EXOTIC SMOOTH STRUCTURES ON $3\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$, PART II

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Abstract. We construct exotic $3\mathbb{CP}^2 \# 11\mathbb{CP}^2$ and $3\mathbb{CP}^2 \# 13\mathbb{CP}^2$ using the surgery techniques of R. Fintushel and R.J. Stern. We show that these 4-manifolds are irreducible by computing their Seiberg-Witten invariants.

1. Introduction

This paper is a sequel to [P]. For some history and general remarks on distinguishing smooth structures on $3\mathbb{CP}^2 \# n\mathbb{CP}^2$, we refer to Section 1 of [P]. Our main result is the following

**Theorem 1.1.** There exists a smooth closed simply-connected irreducible symplectic 4-manifold $X_n$ that is homeomorphic but not diffeomorphic to $3\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ for each integer $10 \leq n \leq 13$.

The even cases $n = 10, 12$ were dealt with in [P]. In this paper we prove the remaining $n = 11, 13$ cases and complete our picture. Together these constructions provide the “smallest” known examples of an exotic closed simply-connected oriented 4-manifold with $b_2^+ > 1$. It remains an open problem whether there is an exotic $3\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ with $n < 10$.

2. Construction of $X_n$

We start out by recalling the following

**Proposition 2.1** (See [T]). Let $X$ be a closed symplectic 4-manifold and suppose that $b_2^+(X) > 1$. Then $X$ does not admit any Riemannian metric of positive scalar curvature.

Our main building block will be a homotopy rational elliptic surface of Fintushel and Stern in [FS2]. First, let us recall some basic properties of the rational elliptic surface $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$. Let $B_{2,1} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ denote the union of four vertical and two horizontal spheres in the direct product $\mathbb{CP}^1 \times \mathbb{CP}^1$. More precisely, we choose six distinct points $p_1, \ldots, p_4, q_1, q_2$ in $\mathbb{CP}^1$ and define the nodal curve

$$B_{2,1} := \bigcup_{i=1}^4 (p_i \times \mathbb{CP}^1) \cup \bigcup_{j=1}^2 (\mathbb{CP}^1 \times \{q_j\}) \subset \mathbb{CP}^1 \times \mathbb{CP}^1.$$
Let $D_{2,1}$ be the double branched cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along $B_{2,1}$. Then $E(1)$ is the desingularization of $D_{2,1}$

$$p : E(1) \to D_{2,1} \to \mathbb{CP}^1 \times \mathbb{CP}^1.$$ 

Let $pr_1 : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ denote the projection onto the first factor and define $\pi = pr_1 \circ p$. Then $\pi : E(1) \to \mathbb{CP}^1$ is a fibration with generic fiber $\mathbb{CP}^1$ since the generic fiber of $pr_1$ meets the branch locus $B_{2,1}$ at two points. On the other hand, note that the composition $pr_2 \circ p$ is the standard elliptic fibration, where $pr_2$ is the projection to the second factor. We will denote the generic torus fiber of $pr_2 \circ p$ by $F$. For more details on these two fibrations, we refer to [GS] (§7.3).

Next we perform a knot surgery on $E(1)$ as in [FS2]. Let $K$ be the trefoil knot in $S^3$ and $m$ a meridional circle to $K$. Perform the 0-framed surgery on $K$ and call the resulting 3-manifold $S^3(K)$. In $S^3(K) \times S^1$ we have the smoothly embedded torus $T = m \times S^1$ of self-intersection 0. Let $E(1)_K$ denote the fiber sum

$$E(1) \#_{F=T}(S^3(K) \times S^1) = \{E(1) - (F \times D^2)\} \cup_{\psi} \{(S^3(K) \times S^1) - (T \times D^2)\},$$

where the two pieces are glued together so as to preserve the homology class $\alpha = \{pt\} \times \partial D^2$. Fintushel and Stern showed that $E(1)_K$ is in fact a symplectic 4-manifold homotopy equivalent to $E(1)$.

