

## NEVANLINNA FUNCTIONS AS QUOTIENTS

EVGUENI DOUBTSOV

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ABSTRACT. Let  $f$  be a holomorphic function in the unit ball. Then  $f$  is a Nevanlinna function if and only if there exist Smirnov functions  $f_+$ ,  $f_-$  such that  $f = f_+/f_-$  and  $f_-$  has no zeros in the ball.

Let  $B = B_n$  be the open unit ball in  $\mathbb{C}^n$  and  $S = \partial B$  be the unit sphere. If  $n = 1$ , then  $\mathbb{D} = B_1$  is the open unit disc in  $\mathbb{C}$ .

The Nevanlinna class  $N(B)$  is the set of all holomorphic functions  $f$  on  $B$  such that

$$\sup_{0 < r < 1} \int_S \log^+ |f_r| d\sigma < +\infty,$$

where  $\sigma$  is the normalized Lebesgue measure on  $S$ ,  $f_r(\zeta) = f(r\zeta)$ , and  $x^+ = \max(x, 0)$ .

The Smirnov class  $N^+(B)$  is the space of all functions  $f \in N(B)$  such that the family  $\{\log^+ |f_r|\}_{0 < r < 1}$  is uniformly integrable on  $S$ . Namely, for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$\int_{\Delta} \log^+ |f_r| d\sigma < \varepsilon$$

for all  $0 < r < 1$  and all sets  $\Delta \subset S$  such that  $\sigma(\Delta) < \delta$ .

Let  $f$  be a holomorphic function on the disc. The classical factorization theorem implies that  $f \in N(\mathbb{D})$  if and only if  $f = f_+/f_-$ , where  $f_+$  and  $f_-$  are holomorphic and bounded by 1 on  $\mathbb{D}$  and  $f_-$  has no zeros on  $\mathbb{D}$ . In particular, the functions  $f$  and  $f_+$  have the same zero sets.

Now, assume  $n \geq 2$ . Then there are many ways to say that the factorization theorem does not hold in the ball  $B_n$ . For example, there exist many functions  $f \in N(B)$ ,  $f \not\equiv 0$ , with the following property: If  $f_+ \in H^\infty(B)$  satisfies  $f_+/f$  is holomorphic, then  $f_+ \equiv 0$ . Hence, the above characterization does not hold for  $n \geq 2$ . Moreover, the zero sets of the Hardy classes  $H^p(B)$  are all different (see [5]). One more negative result in this direction is the following nonfactorization theorem: The set  $\{gh : g, h \in H^2(B)\}$  is a set of first category in  $H^1(B)$  (see [3]).

However, in the present note we obtain the following description of the Nevanlinna class in the ball.

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**Theorem 1.** *Let  $f$  be a holomorphic function on  $B_n$ ,  $n \geq 2$ . Then  $f \in N(B)$  if and only if there exist  $f_+, f_- \in N^+(B)$  such that  $f = f_+/f_-$  and  $f_-(z) \neq 0$  for all  $z \in B$ .*

*Proof.* If  $f_+, f_- \in N^+(B)$  and  $f_+/f_-$  is a holomorphic function, then  $f_+/f_-$  is a Nevanlinna function since

$$\sup_{\frac{1}{2} \leq r < 1} \int_S |\log |(f_-)_r|| \, d\sigma < \infty.$$

So assume  $f \in N(B)$ . The Henkin-Skoda theorem provides a function  $f_0 \in N^+(B)$  which has the same zeros (see [4] and [6]). More precisely,  $f = f_0 F$ , where  $F \in H(B)$  and  $F(z) \neq 0$  for all  $z \in B$ . Without loss of generality  $F(0) \in \mathbb{R}$ .

