GENERICITY OF THE $K$-PROPERTY FOR A CLASS OF TRANSFORMATIONS

ZBIGNIEW S. KOWALSKI AND PIERRE LIARDET

(Communicated by Michael Handel)

ABSTRACT. We exhibit a class of skew products over Bernoulli shifts for which
the $K$-property is generic.

1. Introduction

Skew products over Bernoulli shifts have been considered in many papers. The
particular case of isometric extensions of a two-sided Bernoulli shift with finite
entropy was studied by D. Rudolph [Ru] who showed that such an extension is
Bernoulli as soon as it is mixing. It appears that various results can be obtained if
we only consider one-sided Bernoulli shifts or if we deal with non-isometric exten-
sions. There are examples due to W. Parry [Pa] for which isometric extensions of a
one-sided Bernoulli shift are isomorphic to the underlying shift. Meilijson [Me]
investigated the case of non-isometric extensions over a Bernoulli automorphism
and showed that total ergodicity of so-called power extensions implies the $K$-property.
In this paper we introduce a class of skew products which are non-isometric exten-
sions of one-sided Bernoulli shifts with finite entropy. These transformations are
non-invertible, measure preserving with finite entropy and without any one-sided
generator. Such transformations were first introduced in [Ko1]. Our main focus will
be on the $K$-property. We exhibit dense $G_\delta$-sets in some compact metrizable spaces
of transformations for which this latter property holds. To this end we introduce a
spectral criterion which implies total ergodicity. Finally, an application to uniform
distribution modulo one is given.

2. A family of skew products

In the sequel $a$, $b$ and $p$ will be fixed numbers in $]0, 1[$ with $b > p$. We introduce
the set $G_{a,b}$ of continuous maps $g : [0, 1] \to [0, 1]$ satisfying the following properties:

(i) $g(0) = 0$, $g(1) = 1$;
(ii) $g(x) \leq x$ for any $x \in [0, 1]$;
(iii) for all $(x, y) \in [0, 1]^2$:

$$x \neq y \implies a \leq \frac{g(y) - g(x)}{y - x} \leq \frac{1}{b}.$$
Given a probability \( p \), denote the usual one-sided shift operator. The unit interval \([0, 1]\) is compact and continuous with respect to \( B \)-algebra on \( \Omega \) and the Bernoulli shift \( S \). Given a probability \( \mu \) on \([0, 1]\) by \( \mu(\{0\}) = p \), \( 0 < p < 1 \), we associate the infinite product measure \( \nu = \mu^\infty \) on \( \Omega \) and the Bernoulli shift \( B(\mu) = (\Omega, S, B_\Omega, \nu) \) where \( S \) denotes the usual one-sided shift operator. The unit interval \([0, 1]\) will be endowed with its Borel \( \sigma \)-algebra \( B_{[0,1]} \) and the Lebesgue measure \( \lambda \).

For every fixed homeomorphism \( g \) of \([0, 1]\) such that \( g(0) = 0 \) we define the map \( g_* : [0,1] \to [0,1] \) by the equality
\[
pg + (1 - p)g_* = Id.
\]
If \( g \) is assumed to satisfy (iii), then similar inequalities hold for \( g_* \), namely
\[
(iii.) \quad a_* \leq \frac{g_*(y) - g_*(x)}{y - x} \leq \frac{1}{b_*}
\]
for every \( x, 0 \leq x < y \leq 1 \), with \( a_* = \frac{b - p}{n(1 - p)} \) and \( b_* = \frac{1 - p}{1 - ap} \) (\( > p \)).

**Remark 1.** In general, \( g_* \) is not necessarily one-to-one but if we assume that \( g \) belongs to \( G_{a,b} \), then both \( g \) and \( g_* \) are homeomorphisms of \([0,1]\).

