A REFINEMENT OF THE TORAL RANK CONJECTURE
FOR 2-STEP NILPOTENT LIE ALGEBRAS

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Abstract. It is known that the total (co)-homology of a 2-step nilpotent Lie algebra $g$ is at least $2^{z}$, where $z$ is the center of $g$. We improve this result by showing that a better lower bound is $2^{t}$, where $t = |z| + \left\lfloor \frac{|v|+1}{2} \right\rfloor$ and $v$ is a complement of $z$ in $g$. Furthermore, we provide evidence that this is the best possible bound of the form $2^{t}$.

1. Introduction

An outstanding conjecture, known as the Toral Rank Conjecture (TRC) claims that for any nilpotent Lie algebra $g$ (over $\mathbb{R}$ or $\mathbb{C}$) the total (co)-homology, with trivial coefficients, satisfies the inequality

$$|H_{*}(g)| \geq 2^{z},$$

where $z = \text{center}(g)$.

The TRC is due to S. Halperin ([3], 1987). In 1988, Deninger and Singhof [4] proved it for 2-step nilpotent Lie algebras. Besides this class, a few special cases have been added recently. For example, it was shown in [2] that the TRC holds for $g$ if its center has dimension $\leq 5$ or has codimension $\leq 7$.

It turns out that, in general, $2^{z}$ is quite a lot smaller than $|H_{*}(g)|$, especially when $|z|$ is comparatively small compared to $|g|$.

In this short note we give a new lower bound for $|H_{*}(g)|$, when $g$ is a 2-step nilpotent Lie algebra, that involves both the dimension of the center of $g$ and its codimension. Furthermore, by using existing calculations, we show that actually it is the best possible general lower bound of the form $2^{t}$.

Precisely, we give a direct proof of the following

Theorem. Let $g$ be a 2-step nilpotent Lie algebra of finite dimension over $\mathbb{R}$, or $\mathbb{C}$. Let $v$ be any direct complement, as vector spaces, of $z = \text{center}(g)$. Then

$$|H_{*}(g)| \geq 2^{z} + \left\lfloor \frac{|v|+1}{2} \right\rfloor.$$
2. The proof of the Theorem

We will make use of the following combinatorial result.

**Lemma.** Let $n$ be a positive integer. Then,

$$\left| \sum_{j=0}^{n} (-1)^j \binom{n}{2j} \right| + \left| \sum_{j=0}^{n} (-1)^j \binom{n}{2j+1} \right| = 2^{\left\lceil \frac{n+1}{2} \right\rceil}.$$

**Proof.** Let $P = \sum_{j=0}^{\infty} (-1)^j \binom{n}{2j}$ and $Q = \sum_{j=0}^{\infty} (-1)^j \binom{n}{2j+1}$. Since $(1+i)^n = P + iQ$ and $(1-i)^n = P - iQ$, then $P = \frac{(1+i)^n + (1-i)^n}{2}$ and $Q = i^{-1} \frac{(1+i)^n - (1-i)^n}{2}$. Therefore,

$$|P| = \begin{cases} 2^{n/2}, & \text{if } n \equiv 0 \pmod{4}; \\ 2^{(n-1)/2}, & \text{if } n \equiv 1 \pmod{4}; \\ 0, & \text{if } n \equiv 2 \pmod{4}; \\ 2^{(n-1)/2}, & \text{if } n \equiv 3 \pmod{4}; \end{cases}$$

and

$$|Q| = \begin{cases} 0, & \text{if } n \equiv 0 \pmod{4}; \\ 2^{(n-1)/2}, & \text{if } n \equiv 1 \pmod{4}; \\ 2^{n/2}, & \text{if } n \equiv 2 \pmod{4}; \\ 2^{(n-1)/2}, & \text{if } n \equiv 3 \pmod{4}; \end{cases}$$

from which the lemma follows. \qed

**Proof of the Theorem.** The homology of $\mathfrak{g}$ is the homology of the Koszul complex $(\wedge \mathfrak{g}, \partial)$. Put $\mathfrak{g} = v \oplus \mathfrak{z}$. Hence, we can write $\wedge \mathfrak{g} = \wedge v \otimes \wedge \mathfrak{z}$. It is straightforward to see that

$$\partial : \wedge v \otimes \wedge \mathfrak{z} \rightarrow \wedge v \otimes \wedge \mathfrak{z}.$$

Therefore, it follows that the complex $(\wedge \mathfrak{g}, \partial)$ is the direct sum of an *even* and an *odd* subcomplex, precisely $(\wedge^{2p} v \otimes \wedge \mathfrak{z}, \partial)$ and $(\wedge^{2p+1} v \otimes \wedge \mathfrak{z}, \partial)$. Accordingly, the homology of the Koszul complex is the sum of the homologies of each of the even and odd subcomplexes.

