A WEIGHTED UNIFORM $L^p$–ESTIMATE OF BESSSEL FUNCTIONS: A NOTE ON A PAPER OF GUO

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Abstract. An improved Guo's uniform $L^p$ estimate of Bessel functions is shown by using a uniform pointwise bound of Barcéló and Córdoba.

Recently, Guo has shown, [Guo, Theorem 3.5], the following uniform $L^p$ estimate:

$$
\int_0^\infty |J_\nu(x)|^p x^{\nu} dx \leq C(p - 4)^{-1}, \quad \nu \geq 0, \quad p > 4.
$$

Here $J_\nu(x)$ denotes the Bessel function of the first kind of order $\nu$, cf. [W]. This estimate was proved first for $\nu = 0, 1, \ldots$, by means of a dual form of a Fourier restriction theorem for the plane unit circle and then extended to an arbitrary $\nu \geq 0$. The estimate was crucial in proving the main result of [Guo], Theorem 4.1.

It was quite reasonable to expect a proof of (1) based on intrinsic properties of Bessel functions. Furthermore, it was natural to expect an estimate like (1) for a larger range of $p$'s by adding an appropriate power weight in the integral on the left side of (1). More precisely, it was natural to look for an inequality of the form

$$
\int_0^\infty |J_\nu(x)|^p x^{\alpha} dx \leq C(p, \alpha), \quad \nu \geq 0,
$$

with a constant $C(p, \alpha) > 0$ depending only on $p$ and $\alpha$ (we did not care about making the constant $C(p, \alpha)$ the best possible).

Since $J_\nu(x) = O(x^{-\nu/2})$, $x \to \infty$, the necessary assumption on $\alpha$ to make the integral in (2) convergent at infinity for every single $\nu \geq 0$ is $\alpha < p/2 - 1$. On the other hand $J_\nu(x) = O(x^{\nu})$, $x \to 0$; hence the necessary assumption on $\alpha$ to make the integral in (2) convergent at zero for every $\nu \geq 0$ is $\alpha > -1$.

It is now interesting to note that Guo’s result, (1), shows that the assumption $-1 < \alpha < p/2 - 1$ is also sufficient for (2) to hold in the case $0 < p \leq 4$. Indeed, assume

$$
\int_1^\infty |J_\nu(x)|^q x dx \leq C_q, \quad \nu \geq 0,
$$

holds true for every $q > 4$ and consider $p$ and $\alpha$ such that $0 < p \leq 4$ and $\alpha < p/2 - 1$. Since $2(\alpha + 1) < p \leq 4$, we can choose $s > 1$ satisfying $2(\alpha + 1)s < 4 < ps$. Then,

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because $2(a + 1)s < 4$ implies $(a - 1)s' + 1 < -1$, by Hölder’s inequality we obtain
\[
\int_1^\infty |J_\nu(x)|^p x^a dx \leq \left( \int_1^\infty |J_\nu(x)|^{ps} x dx \right)^{1/s} \left( \int_1^\infty x^{(a-1)s'} x dx \right)^{1/s'} \\
\leq A(C_{ps})^{1/s}.
\]
In a similar way, assuming
\[
\int_0^1 |J_\nu(x)|^q x dx \leq D_q, \quad \nu \geq 0,
\]
is satisfied for every $q > 4$ and taking $p$ and $a$ such that $0 < p \leq 4$ and $a > -1$, we obtain
\[
\int_0^1 |J_\nu(x)|^p x^a dx \leq B(D_p)^{1/s},
\]
where, this time $s > 1$ is chosen in such a way that $ps > 4$ and $(a + 1)s > 2$. The main result of this note claims that, under suitable restrictions on $a$, (2) is valid for any $p$, $0 < p < \infty$.

**Proposition.** Let $0 < p < \infty$ and $-1 < a < \frac{p}{2} - 1$ when $0 < p \leq 4$ or $-1 < a < \frac{p}{2} - \frac{1}{3}$ in the case $4 < p < \infty$. Then the uniform estimate
\[
(3) \quad \int_0^\infty |J_\nu(x)|^p x^a dx \leq C(p, a), \quad \nu \geq 0,
\]
holds true.