**Remark 2.** For every homeomorphism \( g \in G_{a,b} \) and every subinterval \( J \) of \([0,1]\), the equality
\[
\lambda(J) = p\lambda(g(J)) + (1 - p)\lambda(g_*(J))
\]
is an immediate consequence of the definition of \( g_* \). In other words,
\[
\lambda(dx) = p\rho_g(dx) + (1 - p)\rho_{g_*}(dx)
\]
where \( \rho_g \) (resp. \( \rho_{g_*} \)) is the Stieljes measure whose \( g \) (resp. \( g_* \)) is the corresponding distribution function. In particular
\[
\rho_g(\varphi) = \int \varphi(g^{-1})(x)\lambda(dx)
\]
for all continuous maps \( \varphi : [0,1] \to \mathbb{R} \). It follows that \( \rho_g \) and \( \rho_{g_*} \) are absolutely continuous with respect to \( \lambda \) (and in fact equivalent to \( \lambda \)). Therefore, by the Radon-Nikodym Theorem, there exist non-negative functions \( \gamma_g \) and \( \gamma_{g_*} \) in \( L^1(\lambda) \) such that \( \rho_g(dx) = \gamma_g(x)\lambda(dx) \) and \( \rho_{g_*}(dx) = \gamma_{g_*}(x)\lambda(dx) \) (equality being considered up to \( \lambda \)-null sets).
For every fixed $g \in G_{a,b}$, we consider the skew product $\Sigma_g = (\Omega \times [0,1], S_g, B_\Omega \otimes B_{[0,1]}, \nu \otimes \lambda)$ over $B(\mu)$, where $S_g$ is the transformation defined by
\[
S_g(w,x) = (Sw, T_{w_0}(x))
\]
with $T_0 = g^{-1}$ and $T_1 = g_1^{-1}$. It follows from (2) that $S_g$ preserves the product measure $\sigma = \nu \otimes \lambda$. Such dynamical systems were considered by Kowalski in [Ko1] and our aim is to show that the set of maps $g$ in $G_{a,b}$ such that $\Sigma_g$ is totally ergodic, contains a dense $G_\delta$ set. We first need the following weaker result:

**Proposition 2.** The set of $g \in G_{a,b}$ such that $\Sigma_g$ is totally ergodic contains a $G_\delta$-set with respect to the uniform topology on $G_{a,b}$.

The proof will be done in three steps and will use the spectral characterization of total ergodicity. Let $T = (Y, T, \mathcal{B}, m)$ be a dynamical system and let $f \in L^2(m)$. By the Bochner-Herglotz Theorem, there exists a unique Borel measure $m_f$ (called the spectral measure of $f$ with respect to $T$) on the torus $\mathbb{R}/\mathbb{Z}$, whose Fourier coefficients are given by
\[
\hat{m}_f(k) = (f \circ T^r, f \circ T^s)
\]
where $k = r - s$. It is well known that the ergodicity of $T$ is equivalent to
\[
\forall f \in L^2(m) : (f, 1) = 0 \implies m_f(\{0\}) = 0. \quad (4)
\]

**Step 1.** We use the above notations with $T = \Sigma_g$ and $g$ in $G_{a,b}$. Let $\mathcal{M}$ denote the Banach space of signed Borel measures equipped with the total variation norm. The following pair of assertions are well known: the map $f \mapsto \sigma_f$ from $L^2(\sigma)$ to $\mathcal{M}$ is continuous and, for every two functions $f_1$, $f_2$ in $L^2(\sigma)$, the spectral measure $\sigma_{f_1 + f_2}$ is absolutely continuous with respect to $\sigma_{f_1}$ + $\sigma_{f_2}$ (see [Qu] for example). Therefore, in order to prove property (4) we only have to prove it for a family $F$ of functions which generates a dense subspace of the hyperplane $L^2(\sigma)^o = \{ f \in L^2(\sigma) : (f, 1) = 0 \}$. For every fixed non-negative integer $n$, let $\psi_n : \Omega \to \mathcal{R}$ be the map defined by
\[
\psi_n(w) = \sum_{k=0}^{\infty} (-1)^{\varepsilon_k(n)}w_k \left( \frac{p\omega_k + (1-p)(1-\omega_k)}{\sqrt{p(1-p)}} \right)^{\varepsilon_k(n)}
\]
where $n = \sum_{k \geq 0} \varepsilon_k(n)2^k$ is the usual expansion of $n$ in base 2. The family $\{\psi_n : n \in \mathbb{N}\}$ is an orthonormal basis in $L^2(\nu)$ and consequently the family
\[
F = \{ \psi_n \otimes u : n \geq 1 \& u \in L^2(\lambda) \} \cup \{ 1 \otimes u : u \in L^2(\lambda) \& (u, 1) = 0 \}
\]
spans a dense subspace of $L^2(\sigma)^o$.

