as to preserve the homology class $\alpha = \{ pt \} \times \partial D^2$. Fintushel and Stern showed that $E(1)_K$ is in fact a symplectic 4-manifold homotopy equivalent to $E(1)$.

Now let $S$ denote the generic rational fiber of $\pi$. Note that $[S] : [F] = 2$, i.e. $S$ geometrically intersects the elliptic fiber $F$ at two points. Recall that the Seifert surface $\Sigma_0$ of the trefoil knot $K$ is a punctured torus. Since the gluing map $\psi$ sends the homology class $\alpha$ into the homology class $[K \times \{ pt \}] = [\partial \Sigma_0 \times \{ pt \}]$ in $(S^3(K) \times S^1) - (T \times D^2)$, we can glue together two copies of $\Sigma_0$ and the twice-punctured rational fiber $S - (F \times D^2)$, and get a smooth genus 2 surface $\Sigma_2$ in $E(1)_K$. In this way $\pi$ induces a genus 2 fibration $E(1)_K \to \mathbb{CP}^1$ with generic fiber $\Sigma_2$, which we continue to denote by $\pi$. Note that the fibration $\pi : E(1)_K \to \mathbb{CP}^1$ has no multiple fibers. After a small perturbation we can assume that $\pi$ is a Lefschetz fibration (cf. [GS], Chapter 8). We can further assume that the vanishing cycle on the fiber $\Sigma_2$ is homologically nontrivial.

There was nothing special about the trefoil knot in the above construction and we can easily generalize our result to the following

**Lemma 2.2** (See [F]). Suppose $L$ is a fibered knot in $S^3$ of genus $g(L)$. Then there is a genus $2g(L)$ Lefschetz fibration $\pi : E(1)_L \to \mathbb{CP}^1$. 

Our second building block $W$ will be the fiber sum of two copies of the Kodaira-Thurston manifold. This is the manifold $Q_2$ in [G] (p. 570). Recall that $W$ fibers over a genus 2 surface and the fibration admits a section $\Sigma$ of self-intersection 0. Note that $W$ has a symplectic structure and $\Sigma$ is a symplectic submanifold. Also recall from [Sz] (Lemma 2.1, p. 413) that there is an element $g \in \pi_1(\Sigma)$ such that the monodromy relation $g^{-1}a_1g = ab$ holds inside $\pi_1(W)$. Here, $(a,b)$ denotes a set of generators for the fundamental group of the fiber in $W$.

Now we have all the ingredients needed for our construction. Our manifold $X_{11}$ will be the symplectic sum of $W$ and $E(1)_K$ along $\Sigma$ and $\Sigma_2$. We identify the tubular neighborhoods $N_1$ of $\Sigma$ and $N_2$ of $\Sigma_2$ via a diffeomorphism $\varphi : N_1 - \Sigma \to N_2 - \Sigma_2$, which preserves the orientations on the normal disks. We choose the gluing map $\varphi$ in such a way that $\varphi$ maps the generator $g$ to the vanishing cycle on the fiber $\Sigma_2$.
in $E(1)K$. Let

$$X_{11} = W \# \varphi E(1)_K = (W - \Sigma) \cup \varphi (E(1)_K - \Sigma_2),$$

where we use $\varphi$ to identify $N_1 - \Sigma$ and $N_2 - \Sigma_2$. For more on the details of the symplectic sum operation, we refer to [G].

**Lemma 2.3.** (i) $X_{11}$ is a smooth, closed, oriented, symplectic 4-manifold.
(ii) $X_{11}$ is simply-connected.
(iii) $X_{11}$ is homeomorphic to $3\mathbb{CP}^2\# 11\mathbb{CP}^2$.
(iv) $X_{11}$ is not diffeomorphic to $3\mathbb{CP}^2\# 11\mathbb{CP}^2$.

**Proof.** Part (i) is immediate. Since $E(1)K$ is simply-connected, an easy application of van Kampen's theorem says that $\pi_1(E(1)_K - \Sigma_2)/\langle \mu \rangle = 1$, where $\mu$ denotes the meridian of $\Sigma_2$. Now note that $\mu = [a, b]$ in $\pi_1(W \# \varphi E(1)_K)$. But in the group $\pi_1(X_{11})$, we have $g = 1$ which implies that $b = 1$, which in turn implies that $\mu = 1$. We conclude that the homomorphism, $\pi_1(E(1)_K - \Sigma_2) \to \pi_1(X_{11})$, induced by the inclusion map is the zero homomorphism.

