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DISTINCT SUBSET SUMS AND AN INEQUALITY FOR CONVEX FUNCTIONS

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Abstract. In this note we prove an inequality for convex functions which implies a conjecture of P. Erdős about a finite integer set with distinct subset sums.

Let $a_0 < a_1 < \cdots < a_n$ be positive integers with sums $\sum_{i=0}^{n} \epsilon_i a_i$ ($\epsilon_i = 0, 1$) different. P. Erdős conjectured that

$$\sum_{i=0}^{n} \frac{1}{a_i} \leq \sum_{i=0}^{n} \frac{1}{2^i}.$$

This conjecture was proved by Ryavec (see [1]). Several different proofs are given in [4] by A. Bruen and D. Borwein, O.P. Lőssers, M. Edelstein and Esther Szekeres. Hanson, Steele and Stenger [3] proved that

$$\sum_{i=0}^{n} \frac{1}{a_i^{\alpha}} \leq \sum_{i=0}^{n} \frac{1}{2^{\alpha i}}$$

for all $\alpha > 0$. Frenkel [2] further improved this by proving

$$\sum_{i=0}^{n} f(a_i) \leq \sum_{i=0}^{n} f(2^i) \tag{1}$$

for any convex decreasing function. In this note we prove an inequality for convex functions. (1) is a special case of the inequality.

Theorem. *Let f be any given convex decreasing function on $[A, B]$ and $\alpha_0, \alpha_1, \cdots, \alpha_n, \beta_0, \beta_1, \cdots, \beta_n$ real numbers in $[A, B]$ with*

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n, \quad \sum_{i=0}^{k} \alpha_i \geq \sum_{i=0}^{k} \beta_i, \quad k = 0, 1, \cdots, n.$$

Then

$$\sum_{i=0}^{n} f(\alpha_i) \leq \sum_{i=0}^{n} f(\beta_i).$$

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Proof. We transform $\alpha_0, \alpha_1, \cdots, \alpha_n$ into $\alpha'_0, \alpha'_1, \cdots, \alpha'_n$, and then use induction on n. The case $n = 0$ is clear. Let

$$\delta = \min\{\frac{1}{n}(B - \alpha_n), \ \frac{1}{j+1}\left(\sum_{i=0}^{j}\alpha_i - \sum_{i=0}^{j}\beta_i\right) \ : \ j = 0, 1, \cdots, n-1\}$$

and $\alpha'_i = \alpha_i - \delta (i = 0, 1, \cdots, n-1)$, $\alpha'_n = \alpha_n + n\delta$. Then $\alpha'_0, \alpha'_1, \cdots, \alpha'_n$ satisfies the condition of the theorem, and either $\alpha'_n = B$ (in this case we say that $j_0 = n-1$) or $\sum_{i=0}^{j_0}\alpha'_i = \sum_{i=0}^{j_0}\beta_i$ for some j_0, $0 \le j_0 \le n-1$. Thus both $\alpha'_0, \alpha'_1, \cdots, \alpha'_{j_0}; \beta_0, \beta_1, \cdots, \beta_{j_0}$ and $\alpha'_{j_0+1}, \cdots, \alpha'_n; \beta_{j_0+1}, \cdots, \beta_n$ satisfy the condition of the theorem. By the induction hypothesis we have

$$\sum_{i=0}^{j_0} f(\alpha'_i) \le \sum_{i=0}^{j_0} f(\beta_i), \qquad \sum_{i=j_0+1}^{n} f(\alpha'_i) \le \sum_{i=j_0+1}^{n} f(\beta_i).$$

Hence

$$\sum_{i=0}^{n} f(\alpha'_i) \le \sum_{i=0}^{n} f(\beta_i).$$

Further,

$$\sum_{i=0}^{n} f(\alpha_i) \le \sum_{i=0}^{n} f(\alpha'_i) = f(\alpha_0 - \delta) + \cdots + f(\alpha_{n-1} - \delta) + f(\alpha_n + n\delta)$$

follows from the property of the convex function

$$f(\alpha - \gamma) + f(\beta + \gamma) \ge f(\alpha) + f(\beta)$$

where $A \le \alpha - \gamma \le \alpha \le \beta \le \beta + \gamma \le B$. This completes the proof. □

Remark. By taking $g(x) = -f(x), -f(B - x)$ and $f(B - x)$ we may derive similar inequalities for concave increasing, concave decreasing and convex increasing functions respectively.

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