NEVANLINNA FUNCTIONS AS QUOTIENTS

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Abstract. Let \( f \) be a holomorphic function in the unit ball. Then \( f \) is a Nevanlinna function if and only if there exist Smirnov functions \( f_+, f_- \) such that \( f = f_+ / f_- \) and \( f_- \) has no zeros in the ball.

Let \( B = B_n \) be the open unit ball in \( \mathbb{C}^n \) and \( S = \partial B \) be the unit sphere. If \( n = 1 \), then \( \mathbb{D} = B_1 \) is the open unit disc in \( \mathbb{C} \).

The Nevanlinna class \( N(B) \) is the set of all holomorphic functions \( f \) on \( B \) such that
\[
\sup_{0 < r < 1} \int_S \log^+ |f_r| \, d\sigma < +\infty,
\]
where \( \sigma \) is the normalized Lebesgue measure on \( S \), \( f_r(\zeta) = f(r\zeta) \), and \( x^+ = \max(x, 0) \).

The Smirnov class \( N^+(B) \) is the space of all functions \( f \in N(B) \) such that the family \( \{\log^+ |f_r|\}_{0 < r < 1} \) is uniformly integrable on \( S \). Namely, for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that
\[
\int_{\Delta} \log^+ |f_r| \, d\sigma < \varepsilon
\]
for all \( 0 < r < 1 \) and all sets \( \Delta \subset S \) such that \( \sigma(\Delta) < \delta \).

Let \( f \) be a holomorphic function on the disc. The classical factorization theorem implies that \( f \in N(\mathbb{D}) \) if and only if \( f = f_+ / f_- \), where \( f_+ \) and \( f_- \) are holomorphic and bounded by 1 on \( \mathbb{D} \) and \( f_- \) has no zeros on \( \mathbb{D} \). In particular, the functions \( f \) and \( f_+ \) have the same zero sets.

Now, assume \( n \geq 2 \). Then there are many ways to say that the factorization theorem does not hold in the ball \( B_n \). For example, there exist many functions \( f \in N(B) \), \( f \neq 0 \), with the following property: If \( f_+ \in H^\infty(B) \) satisfies \( f_+ / f \) is holomorphic, then \( f_+ \equiv 0 \). Hence, the above characterization does not hold for \( n \geq 2 \). Moreover, the zero sets of the Hardy classes \( H^p(B) \) are all different (see [3]). One more negative result in this direction is the following nonfactorization theorem: The set \( \{gh : g, h \in H^2(B)\} \) is a set of first category in \( H^1(B) \) (see [4]).

However, in the present note we obtain the following description of the Nevanlinna class in the ball.

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Theorem 1. Let \( f \) be a holomorphic function on \( B_n, n \geq 2 \). Then \( f \in N(B) \) if and only if there exist \( f_+, f_- \in N^+(B) \) such that \( f = f_+/f_- \) and \( f_-(z) \neq 0 \) for all \( z \in B \).

Proof. If \( f_+, f_- \in N^+(B) \) and \( f_+/f_- \) is a holomorphic function, then \( f_+/f_- \) is a Nevanlinna function since

\[
\sup_{\frac{1}{2} \leq r < 1} \int_S |\log |(f_-)_r|| \, d\sigma < \infty.
\]

So assume \( f \in N(B) \). The Henkin-Skoda theorem provides a function \( f_0 \in N^+(B) \) which has the same zeros (see [4] and [5]). More precisely, \( f = f_0F \), where \( F \in H(B) \) and \( F(z) \neq 0 \) for all \( z \in B \). Without loss of generality \( F(0) \in \mathbb{R} \).