For every $f = \psi_n \otimes u$ in $F$ with $n \geq 1$ and for every integer $k > \log_2 n$, a straightforward computation shows that
\[
\hat{\sigma}_f(k) = \left( \int_{\Omega} \psi_n \otimes u \right) \sum_{w_0, \ldots, w_{k-1}} p_{w_0} \cdots p_{w_{k-1}} \psi_n(w_0, \ldots, w_{k-1}) \int_0^1 u(T_{w_{k-1}} \cdots T_{w_0} y)u(y)\lambda(dy)
\]
\[
= 0
\]
where the sum runs over all binary strings of length $k$ and $p_0 = p$, $p_1 = 1 - p$. This means that $\sigma_f$ is absolutely continuous with respect to the Haar measure of $\mathbb{R}/\mathbb{Z}$. In particular
\[
\forall n \geq 1, \quad \sigma_{\psi_n \otimes u}(\{0\}) = 0. \quad (5)
\]
Now we claim that

**Lemma 1.** For all $g \in G_{a,b}$ the transformation $\Sigma_g$ is ergodic if and only if $\sigma_{1 \otimes u}(\{0\}) = 0$ for all $u \in L^2(\lambda)$ such that $(u, 1) = 0$.

In fact, from (5), property (4) holds if $\sigma_{1 \otimes u}(\{0\}) = 0$ for every $u \in L^2(\lambda)$ with $(u, 1) = 0$; the ergodicity of $\Sigma_g$ follows. The converse is obvious.

As a by-product we obtain

**Proposition 3.** Assume $g \in G_{a,b}$. Then $f \in L^2(\sigma)$ is invariant under $S_g$ if and only if there exists $u \in L^2(\lambda)$ such that $f = 1 \otimes u$ $\sigma$-a.e. and $u \circ g^{-1} = u \circ g^{-1} = u$ $\lambda$-a.e.

**Proof.** For every fixed $f \in L^2(\sigma)$ there exists a sequence $(u_n)_n$ in $L^2(\lambda)$ such that $f = \sum_{n=0}^{\infty} \psi_n \otimes u_n$, the series being convergent in $L^2$. Assume that $f$ is invariant under $S_g$. Then $\sigma_f = \|f\| \delta_0$ where $\delta_0$ stands for the Dirac probability on $\mathbb{R}/\mathbb{Z}$ at the origin. But $\sigma_f(\{0\}) = \sigma_{1 \otimes u_0}(\{0\})$ from (5). Consequently $\sigma_f = \sigma_{1 \otimes u_0}$ and $u_n = 0$ for all $n \geq 1$. Hence $f = 1 \otimes u_0$ with $u_0 \circ g^{-1} = u_0 \circ g^{-1} = u_0$ $\lambda$-a.e. The converse is obvious. \hfill \square

**Step 2.** Given a Borel measure $\tau$ on $\mathbb{R}/\mathbb{Z}$ we introduce the property

$$(E) \quad \forall n \in \mathbb{N} \setminus \{0\}, \exists L_n, \max_{L_n \leq k < L_n + n} |\hat{\tau}(k)| < \frac{1}{n}.$$ 

This definition has the following application:

**Lemma 2.** Let $\mathcal{T} = (Y, T, \mathcal{B}, m)$ be any dynamical system and assume that for every function $f$ in $L^2(m)$ such that $(f, 1) = 0$ the property (E) holds with $\tau = m_f$. Then $\mathcal{T}$ is totally ergodic.