Now let $S$ denote the generic rational fiber of $\pi$. Note that $[S] \cdot [F] = 2$, i.e. $S$ geographically intersects the elliptic fiber $F$ at two points. Recall that the Seifert surface $\Sigma^0$ of the trefoil knot $K$ is a punctured torus. Since the gluing map $\psi$ sends the homology class $\alpha$ into the homology class $[K \times \{pt\}] = [\partial \Sigma^0 \times \{pt\}]$ in $(S^3(K) \times S^1) - (T \times D^2)$, we can glue together two copies of $\Sigma^0$ and the twice-punctured rational fiber $S - (F \times D^2)$, and get a smooth genus 2 surface $\Sigma_2$ in $E(1)_K$. In this way $\pi$ induces a genus 2 fibration $E(1)_K \to \mathbb{CP}^1$ with generic fiber $\Sigma_2$, which we continue to denote by $\pi$. Note that the fibration $\pi : E(1)_K \to \mathbb{CP}^1$ has no multiple fibers. After a small perturbation we can assume that $\pi$ is a Lefschetz fibration (cf. [GS], Chapter 8). We can further assume that the vanishing cycle on the fiber $\Sigma_2$ is homologically nontrivial.

There was nothing special about the trefoil knot in the above construction and we can easily generalize our result to the following

**Lemma 2.2** (See [F]). Suppose $L$ is a fibered knot in $S^3$ of genus $g(L)$. Then there is a genus $2g(L)$ Lefschetz fibration $\pi : E(1)_L \to \mathbb{CP}^1$. \hfill \Box

Our second building block $W$ will be the fiber sum of two copies of the Kodaira-Thurston manifold. This is the manifold $Q_2$ in [G] (p. 570). Recall that $W$ fibers over a genus 2 surface and the fibration admits a section $\Sigma$ of self-intersection 0. Note that $W$ has a symplectic structure and $\Sigma$ is a symplectic submanifold. Also recall from [SZ] (Lemma 2.1, p. 413) that there is an element $g \in \pi_1(\Sigma)$ such that the monodromy relation $g^{-1}ag = ab$ holds inside $\pi_1(W)$. Here, $(a, b)$ denotes a set of generators for the fundamental group of the fiber in $W$.

Now we have all the ingredients needed for our construction. Our manifold $X_{11}$ will be the symplectic sum of $W$ and $E(1)_K$ along $\Sigma$ and $\Sigma_2$. We identify the tubular neighborhoods $N_1$ of $\Sigma$ and $N_2$ of $\Sigma_2$ via a diffeomorphism $\varphi : N_1 - \Sigma \to N_2 - \Sigma_2$, which preserves the orientations on the normal disks. We choose the gluing map $\varphi$ in such a way that $\varphi$ maps the generator $g$ to the vanishing cycle on the fiber $\Sigma_2$.
in $E(1)_K$. Let
\[ X_{11} = W \# E(1)_K = (W - \Sigma) \cup_{\varphi} (E(1)_K - \Sigma_2), \]
where we use $\varphi$ to identify $N_1 - \Sigma$ and $N_2 - \Sigma_2$. For more on the details of the symplectic sum operation, we refer to [G].

**Lemma 2.3.** (i) $X_{11}$ is a smooth, closed, oriented, symplectic 4-manifold.

(ii) $X_{11}$ is simply-connected.

(iii) $X_{11}$ is homeomorphic to $3\mathbb{C}P^2 \# 11\mathbb{C}P^2$.

(iv) $X_{11}$ is not diffeomorphic to $3\mathbb{C}P^2 \# 11\mathbb{C}P^2$.

**Proof.** Part (i) is immediate. Since $E(1)_K$ is simply-connected, an easy application of van Kampen’s theorem says that $\pi_1(E(1)_K - \Sigma_2)/\langle \mu \rangle = 1$, where $\mu$ denotes the meridian of $\Sigma_2$. Now note that $\mu = [a, b]$ in $\pi_1(W \# E(1)_K)$. But in the group $\pi_1(X_{11})$, we have $g = 1$ which implies that $b = 1$, which in turn implies that $\mu = 1$.

We conclude that the homomorphism, $\pi_1(E(1)_K - \Sigma_2) \to \pi_1(X_{11})$, induced by the inclusion map is the zero homomorphism.

It can be shown (cf. [G], p. 571) that $\pi_1(W - \Sigma)/\langle \pi_1(\Sigma) \rangle = 1$, where $\Sigma$ is a parallel copy of $\Sigma$ in $(W - \Sigma)$. Since $\Sigma$ gets identified with $\Sigma_2$ in $E(1)_K$ and the composition of inclusions, $\Sigma \to (E(1)_K - \Sigma_2) \to X_{11}$, induces the zero map on the fundamental groups, an easy application of van Kampen’s theorem gives (ii).