It is well known that if $(\mathcal{C} = (C_t), \partial)$ is a finite complex of finite dimensional vector spaces, then

$$|H_\ast (\mathcal{C})| \geq \left| \sum_i (-1)^i \dim (C_i) \right|.$$

By applying this to each of the even and odd subcomplexes we get that

$$|H_\ast (\mathfrak{g})| \geq \left| \sum_{j=0}^{n} (-1)^j \dim \left( \wedge^{2j} v \otimes \wedge \mathfrak{z} \right) \right| + \left| \sum_{j=0}^{n} (-1)^j \dim \left( \wedge^{2j+1} v \otimes \wedge \mathfrak{z} \right) \right| = 2^{j} \left| \sum_{j=0}^{n} (-1)^j \binom{|v|}{2j} \right| + 2^{j} \left| \sum_{j=0}^{n} (-1)^j \binom{|v|}{2j+1} \right|.

Now the Theorem follows from the Lemma. \qed

**Remark.** The bound obtained in the Theorem is the best possible, of the form $2^t$ ($t$ an integer), that can be given in general. In fact, among the few available computations there are examples that support this claim.

\footnote{2^\partial(x_1 \wedge \ldots \wedge x_p) = \sum_{i \leq j} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_p.}
Example 1. Let $g_2$ denote the Lie algebra with basis $\{x_1, x_2, y_1, y_2, z\}$ and non-zero brackets $[x, x_i] = y_i$ for each $1 \leq i \leq 2$. It was shown in [1] that
\[
 b_0 = 1, \quad b_1 = 3, \quad b_2 = 6, \quad b_3 = 6, \quad b_4 = 3, \quad b_5 = 1,
\]
where $b_i = |H^i(g_2)|$. Therefore, the total cohomology of $g_2$ is equal to 20 and our bound is $2^{2+2} = 16$.

Example 2. Let $g$ denote the Lie algebra with basis $\{a, b, c, d, e, f, g\}$ and non-zero brackets $[a, b] = e$, $[b, d] = g$, $[c, d] = e$ and $[a, c] = f$. This Lie algebra appears as 3, 7D in Seeley's classification [6]. In [3] one can explicitly find that
\[
 b_0 = 1, \quad b_1 = 4, \quad b_2 = 11, \quad b_3 = 14, \quad b_4 = 14, \quad b_5 = 11, \quad b_6 = 4, \quad b_7 = 1,
\]
where $b_i = |H^i(g)|$. Hence $|H^*(g)| = 60$, while our bound is $2^{3+2} = 32$.

Example 3. We consider here a family of examples. For each $r \geq 2$ let $E$ be an $r$-dimensional vector space; then $g_r = E \oplus \bigwedge^2 E$ is the rank $r$ 2-step free nilpotent Lie algebra. For $e, f \in E$, $[e, f] = e \wedge f$, these being all the non-zero brackets.

Their homology has been computed by Sigg [7]. Using his result one can compute, for each $r$, the total homology for the rank $r$ algebra. With a simple computer program written in Maple V we have done it for small $r$’s. On the other hand, it is clear that our bound is $2^{(2r+1)}$. In the following table we show the results for $2 \leq r \leq 13$. ($t = \left(\frac{2}{3}\right) + \left[\frac{r}{3}\right]$)

| $r$ | $|g_r|$ | Total homology | $t$ | $2^t$ |
|-----|--------|----------------|----|-------|
| 2   | 3      | 6              | 2  | 4     |
| 3   | 6      | 36             | 5  | 32    |
| 4   | 10     | 420            | 8  | 256   |
| 5   | 15     | 9800           | 13 | 8192  |
| 6   | 21     | 452760         | 18 | 262144|
| 7   | 28     | $4.1835024 \times 10^7$ | 25 | $3.3554432 \times 10^7$ |
| 8   | 36     | $7.691067984 \times 10^9$ | 32 | $4.294967296 \times 10^9$ |
| 9   | 45     | $2.828336198888 \times 10^{12}$ | 41 | $2.19902325552 \times 10^{12}$ |
| 10  | 55     | $2.073619375892064 \times 10^{15}$ | 50 | $1.125899906842624 \times 10^{15}$ |
| 11  | 66     | $3.040584296923128384 \times 10^{18}$ | 61 | $2.305843009213693952 \times 10^{18}$ |
| 12  | 78     | $8.89850922924075664896 \times 10^{21}$ | 72 | $4.72366482869645213696 \times 10^{21}$ |
| 13  | 91     | $5.2084270468101585237918848 \times 10^{25}$ | 85 | $3.8685626227668133590597632 \times 10^{25}$ |

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References


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