**Proof.** We use notations in (E) with $\tau = m_f$. An immediate application of the Lebesgue’s dominated convergence theorem gives

$$\lim_{n \to \infty} \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{n} \sum_{0 \leq k < n} e^{2\pi i (L_n + k)t} m_f(dt) = m_f(\{0\}).$$

On the other hand, Property (E) implies

$$|\int_{\mathbb{R}/\mathbb{Z}} \frac{1}{n} \sum_{0 \leq k < n} e^{2\pi i (L_n + k)t} m_f(dt)| \leq \frac{1}{n}$$

and passing to the limit, we get $m_f(\{0\}) = 0$. Therefore, $\mathcal{T}$ is ergodic. Due to the obvious fact that the Fourier coefficient at $k, k \in \mathbb{Z}$, of the spectral measure of $f$ with respect to $T^n$ is $m_f(nk)$, a similar argument shows that $T^n$ is also ergodic. This proves Lemma 2. \hfill \square

**Step 3.** In this last step, we assume that $g$ belongs to $G_{a,b}$ and we denote by $\sigma_q$ the spectral measure of the function $(w, y) \mapsto e^{2\pi i q y}$ $(q \in \mathbb{Z})$ with respect to $S_g$. For $n \geq 1$ given, let us introduce the subset

$$\Gamma(q, n, L) = \{ g \in G_{a,b}; |\hat{\sigma}_q(L + k)| < \frac{1}{n} \text{ for } 0 \leq k < n \}.$$ 

The next lemma will be useful to prove that the set $\Gamma(q, n, L)$ is open in $G_{a,b}$. 


License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Lemma 3. Let $g$ and $h$ be elements of $G_{a,b}$ and put $g_0 = g$, $h_0 = h$, $g_1 = g_*$, $h_1 = h_*$. Then, for every binary string $w_0 \ldots w_{k-1}$ of length $k$, one has

$$
\|g_{w_{k-1}}^{-1} \ldots g_{w_0}^{-1} - h_{w_{k-1}}^{-1} \ldots h_{w_0}^{-1}\|_\infty \leq d \left( \frac{d^k - 1}{d - 1} \right) \|g - h\|_\infty
$$

where $d = \max \{ \frac{1}{a}, \frac{b}{b-p}, \frac{b(1-p)}{b-p} \}$. 

Proof. We prove the assertion by induction on $k$. The case $k = 1$ follows immediately from just properties (iii) or (iii$^*$) and the fact that $p(g-h) + (1-p)(g_*-h_*) = 0$. Now assume that inequality (6) holds for a given $k \geq 1$, so we have successively

$$
\|g_{w_{k-1}}^{-1} \ldots g_{w_0}^{-1} - h_{w_{k-1}}^{-1} \ldots h_{w_0}^{-1}\|_\infty \leq \|g_{w_{k-1}}^{-1} \circ (g_{w_{k-2}}^{-1} \ldots g_{w_0}^{-1}) - h_{w_{k-1}}^{-1} \circ (h_{w_{k-2}}^{-1} \ldots h_{w_0}^{-1})\|_\infty
$$

and by means of a standard argument, (6) holds for every binary string $w_0 \ldots w_{k-1}$ and consider the family $\{ \phi_n \}$ which is clearly a $G_\delta$-set. Let $\chi_q$ denote the exponential map $y \mapsto e^{2i\pi qy}$ defined on $[0,1]$, choose $g \in \mathcal{E}$ and consider the family

$$
F = \{ \psi_n \otimes \chi_q : (n,q) \in \mathbb{N} \times \mathbb{Z} \setminus \{(0,0)\} \}.
$$