Various topological invariants behave nicely under the symplectic sum operation (cf. [G], p. 535):
\[ \text{sign}(X_{11}) = \text{sign}(W) + \text{sign}(E(1)_K) = \text{sign}(E(1)_K) = -8, \]
\[ e(X_{11}) = e(W) + e(E(1)_K) - 2e(\Sigma) = e(E(1)_K) + 4 = 16. \]
It follows that $b_1^+(X_{11}) = 3$ and $b_2^+(X_{11}) = 11$. Hence $X_{11}$ has an odd intersection form $3(1) \otimes 11(-1)$. Since $X_{11}$ is smooth, (iii) follows from Freedman’s famous classification theorem (cf. [FQ]). Part (iv) follows from Proposition 2.1 since $3\mathbb{C}P^2 \# 11\mathbb{C}P^2$ admits a metric of positive scalar curvature (see e.g. [Sa]).

Now we proceed with the construction of $X_{13}$. This time around, we keep the homotopy rational elliptic surface summand $E(1)_K$ in the construction of $X_{11}$, but replace the $W$ summand with $T^4 \# 2\mathbb{C}P^2$. Choose four distinct points $z_i \in S^1$, $i = 1, \ldots, 4$. Let us define smoothly embedded 2-tori in the 4-torus $T_{12} \subset T^4$, $T_{34} \subset T^4$ by
\[ T_{12} = S^1 \times S^1 \times \{z_1\} \times \{z_2\}, \quad T_{34} = \{z_3\} \times \{z_4\} \times S^1 \times S^1. \]
We choose a symplectic form on $T^4$ for which $T_{12}$, $T_{34}$ are symplectic submanifolds. Note that $[T_{12}] : [T_{34}] = 1$, and $[T_{12}]^2 = [T_{34}]^2 = 0$. By symplectically resolving the intersection of $T_{12}$ and $T_{34}$, and then blowing up twice to reduce the self-intersection, we obtain a symplectic genus 2 surface $\tilde{\Sigma} \hookrightarrow T^4 \# 2\mathbb{C}P^2$ with self-intersection 0. We denote $T^4 \# 2\mathbb{C}P^2$ by $V$. Note that $\pi_1(V) \cong \pi_1(T^4) \cong \mathbb{Z}^4$, $e(V) = 2$, and $\text{sign}(V) = -2$.

As before, we symplectically sum $V$ and $E(1)_K$ along $\tilde{\Sigma}$ and $\Sigma_2$, and define
\[ X_{13} = V \# E(1)_K = (V - \tilde{\Sigma}) \cup_{\varphi} (E(1)_K - \Sigma_2), \]
where $\varphi$ is some suitably chosen diffeomorphism as before.
Lemma 2.4.  
(i) $X_{13}$ is a smooth, closed, oriented, symplectic 4-manifold.  
(ii) $X_{13}$ is simply-connected.  
(iii) $X_{13}$ is homeomorphic to $3\mathbb{CP}^2 \# 13\mathbb{CP}^2$.  
(iv) $X_{13}$ is not diffeomorphic to $3\mathbb{CP}^2 \# 13\mathbb{CP}^2$.  

Proof.  
Part (i) is immediate. The proofs of the other parts mirror the proofs of the corresponding statements for $X_{11}$. For (ii), we note that if $\tilde{\Sigma}^i$ is a parallel copy of $\Sigma$ in $(V - \Sigma)$, then the inclusion induces a surjection $\pi_1(\Sigma^i) \to \pi_1(V - \Sigma)$. In particular, the meridian of $\Sigma$ is killed by an embedded disk coming from the exceptional divisor of a blow-up, and hence we can proceed as before. For (iii), we calculate that $e(X_{13}) = 18$, $\text{sign}(X_{13}) = 10$, and then invoke Freedman’s theorem. Since $3\mathbb{CP}^2 \# 13\mathbb{CP}^2$ admits a metric of positive scalar curvature, Proposition 2.1 implies (iv). 

We have thus proved Theorem 1.1 for the odd cases $n = 11, 13$ except for the irreducibility condition which will be proved in the next section.

3. Irreducibility and SW-invariants

Recall that a smooth closed simply-connected 4-manifold $X$ is called irreducible if each connected sum decomposition of $X$ as $X = Y \# Z$ satisfies that either $Y$ or $Z$ is a homotopy $S^4$. Irreducibility of $X_n$ will follow easily from the computation of SW-invariants of $X_n$ using the product formula of [MST]. In order to use the product formula we must first compute the SW-invariants of each summand. We keep the conventions in Section 3 of [P] regarding the choice involved in the definition of SW-invariant for the $b_2^+ = 1$ case. A lot of times, we abuse notation and use the same capital letter to denote a surface, its homology class, or the Poincaré dual of its homology class.