It can be shown (cf. [G], p. 571) that $\pi_1(W - \Sigma)/\langle \pi_1(\Sigma) \rangle = 1$, where $\Sigma$ is a parallel copy of $\Sigma$ in $(W - \Sigma)$. Since $\Sigma$ gets identified with $\Sigma_2$ in $E(1)_K$ and the composition of inclusions, $\Sigma \hookrightarrow (E(1)_K - \Sigma_2) \hookrightarrow X_{11}$, induces the zero map on the fundamental groups, an easy application of van Kampen’s theorem gives (ii).

Various topological invariants behave nicely under the symplectic sum operation (cf. [G], p. 355):

$$\text{sign}(X_{11}) = \text{sign}(W) + \text{sign}(E(1)_K) = \text{sign}(E(1)_K) = -8,$$

$$e(X_{11}) = e(W) + e(E(1)_K) - 2e(\Sigma) = e(E(1)_K) + 4 = 16.$$

It follows that $b_7^+(X_{11}) = 3$ and $b_8^-(X_{11}) = 11$. Hence $X_{11}$ has an odd intersection form $3(1) \oplus 11(-1)$. Since $X_{11}$ is smooth, (iii) follows from Freedman’s famous classification theorem (cf. [FQ]). Part (iv) follows from Proposition 2.4, since $3\mathbb{CP}^2\# 11\mathbb{CP}^2$ admits a metric of positive scalar curvature (see e.g. [Sa]).

Now we proceed with the construction of $X_{13}$. This time around, we keep the homotopy rational elliptic surface summand $E(1)_K$ in the construction of $X_{11}$, but replace the $W$ summand with $T^4\# 2\mathbb{CP}^2$. Choose four distinct points $z_i \in S^1$, $i = 1, \ldots, 4$. Let us define smoothly embedded 2-tori in the 4-torus $T_{12} \subset T^4$, $T_{34} \subset T^4$ by

$$T_{12} = S^1 \times S^1 \times \{z_1\} \times \{z_2\}, \quad T_{34} = \{z_3\} \times \{z_4\} \times S^1 \times S^1.$$

We choose a symplectic form on $T^4$ for which $T_{12}$, $T_{34}$ are symplectic submanifolds. Note that $[T_{12}] \cdot [T_{34}] = 1$, and $[T_{12}]^2 = [T_{34}]^2 = 0$. By symplectically resolving the intersection of $T_{12}$ and $T_{34}$, and then blowing up twice to reduce the self-intersection, we obtain a symplectic genus 2 surface $\tilde{\Sigma} \hookrightarrow T^4\# 2\mathbb{CP}^2$ with self-intersection 0. We denote $T^4\# 2\mathbb{CP}^2$ by $V$. Note that $\pi_1(V) \cong \pi_1(T^4) \cong \mathbb{Z}^4$, $e(V) = 2$, and $\text{sign}(V) = -2$.

As before, we symplectically sum $V$ and $E(1)_K$ along $\tilde{\Sigma}$ and $\Sigma_2$, and define

$$X_{13} = V \# \varphi E(1)_K = (V - \tilde{\Sigma}) \cup \varphi (E(1)_K - \Sigma_2),$$

where $\varphi$ is some suitably chosen diffeomorphism as before.
Lemma 2.4.  
(i) \(X_{13}\) is a smooth, closed, oriented, symplectic 4-manifold.
(ii) \(X_{13}\) is simply-connected.
(iii) \(X_{13}\) is homeomorphic to \(3\text{CP}^2 \# 13\text{CP}^2\).
(iv) \(X_{13}\) is not diffeomorphic to \(3\text{CP}^2 \# 13\text{CP}^2\).

Proof. Part (i) is immediate. The proofs of the other parts mirror the proofs of the corresponding statements for \(X_{11}\). For (ii), we note that if \(\tilde{\Sigma}\) is a parallel copy of \(\Sigma\) in \((V - \tilde{\Sigma})\), then the inclusion induces a surjection \(\pi_1(\tilde{\Sigma}) \rightarrow \pi_1(V - \tilde{\Sigma})\). In particular, the meridian of \(\Sigma\) is killed by an embedded disk coming from the exceptional divisor of a blow-up, and hence we can proceed as before. For (iii), we calculate that \(e(X_{13}) = 18, \text{sign}(X_{13}) = 10\), and then invoke Freedman’s theorem. Since \(3\text{CP}^2 \# 13\text{CP}^2\) admits a metric of positive scalar curvature, Proposition 2.1 implies (iv).

We have thus proved Theorem 1.1 for the odd cases \(n = 11, 13\) except for the irreducibility condition which will be proved in the next section.