For every $f \in F$, the spectral measure $\sigma_f^{(g)}$ satisfies (E). This is clear by the choice of $g$ for $f = 1 \otimes \chi_q$. Otherwise, $f = \psi_n \otimes \chi_q$ for suitable $n$ and $q$, but in that case $\lim_{k \to \infty} \hat{\sigma}_f^{(g)}(k) = 0$; hence $\sigma_f^{(g)}$ still satisfies (E). Now notice that $F \cup \{1\}$ is an orthonormal basis in $L^2(\sigma)$ and, if $(f_n)$ is a sequence in $L^2(\sigma)$ which converges to $f$ such that all the spectral measures $\sigma_{f_n}^{(g)}$ satisfy property (E), then $\sigma_f^{(g)}$ also satisfies property (E). In fact, $\limsup_{n \to \infty} \sup_{k \in \mathbb{Z}} |\hat{\sigma}_{f_n}^{(g)}(k) - \hat{\sigma}_f^{(g)}(k)| \leq \lim_{n \to \infty} ||\sigma_{f_n}^{(g)} - \sigma_f^{(g)}||_1 = 0$. Here, || || denotes the total variation norm. Lemma 2 finishes the proof of Proposition 2.

Remark 3. There exist many functions $g$ in $G_{a,b}$ such that $S_g$ is not ergodic. For example, if $g$ has a fixed point in $[0,1]$, then $S_g$ cannot be ergodic.
3. Genericity of the $K$-property

Let $\Phi_g : L^1(\sigma) \to L^1(\sigma)$ be the Perron-Frobenius operator associated to $\Sigma_g$. For every $F \in L^1(\sigma)$, the following formula

$$\Phi_g(F)(u, y) = p\gamma_g(y)F(0u, g(y)) + (1 - p)\gamma_{g_*}(y)F(1u, g_*(y))$$

is classical. We denote by $T_g$ the operator defined on the space $L^1([0, 1], B_{[0,1]}, \lambda)$ by

$$T_g(f) = \Phi_g(1 \otimes f).$$

Notice that $T_g$ is doubly stochastic and the proof of the next proposition shows that the mixing property of $\Phi_g$ depends on that of $T_g$.

Proposition 4. For any $g \in G_{a,b}$, ergodicity of $\Sigma_g$ implies strong mixing.

Proof. Let $A : L^1(\lambda) \to L^1(\lambda)$ be the linear positive operator defined by

$$A(h) = ph \circ g + (1 - p)h \circ g_*.$$

If $h \in C^1([0, 1])$, then

$$\frac{d}{dx}Ah = T_g \left( \frac{d}{dx}h \right) \lambda \text{ a.e.}$$

Therefore, we can apply the argument of Theorem 1 in [Ko2] to deduce the strong mixing property of $\Sigma_g$.

Using Theorem 1 in [Ko3] we obtain from Proposition 4 a similar result to the classification of D. Rudolph [Ru]:

Proposition 5. For every $g \in G_{a,b}$, the ergodicity of $\Sigma_g$ implies the $K$-property of $\Sigma_g$.

We are ready to study the genericity of the $K$-property from a topological point of view.

Theorem 1. Assume that $a, b, p \in [0, 1[, p < b$. Then the set of $g$ in $G_{a,b}$ such that $\Sigma_g$ has the $K$-property contains a dense $G_\delta$-set with respect to the uniform topology.

Proof. By Propositions 1 and 5, it is enough to find a dense subset of elements in $G_{a,b}$ such that $\Sigma_g$ is ergodic. Let $D$ be the set of functions $g \in G_{a,b}$ such that there exists $x_0 \in [0, 1[\text{ with the following properties:}$

\begin{itemize}
  \item[(j)] The restriction of $g$ to the interval $[g(x_0), g_*(x_0)]$ is twice continuously differentiable;
  \item[(jj)] $g'(x_0) = 1, g'(x) < 1$ for all $x$ in $[g(x_0), x_0[$ and $g'(x) > 1$ for all $x \in ]x_0, g_*(x_0)];$
  \item[(jjj)] $g(x) < x$ for all $x \in [0, 1[$.
\end{itemize}

It is easily checked that $D$ is dense in $G_{a,b}$ with respect to the uniform topology. Moreover, for every $g \in D$ the skew product $\Sigma_g$ is ergodic. This fact is a direct consequence of the proof of Theorem 3 in [Ko1].