Lemma 3.1. Let $T$ denote the Poincaré dual of the homology class of the torus fiber in $E(1)$. Then the only SW-basic classes of $E(1)_K$ are $\pm T$.

Proof. We refer the reader to the last section of [FS2], where a more general statement is proved for the case when $K$ is an arbitrary twist knot. Note that the trefoil is $(-1)$-twist knot.

Lemma 3.2. Let $F^*$ denote the Poincaré dual of the homology class of the fiber in $W$. Then the only possible SW-basic classes of $W$ are $0, \pm 2F^*$.

Proof. This is an easy application of the generalized adjunction inequality for the SW-basic classes (cf. [OS]), using the fact that $H_2(W; \mathbb{Z}) \cong \mathbb{Z}^6$ (cf. [Sz], p. 413) has generators consisting of the section $\Sigma$ and five tori (one of which is the fiber $F$).

Lemma 3.3. The only SW-basic classes of $V$ are $\pm D_1 \pm D_2$, where $D_i$ are Poincaré-dual to the exceptional divisors of the blow-ups.

Proof. By the generalized adjunction inequality, 0 is the only SW-basic class of $T^4$. Now apply the blow-up formula for SW-invariants (cf. [FS1]).

Theorem 3.4. Let $K_{X_{11}}$ denote the canonical class of the symplectic structure on $X_{11}$. Then $SW_{X_{11}}(\pm K_{X_{11}}) = \pm 1$, and $SW_{X_{11}}(L) = 0$ if $L \neq \pm K_{X_{11}}$. 


Proof. The first statement is proved in [T]. It remains to prove that there are no other basic classes. Choose a horizontal sphere in $B_{2,1}$, say $\mathbb{CP}^1 \times \{q_1\}$, and recall that there is a curve $R \subset E(1)$ of self-intersection $R^2 = -2$ such that $p(R) = \mathbb{CP}^1 \times \{q_1\}$ (cf. [GS], §7.3). Note that $R \cdot \Sigma_2 = 1$ in $E(1)_K$. If $F$ denotes the fiber of $W$, then we have $F \cdot \Sigma = 1$. Since the identification $\Sigma = \Sigma_2$ is made in $X_{11}$, we can cut out a small disk from both $F$ and $R$, centered at their intersection with $\Sigma$ and $\Sigma_2$ respectively, and then glue together the resulting open surfaces $F^0$ and $R^0$ along a meridional tube of $\Sigma = \Sigma_2$ to get a new smooth surface $\Gamma$. Inside $X_{11}$, we have $\Gamma \cdot \Gamma = -2$, and $\Gamma \cdot \Sigma = 1$. Now $b_2(X_{11}) = 14$ and we easily get the following orthogonal decomposition of $H_2(X_{11})$ with respect to the intersection pairing

$$H_2(X_{11}; \mathbb{Z}) \cong T \oplus \langle \Sigma, \Gamma \rangle \oplus N.$$  

Here, $T \cong \mathbb{Z}^4$ comes from $(\Sigma, F)^{\perp} \subset H_2(W)$, and has generators consisting of four tori of self-intersection 0. $N \cong \mathbb{Z}^8$ is negative definite and comes from $(\Sigma_2, R)^{\perp} \subset H_2(E(1)_K)$. Note that the generators of $T$ and $N$ lie away from the gluing area of the symplectic sum operation $\#_\varphi$.

Now suppose $SW_{X_{11}}(L) \neq 0$. Then by repeated applications of the generalized adjunction inequality, $L$ is orthogonal to $T$, i.e.

$$L = a\Sigma + b\Gamma + \nu,$$

where $\nu \in N$ and $b = 0, \pm 2$. But $X_{11}$ is symplectic, hence of $SW$-simple type. This implies that $L^2 = 2e(X_{11}) + 3\text{sign}(X_{11}) = 8$. Since $2ab \geq L^2$, we must have $b \neq 0$. Now

$$\langle \pm L, \Sigma \rangle = \pm b = 2 = 2g(\Sigma) - 2$$

so we can apply the product formula in [MST]. From the computations in two previous lemmas applied to the product formula, we easily see that $L$ must now be orthogonal to $N$, i.e. $L = a\Sigma \pm 2\Gamma$. From $L^2 = 2ab - 2b^2 = 8$, we conclude that $L = \pm (4\Sigma + 2\Gamma)$. By the pigeonhole principle,

$$L = \pm K_{X_{11}} = \pm (4\Sigma + 2\Gamma).$$

\[ \square \]

**Theorem 3.5.** Let $K_{X_{13}}$ denote the canonical class of the symplectic structure on $X_{13}$. Then $SW_{X_{13}}(\pm K_{X_{13}}) = \pm 1$, and $SW_{X_{13}}(L) = 0$ if $L \neq \pm K_{X_{13}}$.

