NEVANLINNA FUNCTIONS AS QUOTIENTS

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Abstract. Let \( f \) be a holomorphic function in the unit ball. Then \( f \) is a Nevanlinna function if and only if there exist Smirnov functions \( f_+, f_- \) such that \( f = f_+/f_- \) and \( f_- \) has no zeros in the ball.

Let \( B = B_n \) be the open unit ball in \( \mathbb{C}^n \) and \( S = \partial B \) be the unit sphere. If \( n = 1 \), then \( \mathbb{D} = B_1 \) is the open unit disc in \( \mathbb{C} \).

The Nevanlinna class \( N(B) \) is the set of all holomorphic functions \( f \) on \( B \) such that

\[
\sup_{0 < r < 1} \int_S \log^+ |f_r| \, d\sigma < +\infty,
\]

where \( \sigma \) is the normalized Lebesgue measure on \( S \), \( f_r(\zeta) = f(r\zeta) \), and \( x^+ = \max(x, 0) \).

The Smirnov class \( N^+(B) \) is the space of all functions \( f \in N(B) \) such that the family \( \{\log^+ |f_r|\}_{0 < r < 1} \) is uniformly integrable on \( S \). Namely, for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
\int_{\Delta} \log^+ |f_r| \, d\sigma < \varepsilon
\]

for all \( 0 < r < 1 \) and all sets \( \Delta \subset S \) such that \( \sigma(\Delta) < \delta \).

Let \( f \) be a holomorphic function on the disc. The classical factorization theorem implies that \( f \in N(\mathbb{D}) \) if and only if \( f = f_+/f_- \), where \( f_+ \) and \( f_- \) are holomorphic and bounded by 1 on \( \mathbb{D} \) and \( f_- \) has no zeros on \( \mathbb{D} \). In particular, the functions \( f \) and \( f_+ \) have the same zero sets.

Now, assume \( n \geq 2 \). Then there are many ways to say that the factorization theorem does not hold in the ball \( B_n \). For example, there exist many functions \( f \in N(B), \ f \neq 0 \), with the following property: If \( f_+ \in H^\infty(B) \) satisfies \( f_+/f \) is holomorphic, then \( f_+ \equiv 0 \). Hence, the above characterization does not hold for \( n \geq 2 \). Moreover, the zero sets of the Hardy classes \( H^p(B) \) are all different (see [5]). One more negative result in this direction is the following nonfactorization theorem: The set \( \{gh : g, h \in H^2(B)\} \) is a set of first category in \( H^1(B) \) (see [6]).

However, in the present note we obtain the following description of the Nevanlinna class in the ball.
Theorem 1. Let $f$ be a holomorphic function on $B_n$, $n \geq 2$. Then $f \in N(B)$ if and only if there exist $f_+, f_- \in N^+(B)$ such that $f = f_+/f_-$ and $f_-(z) \neq 0$ for all $z \in B$.

Proof. If $f_+, f_- \in N^+(B)$ and $f_+/f_-$ is a holomorphic function, then $f_+/f_-$ is a Nevanlinna function since

$$\sup_{\frac{1}{2} < r < 1} \int_S |\log |(f_-)_r|| \, d\sigma < \infty.$$ 

So assume $f \in N(B)$. The Henkin-Skoda theorem provides a function $f_0 \in N^+(B)$ which has the same zeros (see [4] and [5]). More precisely, $f = f_0F$, where $F \in H(B)$ and $F(z) \neq 0$ for all $z \in B$. Without loss of generality $F(0) \in \mathbb{R}$.