Observe that \( F \in N(B) \). Therefore \( u = \log |F| \) is a pluriharmonic function and \( \sup_{0<r<1} \int_S |u_r|d\sigma < \infty \). Hence, there exists a real measure \( \mu \) on the sphere such that \( u \) is the Poisson integral \( P[\mu] \). Let \( \mu = \mu_+ + \mu_0 \) be the Lebesgue decomposition of \( \mu \) (here \( \mu_+ \perp \sigma \) and \( \mu_0 \ll \sigma \)). Also, let \( \mu_+ = \mu + \mu_- \), \( \mu \geq 0 \), be the Jordan decomposition of \( \mu_+ \). We have \( \mu_+ \ll \mu_- \); thus \( \mu_+ \) is the singular part of a measure \( \nu \) such that the Poisson integral \( P[\nu] \) is a pluriharmonic function in the ball ([2], Corollary 2.7). In other words, there exists a function \( g \in L^1(\sigma) \) such that \( P[g\sigma - \mu_+] \) is pluriharmonic. Denote by \( F_- \) the holomorphic function such that \( \text{Re } F_- = P[g\sigma - \mu_+] \) and \( F_-(0) \in \mathbb{R} \). Respectively, the Poisson integral of the measure \( \mu_+ + g\sigma - \mu_- = \mu - (\mu_+ - g\sigma) \) is a pluriharmonic function. So let \( F_+ \) be the holomorphic function such that \( \text{Re } F_+ = P[\mu_+ + g\sigma - \mu_-] \) and \( F_+(0) \in \mathbb{R} \).

Clearly, the definitions of \( F_- \) and \( F_+ \) yield the identity \( F = \exp(F_-) \). Put \( f_- = \exp F_- \). Then \( \log |(f_-)_r| = P_r[g\sigma - \mu_+] \leq P_r[g\sigma] \) since \( \mu_+ \) is a nonnegative measure. So \( \log^+ |(f_-)_r| \leq P_r[g\sigma] \), hence, the family \( \{\log^+ |(f_-)_r|\}_{0<r<1} \) is uniformly integrable. In other words \( f_- \in N^+(B) \).

Analogously \( \exp F_+ \in N^+(B) \), and we put \( f_+ = f_0 \exp F_+ \).

Observe that even a nonvanishing Nevanlinna function is not necessarily a quotient \( f_+/f_- \), where \( f_\pm \in H^\infty(B) \) and \( f_- \) has no zeros in the ball. Moreover, one has the following negative example.

Proposition 2. There exists \( f \in N(B) \) such that \( f(z) \neq 0 \) for all \( z \in B \) and \( f \) cannot be represented as \( f_+/f_- \), where \( f_+, f_- \in N^+(B) \), \( f_- \in H^\infty(B) \) and \( f_- \) has no zeros in the ball.

Proof. Fix a point \( \zeta \in S \). Let \( m \) be the normalized Lebesgue measure on the circle \( T_\zeta = \{\lambda \zeta : \lambda \in T\} \subset S \). Choose a strictly positive lower semicontinuous function \( g \in L^1(\sigma) \) such that

\[
\sup_{0<r<1} \int_S P_r[m]g \, d\sigma = +\infty.
\]

Since \( g \) is lower semicontinuous, there exists a singular positive measure \( \mu \) on \( S \) such that \( P[g\sigma - \mu] = \text{Re } F \) for certain \( F \in H(B) \) (see [1], Chapter 5, 4.2). Put \( f = \exp(-F) \in N(B) \).

Now, suppose \( f = f_+/f_- \), where \( f_\pm \) are nonvanishing, \( f_+ \in N^+(B) \), and \( f_- \in H^\infty(B) \). Since \( f_\pm \) are zero-free, there exist measures \( \nu_\pm \in M(S) \) such that \(- \log |f_\pm| = P[\nu_\pm] \). Without loss of generality, let \( \log |f_-| \leq 0 \), in other words \( \nu_- \geq 0 \). Let \( (\nu_\pm)_s \) be the singular parts of \( \nu_\pm \); then \( \mu = (\nu_-)_s - (\nu_+)_s \). Recall that
f_+ \in N^+(B)$, so $(\nu_+)_s \geq 0$. Thus $\nu_- \geq (\nu_-)_s \geq \mu$. Therefore $g\sigma \leq g\sigma + \nu_- - \mu$ and the Poisson integral $u = P[(g\sigma - \mu) + \nu_-]$ is pluriharmonic, hence

$$u(0) = \sup_{0 < r < 1} \int_S u_r \, dm \geq \sup_{0 < r < 1} \int_S P_r[g\sigma] \, dm = +\infty.$$ 

A contradiction.

**References**


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