Open problem: In connection with the example of an exact transformation without a one-sided generator with finite entropy given in [KaKoLi], show that the exactness property is generic in $G_{a,b}$.
4. A FAMILY OF UNIFORMLY DISTRIBUTED SEQUENCES

Let \( g_0 = g \) be a homeomorphism of \([0, 1]\) such that \( g_1 = g_* \) (the map defined by 
(1), section 2) is also a homeomorphism of \([0, 1]\) and the sequence \( n \mapsto g_{\omega_n}^{-1} \ldots g_{\omega_0}^{-1} y \)
is well defined for each \( \omega \in \Omega \).

**Theorem 2.** If \( \Sigma_g \) is ergodic, then, for almost every point \( \omega \in \Omega \), the sequence

\[
(7) \quad n \mapsto g_{\omega_n}^{-1} \ldots g_{\omega_0}^{-1} y
\]
is uniformly distributed in \([0, 1]\) for every \( y \in [0, 1] \).

**Proof.** Let \( s(\omega, y) \) denote the sequence defined by (7). Assume that for \( \omega \in \Omega \) there exist \( u \) and \( v \) in \([0, 1]\), \( u < v \), such that both sequences \( s(\omega, u) \) and \( s(\omega, v) \) are uniformly distributed. Now, the monotonicity of \( g \) and \( g_* \) implies that, for every \( t \in [0, 1] \) and every \( y \in [u, v] \), one has

\[
1_{[0, t]}(s(\omega, v)_n) \leq 1_{[0, t]}(s(\omega, y)_n) \leq 1_{[0, t]}(s(\omega, u)_n).
\]

By our assumption on \( u \) and \( v \) we obtain

\[
\forall t \in [0, 1], \quad \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} 1_{[0, t]}(s(\omega, y)_n) = t.
\]

This means that the sequence \( s(\omega, y) \) is uniformly distributed in \([0, 1]\). But, because \( S_g \) is ergodic, for almost all \( \omega \), the set \( I(\omega) \) of elements \( x \) in \([0, 1]\) such that \( s(\omega, x) \)
is uniformly distributed in \([0, 1]\), has measure 1. Therefore, the above result shows that in fact \( I(\omega) = [0, 1] \).

**Remark 4.** The uniformly distributed sequence \( s(\omega, y) \) in Theorem 2 is not completely uniformly distributed (see [KuNi] for the definition). In fact, for every given continuous function \( f : [0, 1] \times [0, 1] \to \mathbb{C} \), a straightforward computation shows that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(s(\omega, y)_n, s(\omega, y)_{n+1})
\]

\[
= p \int_0^1 f(x, g^{-1}x) \lambda(dx) + (1 - p) \int_0^1 f(x, g_*^{-1}x) \lambda(dx)
\]

for all \( y \in [0, 1] \) and almost all \( \omega \). In particular, for such \( \omega \) the sequence

\[
n \mapsto (s(\omega, y)_n, s(\omega, y)_{n+1})
\]
is distributed according to a measure carried by the union of the graphs of \( g^{-1} \) and \( g_*^{-1} \).

**References**


[Pa] W. Parry, Automorphisms of the Bernoulli endomorphism and a class of skew-products, Ergodic theory and dynamical systems 16 (1996), 519-530. [MR 97h:28006]


Institute of Mathematics, Wrocław University of Technology, Wybrzeże St. Wyspiańskiego 27, 50–370 Wrocław, Poland
E-mail address: kowalski@im.pwr.wroc.pl

Université de Provence, CMI, 39 rue Joliot-Curie, F-13453 Marseille cedex 13, France
E-mail address: liardet@gyptis.univ-mrs.fr