Observe that  $F \in N(B)$ . Therefore  $u = \log |F|$  is a pluriharmonic function and  $\sup_{0 < r < 1} \int_S |u_r| \, d\sigma < \infty$ . Hence, there exists a real measure  $\mu$  on the sphere such that  $u$  is the Poisson integral  $P[\mu]$ . Let  $\mu = \mu_s + \mu_a$  be the Lebesgue decomposition of  $\mu$  (here  $\mu_s \perp \sigma$  and  $\mu_a \ll \sigma$ ). Also, let  $\mu_s = \mu_+ - \mu_-$ ,  $\mu_{\pm} \geq 0$ , be the Jordan decomposition of  $\mu_s$ . We have  $\mu_+ \ll \mu_s$ ; thus  $\mu_+$  is the singular part of a measure  $\nu$  such that the Poisson integral  $P[\nu]$  is a pluriharmonic function in the ball ([2], Corollary 2.7). In other words, there exists a function  $g \in L^1(\sigma)$  such that  $P[g\sigma - \mu_+]$  is pluriharmonic. Denote by  $F_-$  the holomorphic function such that  $\operatorname{Re} F_- = P[g\sigma - \mu_+]$  and  $F_-(0) \in \mathbb{R}$ . Respectively, the Poisson integral of the measure  $\mu_a + g\sigma - \mu_- = \mu - (\mu_+ - g\sigma)$  is a pluriharmonic function. So let  $F_+$  be the holomorphic function such that  $\operatorname{Re} F_+ = P[\mu_a + g\sigma - \mu_-]$  and  $F_+(0) \in \mathbb{R}$ .

Clearly, the definitions of  $F_-$  and  $F_+$  yield the identity  $F = \exp F_+ \exp(-F_-)$ . Put  $f_- = \exp F_-$ . Then  $\log |(f_-)_r| = P_r[g\sigma - \mu_+] \leq P_r[g\sigma]$  since  $\mu_+$  is a non-negative measure. So  $\log^+ |(f_-)_r| \leq P_r[g^+\sigma]$ , hence, the family  $\{\log^+ |(f_-)_r|\}_{0 < r < 1}$  is uniformly integrable. In other words  $f_- \in N^+(B)$ .

Analogously  $\exp F_+ \in N^+(B)$ , and we put  $f_+ = f_0 \exp F_+$ .  $\square$

Observe that even a nonvanishing Nevanlinna function is not necessarily a quotient  $f_+/f_-$ , where  $f_{\pm} \in H^\infty(B)$  and  $f_-$  has no zeros in the ball. Moreover, one has the following negative example.

**Proposition 2.** *There exists  $f \in N(B)$  such that  $f(z) \neq 0$  for all  $z \in B$  and  $f$  cannot be represented as  $f_+/f_-$ , where  $f_+ \in N^+(B)$ ,  $f_- \in H^\infty(B)$  and  $f_-$  has no zeros in the ball.*

*Proof.* Fix a point  $\zeta \in S$ . Let  $m$  be the normalized Lebesgue measure on the circle  $\mathbb{T}_\zeta = \{\lambda\zeta : \lambda \in \mathbb{T}\} \subset S$ . Choose a strictly positive lower semicontinuous function  $g \in L^1(\sigma)$  such that

$$\sup_{0 < r < 1} \int_S P_r[m]g \, d\sigma = +\infty.$$

Since  $g$  is lower semicontinuous, there exists a singular positive measure  $\mu$  on  $S$  such that  $P[g\sigma - \mu] = \operatorname{Re} F$  for certain  $F \in H(B)$  (see [1], Chapter 5, 4.2). Put  $f = \exp(-F) \in N(B)$ .

Now, suppose  $f = f_+/f_-$ , where  $f_{\pm}$  are nonvanishing,  $f_+ \in N^+(B)$ , and  $f_- \in H^\infty(B)$ . Since  $f_{\pm}$  are zero-free, there exist measures  $\nu_{\pm} \in M(S)$  such that  $-\log |f_{\pm}| = P[\nu_{\pm}]$ . Without loss of generality, let  $\log |f_-| \leq 0$ , in other words  $\nu_- \geq 0$ . Let  $(\nu_{\pm})_s$  be the singular parts of  $\nu_{\pm}$ ; then  $\mu = (\nu_-)_s - (\nu_+)_s$ . Recall that

$f_+ \in N^+(B)$ , so  $(\nu_+)_s \geq 0$ . Thus  $\nu_- \geq (\nu_-)_s \geq \mu$ . Therefore  $g\sigma \leq g\sigma + \nu_- - \mu$  and the Poisson integral  $u = P[(g\sigma - \mu) + \nu_-]$  is pluriharmonic, hence

$$u(0) = \sup_{0 < r < 1} \int_S u_r dm \geq \sup_{0 < r < 1} \int_S P_r[g\sigma] dm = +\infty.$$

A contradiction. □

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DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48824

*E-mail address:* `dubtsov@math.msu.edu`