3. Irreducibility and SW-invariants

Recall that a smooth closed simply-connected 4-manifold \(X\) is called irreducible if each connected sum decomposition of \(X\) as \(X = Y \# Z\) satisfies that either \(Y\) or \(Z\) is a homotopy \(S^4\). Irreducibility of \(X_n\) will follow easily from the computation of SW-invariants of \(X_n\) using the product formula of [MST]. In order to use the product formula we must first compute the SW-invariants of each summand. We keep the conventions in Section 3 of [P] regarding the choice involved in the definition of SW-invariant for the \(b_+=1\) case. A lot of times, we abuse notation and use the same capital letter to denote a surface, its homology class, or the Poincaré dual of its homology class.

Lemma 3.1. Let \(T\) denote the Poincaré dual of the homology class of the torus fiber in \(E(1)\). Then the only SW-basic classes of \(E(1)_K\) are \(\pm T\).

Proof. We refer the reader to the last section of [FS2], where a more general statement is proved for the case when \(K\) is an arbitrary twist knot. Note that the trefoil is \((-1)\)-twist knot.

Lemma 3.2. Let \(F^*\) denote the Poincaré dual of the homology class of the fiber in \(W\). Then the only possible SW-basic classes of \(W\) are \(0, \pm 2F^*\).

Proof. This is an easy application of the generalized adjunction inequality for the SW-basic classes (cf. [OS]), using the fact that \(H_2(W; Z) \cong Z^6\) (cf. [S2], p. 413) has generators consisting of the section \(\Sigma\) and five tori (one of which is the fiber \(F\)).

Lemma 3.3. The only SW-basic classes of \(V\) are \(\pm D_1 \pm D_2\), where \(D_i\) are Poincaré-dual to the exceptional divisors of the blow-ups.

Proof. By the generalized adjunction inequality, 0 is the only SW-basic class of \(T^4\). Now apply the blow-up formula for SW-invariants (cf. [FS1]).

Theorem 3.4. Let \(K_{X_{11}}\) denote the canonical class of the symplectic structure on \(X_{11}\). Then \(SW_{X_{11}}(\pm K_{X_{11}}) = \pm 1\), and \(SW_{X_{11}}(L) = 0\) if \(L \neq \pm K_{X_{11}}\).
Proof. The first statement is proved in [T]. It remains to prove that there are no other basic classes. Choose a horizontal sphere in $B_{2,1}$, say $\mathbb{CP}^1 \times \{ q_1 \}$, and recall that there is a curve $R \subset E(1)$ of self-intersection $R^2 = -2$ such that $p(R) = \mathbb{CP}^1 \times \{ q_1 \}$ (cf. [GS], §7.3). Note that $R \cdot \Sigma_2 = 1$ in $E(1)_K$. If $F$ denotes the fiber of $W$, then we have $F \cdot \Sigma = 1$. Since the identification $\Sigma = \Sigma_2$ is made in $X_{11}$, we can cut out a small disk from both $F$ and $R$, centered at their intersection with $\Sigma$ and $\Sigma_2$ respectively, and then glue together the resulting open surfaces $F^0$ and $R^0$ along a meridional tube of $\Sigma = \Sigma_2$ to get a new smooth surface $\Gamma$. Inside $X_{11}$, we have $\Gamma \cdot \Sigma = -2$, and $\Gamma \cdot \Sigma = 1$. Now $b_2(X_{11}) = 14$ and we easily get the following orthogonal decomposition of $H_2(X_{11})$ with respect to the intersection pairing

$$H_2(X_{11}; \mathbb{Z}) \cong T \oplus \langle \Sigma, \Gamma \rangle \oplus \mathcal{N}.$$ 

Here, $T \cong \mathbb{Z}^4$ comes from $\langle \Sigma, F \rangle^\perp \subset H_2(W)$, and has generators consisting of four tori of self-intersection $0$. $\mathcal{N} \cong \mathbb{Z}^8$ is negative definite and comes from $(\Sigma_2, R)^\perp \subset H_2(E(1)_K)$. Note that the generators of $T$ and $\mathcal{N}$ lie away from the gluing area of the symplectic sum operation $\#_\varphi$.

Now suppose $SW_{X_{11}}(L) \neq 0$. Then by repeated applications of the generalized adjunction inequality, $L$ is orthogonal to $T$, i.e.

$$L = a\Sigma + b\Gamma + \nu,$$

where $\nu \in \mathcal{N}$ and $b = 0, \pm 2$. But $X_{11}$ is symplectic, hence of $SW$-simple type. This implies that $L^2 = 2e(X_{11}) + 3\text{sign}(X_{11}) = 8$. Since $2ab \geq L^2$, we must have $b \neq 0$. Now

$$\langle \pm L, \Sigma \rangle = \pm b = 2 = 2g(\Sigma) - 2$$

so we can apply the product formula in [MST]. From the computations in two previous lemmas applied to the product formula, we easily see that $L$ must now be orthogonal to $\mathcal{N}$, i.e. $L = a\Sigma \pm 2\Gamma$. From $L^2 = 2ab - 2b^2 = 8$, we conclude that $L = \pm(4\Sigma + 2\Gamma)$. By the pigeonhole principle,

$$L = \pm K_{X_{11}} = \pm(4\Sigma + 2\Gamma).$$

\[\square\]

**Theorem 3.5.** Let $K_{X_{13}}$ denote the canonical class of the symplectic structure on $X_{13}$. Then $SW_{X_{13}}(\pm K_{X_{13}}) = \pm 1$, and $SW_{X_{13}}(L) = 0$ if $L \neq \pm K_{X_{13}}$.