Observe that $F \in N(B)$. Therefore $u = \log |F|$ is a pluriharmonic function and

$$\sup_{0 < r < 1} \int_S |u_r| \, d\sigma < \infty.$$ 

Hence, there exists a real measure $\mu$ on the sphere such that $u$ is the Poisson integral $P[\mu]$. Let $\mu = \mu_s + \mu_0$ be the Lebesgue decomposition of $\mu$ (here $\mu_s \perp \sigma$ and $\mu_0 \ll \sigma$). Also, let $\mu_s = \mu_+ - \mu_-$, $\mu_0 \geq 0$, be the Jordan decomposition of $\mu_s$. We have $\mu_+ \ll \mu_-$; thus $\mu_+$ is the singular part of a measure $\nu$ such that the Poisson integral $P[\nu]$ is a pluriharmonic function in the ball ([2], Corollary 2.7). In other words, there exists a function $g \in L^1(\sigma)$ such that $P(g\sigma - \mu_+)$ is pluriharmonic. Denote by $F_-$ the holomorphic function such that $\text{Re}F_- = P(g\sigma - \mu_+)$ and $F_-(0) \in \mathbb{R}$. Respectively, the Poisson integral of the measure $\mu_s + g\sigma - \mu_- = \mu - (\mu_+ - g\sigma)$ is a pluriharmonic function. So let $F_+$ be the holomorphic function such that $\text{Re}F_+ = P(\mu_s + g\sigma - \mu_-)$ and $F_+(0) \in \mathbb{R}$.

Clearly, the definitions of $F_-$ and $F_+$ yield the identity $F = \exp F_+ \exp (-F_-)$. Put $f_- = \exp F_-$. Then $\log |(f_-)_r| = P[r(g\sigma - \mu_+)] \leq P[r(g\sigma)]$ since $\mu_+$ is a non-negative measure. So $\log^+ |(f_-)_r| \leq P[r(g\sigma)]$, hence, the family $\{\log^+ |(f_-)_r|\}_{0 < r < 1}$ is uniformly integrable. In other words $f_- \in N^+(B)$.

Analogously $\exp F_+ \in N^+(B)$, and we put $f_+ = f_0 \exp F_+$. $\square$

Observe that even a nonvanishing Nevanlinna function is not necessarily a quotient $f_+/f_-$, where $f_\pm \in H^\infty(B)$ and $f_-$ has no zeros in the ball. Moreover, one has the following negative example.

Proposition 2. There exists $f \in N(B)$ such that $f(z) \neq 0$ for all $z \in B$ and $f$ cannot be represented as $f_+/f_-$, where $f_+, f_- \in N^+(B)$, $f_- \in H^\infty(B)$ and $f_-$ has no zeros in the ball.

Proof. Fix a point $z \in S$. Let $m$ be the normalized Lebesgue measure on the circle $T_z = \{\lambda z : \lambda \in T\} \subset S$. Choose a strictly positive lower semicontinuous function $g \in L^1(\sigma)$ such that

$$\sup_{0 < r < 1} \int_S P[r]g \, d\sigma = +\infty.$$ 

Since $g$ is lower semicontinuous, there exists a singular positive measure $\mu$ on $S$ such that $P[g\sigma - \mu] = \text{Re} F$ for certain $F \in H(B)$ (see [1], Chapter 5, 4.2). Put $f = \exp (-F) \in N(B)$.

Now, suppose $f = f_+/f_-$, where $f_\pm$ are nonvanishing, $f_+ \in N^+(B)$, and $f_- \in H^\infty(B)$. Since $f_\pm$ are zero-free, there exist measures $\nu_\pm \in M(S)$ such that $-\log |f_\pm| = P[\nu_\pm]$. Without loss of generality, let $\log |f_-| \leq 0$, in other words $\nu_- \geq 0$. Let $(\nu_\pm)_s$ be the singular parts of $\nu_\pm$; then $\mu = (\nu_-)_s - (\nu_+)_s$. Recall that
$f_+ \in N^+(B)$, so $(\nu_+)_s \geq 0$. Thus $\nu_- \geq (\nu_-)_s \geq \mu$. Therefore $g \sigma \leq g \sigma + \nu_- - \mu$ and the Poisson integral $u = P[(g \sigma - \mu) + \nu_-]$ is pluriharmonic, hence

$$u(0) = \sup_{0 < r < 1} \int_S u_r \, dm \geq \sup_{0 < r < 1} \int_S P_r[g \sigma] \, dm = +\infty.$$  

A contradiction.

REFERENCES


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