The proof of the proposition is based on the following, uniform on $\nu \geq 2$, pointwise bounds for the Bessel functions ($C$ and $d$ are positive constants):
\[
(4) \quad |J_\nu(x)| \leq C \begin{cases} 
\exp(-d\nu), & 0 < x < \nu/2, \\
\nu^{-1/4}(|x - \nu| + \nu^{1/3})^{-1/4}, & \nu/2 < x < 2\nu, \\
x^{-1/2}, & 2\nu < x < \infty.
\end{cases}
\]
The estimate (4) on the interval $0 < x < \nu/2$ is a consequence of
\[
\Gamma(\nu + 1)(x/2)^{-\nu} |J_{\nu}(x)| \leq 1, \quad x > 0,
\]
(and Stirling’s formula) that holds for every $\nu \geq -1/2$ [W p. 49 (1)], while on the two other intervals it is a consequence of bounds done by Barceló and Córdoba (see [BC p. 66] or [C p. 24]; cf. also [Va p. 70]).

**Proof of the Proposition.** The left side of (3) is a continuous function of the variable $\nu \geq 0$; hence we can assume $\nu$ to be large, say $\nu \geq 2$. Given $\nu \geq 2$ we split the integration in (3) onto the intervals $(0, \nu/2)$, $(\nu/2, 2\nu)$ and $(2\nu, \infty)$. Then
\[
\int_{2\nu}^\infty |J_\nu(x)|^p x^a dx \leq C \int_{2\nu}^\infty x^{a-p/2} dx = C_1 \nu^{a-p/2+1} \leq C_2
\]
for $\nu \geq 2$ and $p$ and $a$ satisfying $a < p/2 - 1$. Also,
\[
\int_{0}^{\nu/2} |J_\nu(x)|^p x^a dx \leq C \exp(-pd\nu) \int_{0}^{\nu/2} x^a dx \leq C_3
\]
for $\nu \geq 2$ when $a$ satisfies $a > -1$. On the interval $(\nu/2, 2\nu)$ we consider only the integration over $(\nu/2, 2\nu)$; the integration over $(\nu/2, \nu)$ can be treated analogously. We have

$$
(5) \quad \int_{\nu}^{2\nu} |J_{\nu}(x)|^p x^a dx \leq C\nu^{a-p/4} \int_{\nu}^{2\nu} (x - \nu + \nu^{1/3})^{-p/4} dx.
$$

If $0 < p < 4$, we evaluate the last integral and bound the right side of (5) by $C\nu^{a-p/2+1}$ when $p < 4$ or, by $C\nu^{a-1} \log \nu$ when $p = 4$. Both bounds are small for large $\nu$ by the assumption made on $a$. If $p > 4$, evaluating the last integral gives the bound $C\nu^{a-p/3+1/3}$ for the right side of (5) which is also correct by the assumption made on $a$. This finishes the proof of the proposition.

Remark. In fact, using the asymptotics of [BC, p. 66] leads to precise asymptotics of weighted $L^p$ norms of the Bessel functions. Let $1 \leq p \leq \infty$, $\alpha < \frac{1}{2} - \frac{1}{p}$ and $\nu \to \infty$. Then (for $p = \infty$ one has to take $\sup_{x>0} |J_{\nu}(x)x^\alpha|$ as the $L^\infty$ norm)

$$
(6) \quad \left( \int_0^\infty \left| J_{\nu}(x)x^\alpha \right|^p dx \right)^{1/p} \sim \begin{cases} \\
\nu^{a-1/2+1/p}, & 1 \leq p < 4, \\
\nu^{a-1/4}(\log \nu)^{1/4}, & p = 4, \\
\nu^{a-1/3+1/(3p)}, & 4 < p \leq \infty.
\end{cases}
$$

Here $f(\nu) \sim g(\nu)$ as $\nu \to \infty$ stands for $f(\nu) = O(g(\nu))$ and $g(\nu) = O(f(\nu))$ as $\nu \to \infty$. The upper bound in (6) is obtained, as in the proof of the proposition, by dividing $(0, \infty)$ into three different subintervals, majorizing the integrand and comparing the occurring bounds. The lower bound in (6) is a consequence of the aforementioned precise asymptotics of [BC].

References


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