**Proof.** This proof is completely analogous to the argument made for $X_{11}$. We use the previous cut-and-paste method to make sense out of the expressions $[T_{12}\# R], [T_{34}\# R], [(D_1 + D_2)\# T]$, etc. (Here, $T$ is the elliptic fiber of $E(1)$.) For example, $[T_{12}\# R]$ and $[T_{34}\# R]$ are represented by smooth tori of square $(-2)$ inside $X_{13}$. If $L$ is a basic class of $X_{13}$, then $L^2 = 6$. From the generalized adjunction inequality [OS] and the product formula [MST], it easily follows that

$$L = \pm(3|\Sigma| + [T_{12}\# R] + [T_{34}\# R]) = \pm K_{X_{13}}.$$ 

\[\square\]

**Corollary 3.6.** $X_{11}$ and $X_{13}$ are irreducible.

**Proof.** Since $X_n$ has nontrivial Seiberg-Witten invariants, it follows that every connected sum decomposition of $X_n$ as $X_n = Y \# Z$ satisfies that one of the pieces,
say $Z$, is a homotopy $k\mathbb{CP}^2$ with some $k \geq 0$. If $k > 0$, then the blow-up formula for $SW$-invariants implies the existence of $SW$-basic classes $L, L'$ of $X_n$ with $(L - L')^2 = -4$. But this is easily seen to be impossible.

Remark 3.7. The connected sum theorem for the Seiberg-Witten invariant (cf. [Sa]) can be invoked to say that the $SW$-invariant of $3\mathbb{CP}^2 \# n\mathbb{CP}^2$ is 0. Combined with Theorems 3.4 and 3.5 this gives an alternative proof that $X_n$ is not diffeomorphic to $3\mathbb{CP}^2 \# n\mathbb{CP}^2$.

4. Appendix: Nonexistence of certain holomorphic curves

In the preliminary draft of this paper, the author constructed $X_n$ on the assumption that the Barlow surface (cf. [B]) contains a genus 2 holomorphic curve of self-intersection 1. It turns out that this assumption is false and one has the following

Proposition 4.1. There is no holomorphic genus 2 curve of square 1 inside the Barlow surface $B$.

Proof. Recall that $B$ is homeomorphic to $\mathbb{CP}^2 \# 8\mathbb{CP}^2$ with $K^2 = 1$ and $q = p_g = 0$. Choose an orthogonal basis \{ $H, E_1, \ldots, E_8$ \} of $H_2(B; \mathbb{Z})$ such that $H^2 = 1$ and $E_i^2 = -1$. After a change of basis we can assume that the canonical class $K$ is a reduced class (cf. [LL], p. 576). Since $K^2 = 1$, there are only two possibilities, namely $K = H$ or $K = 3H - \sum_{i=1}^{8} E_i$. Now suppose $C = aH - \sum_{i=1}^{8} b_i E_i$ is a genus 2 holomorphic curve of square $C^2 = 1$. By the adjunction formula, we must have $K \cdot C = 1$. If $K = H$, then we immediately see that $C = H$. If $K = 3H - \sum_{i=1}^{8} E_i$, then we must have

\[
\begin{cases}
3a - \sum_{i=1}^{8} b_i = 1, \\
a^2 - \sum_{i=1}^{8} b_i^2 = 1.
\end{cases}
\]

By the Cauchy-Schwartz inequality, $(3a - 1)^2 \leq 8(a^2 - 1)$, which implies $a = 3$. It easily follows that $C = K$ in this case as well. But $p_g = \dim_{\mathbb{C}} \Gamma(B, \Lambda^{2,0}T^*B) = 0$, hence there cannot be a holomorphic curve representing $K = c_1(\Lambda^{2,0}T^*B)$.

Remark 4.2. In fact the above proposition continues to hold for an arbitrary simply-connected numerical Godeaux surface.

Acknowledgment

The author would like to thank his advisor Zoltán Szabó for generously providing much-needed guidance and inspiration. He also thanks Ronald Fintushel, Tian-Jun Li and K. Soundararajan for helpful conversations. A part of this work was done while the author was visiting The University of Arhus.

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