**Proof.** This proof is completely analogous to the argument made for $X_{11}$. We use the previous cut-and-paste method to make sense out of the expressions $[T_{12}\#R]$, $[T_{34}\#R]$, $[(D_1 + D_2)\#T]$, etc. (Here, $T$ is the elliptic fiber of $E(1)$.) For example, $[T_{12}\#R]$ and $[T_{34}\#R]$ are represented by smooth tori of square $(-2)$ inside $X_{13}$. If $L$ is a basic class of $X_{13}$, then $L^2 = 6$. From the generalized adjunction inequality [OS] and the product formula [MST], it easily follows that

$$L = \pm (9\Sigma) + [T_{12}\#R] + [T_{34}\#R] = \pm K_{X_{13}}.$$

\[ \square \]

**Corollary 3.6.** $X_{11}$ and $X_{13}$ are irreducible.

**Proof.** Since $X_n$ has nontrivial Seiberg-Witten invariants, it follows that every connected sum decomposition of $X_n$ as $X_n = Y \# Z$ satisfies that one of the pieces,
say $Z$, is a homotopy $k\mathbb{CP}^2$ with some $k \geq 0$. If $k > 0$, then the blow-up formula for $SW$-invariants implies the existence of $SW$-basic classes $L, L'$ of $X_n$ with $(L - L')^2 = -4$. But this is easily seen to be impossible.

**Remark 3.7.** The connected sum theorem for the Seiberg-Witten invariant (cf. [Sa]) can be invoked to say that the $SW$-invariant of $3\mathbb{CP}^2 \# n\mathbb{CP}^2$ is 0. Combined with Theorems 3.4 and 3.5 this gives an alternative proof that $X_n$ is not diffeomorphic to $3\mathbb{CP}^2 \# n\mathbb{CP}^2$.

4. **Appendix: Nonexistence of certain holomorphic curves**

In the preliminary draft of this paper, the author constructed $X_n$ on the assumption that the Barlow surface (cf. [B]) contains a genus 2 holomorphic curve of self-intersection 1. It turns out that this assumption is false and one has the following

**Proposition 4.1.** There is no holomorphic genus 2 curve of square 1 inside the Barlow surface $B$.

**Proof.** Recall that $B$ is homeomorphic to $\mathbb{CP}^2 \# 8\mathbb{CP}^2$ with $K^2 = 1$ and $q = p_g = 0$. Choose an orthogonal basis $\{H, E_1, \ldots, E_8\}$ of $H_2(B; \mathbb{Z})$ such that $H^2 = 1$ and $E_i^2 = -1$. After a change of basis we can assume that the canonical class $K$ is a reduced class (cf. [LL], p. 576). Since $K^2 = 1$, there are only two possibilities, namely $K = H$ or $K = 3H - \sum_{i=1}^{8} E_i$. Now suppose $C = aH - \sum_{i=1}^{8} b_i E_i$ is a genus 2 holomorphic curve of square $C^2 = 1$. By the adjunction formula, we must have $K \cdot C = 1$. If $K = H$, then we immediately see that $C = H$. If $K = 3H - \sum_{i=1}^{8} E_i$, then we must have

$$\begin{align*}
3a - \sum_{i=1}^{8} b_i &= 1, \\
a^2 - \sum_{i=1}^{8} b_i^2 &= 1.
\end{align*}$$

By the Cauchy-Schwartz inequality, $(3a - 1)^2 \leq 8(a^2 - 1)$, which implies $a = 3$. It easily follows that $C = K$ in this case as well. But $p_g = \dim_{\mathbb{C}} \Gamma(B, \Lambda^{2,0}T^*B) = 0$, hence there cannot be a holomorphic curve representing $K = c_1(\Lambda^{2,0}T^*B)$.

**Remark 4.2.** In fact the above proposition continues to hold for an arbitrary simply-connected numerical Godeaux surface.

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