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GENERICITY OF THE K-PROPERTY FOR A CLASS OF TRANSFORMATIONS

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Abstract. We exhibit a class of skew products over Bernoulli shifts for which the K-property is generic.

1. Introduction

Skew products over Bernoulli shifts have been considered in many papers. The particular case of isometric extensions of a two-sided Bernoulli shift with finite entropy was studied by D. Rudolph [Ru] who showed that such an extension is Bernoulli as soon as it is mixing. It appears that various results can be obtained if we only consider one-sided Bernoulli shifts or if we deal with non-isometric extensions. There are examples due to W. Parry [Pa] for which isometric extensions of a one-sided Bernoulli shift are isomorphic to the underlying shift. Meilijson [Me] investigated the case of non-isometric extensions over a Bernoulli automorphism and showed that total ergodicity of so-called power extensions implies the K-property. In this paper we introduce a class of skew products which are non-isometric extensions of one-sided Bernoulli shifts with finite entropy. These transformations are non-invertible, measure preserving with finite entropy and without any one-sided generator. Such transformations were first introduced in [Ko1]. Our main focus will be on the K-property. We exhibit dense G_δ-sets in some compact metrizable spaces of transformations for which this latter property holds. To this end we introduce a spectral criterion which implies total ergodicity. Finally, an application to uniform distribution modulo one is given.

2. A family of skew products

In the sequel a, b and p will be fixed numbers in $]0,1[$ with $b > p$. We introduce the set $G_{a,b}$ of continuous maps $g : [0,1] \to [0,1]$ satisfying the following properties:

(i) $g(0) = 0$, $g(1) = 1$;
(ii) $g(x) \le x$ for any $x \in [0,1]$;
(iii) for all $(x,y) \in [0,1]^2$:

$$x \neq y \Longrightarrow a \le \frac{g(y) - g(x)}{y - x} \le \frac{1}{b}.$$

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Notice that $G_{a,b}$ contains two particular maps, namely

$$I(x) = \sup_{g \in G_{a,b}} g(x)$$

which is nothing but the identity map, and

$$\Lambda(x) = \inf_{g \in G_{a,b}} g(x) = \begin{cases} ax & \text{if } 0 \le x \le \frac{1-b}{1-ab}; \\ \frac{1}{b}x + \left(1 - \frac{1}{b}\right) & \text{otherwise.} \end{cases}$$

Proposition 1. *The set $G_{a,b}$ is a compact convex subset of the Banach space $\mathcal{C}_{\mathbb{R}}([0,1])$.*

Proof. It is immediate that $G_{a,b}$ is a convex closed subset of $\mathcal{C}_{\mathbb{R}}([0,1])$ and is equicontinuous because of Property (iii). Hence $G_{a,b}$ is compact by the Ascoli Theorem. □

Let Ω be the product space $\{0,1\}^{\mathbb{N}}$ endowed with the usual metrizable compact product topology and the Borel σ-algebra $\mathcal{B}_\Omega$. An element w in Ω will be viewed as an infinite binary string $w_0w_1w_2\dots$ and to every fixed finite binary string $v_0\dots v_n$ we defined the open-closed cylinder set $C_{v_0\dots v_n} = \{w \in \Omega\,;\, w_0\dots w_n = v_0\dots v_n\}$. Given a probability μ on $\{0,1\}$ by $\mu(\{0\}) = p$, $0 < p < 1$, we associate the infinite product measure $\nu = \mu^\infty$ on Ω and the Bernoulli shift $B(\mu) = (\Omega, S, \mathcal{B}_\Omega, \nu)$ where S denotes the usual one-sided shift operator. The unit interval $[0,1]$ will be endowed with its Borel σ-algebra $\mathcal{B}_{[0,1]}$ and the Lebesgue measure λ.

For every fixed homeomorphism g of $[0,1]$ such that $g(0) = 0$ we define the map $g_* : [0,1] \to [0,1]$ by the equality

$$pg + (1-p)g_* = Id\,. \tag{1}$$

If g is assumed to satisfy (iii), then similar inequalities hold for g_*, namely

$$a_* \le \frac{g_*(y) - g_*(x)}{y - x} \le \frac{1}{b_*} \tag{iii$_*$}$$

for every x, $0 \le x < y \le 1$, with $a_* = \frac{b-p}{b(1-p)}$ and $b_* = \frac{1-p}{1-ap}$ $(> p)$.

Remark 1. In general, g_* is not necessarily one-to-one but if we assume that g belongs to $G_{a,b}$, then both g and g_* are homeomorphisms of $[0,1]$.

Remark 2. For every homeomorphism $g \in G_{a,b}$ and every subinterval J of $[0,1]$, the equality

$$\lambda(J) = p\lambda(g(J)) + (1-p)\lambda(g_*(J)) \tag{2}$$

is an immediate consequence of the definition of g_*. In other words,

$$\lambda(dx) = p\rho_g(dx) + (1-p)\rho_{g_*}(dx)$$

where ρ_g (resp. ρ_{g_*}) is the Stieljes measure whose g (resp. g_*) is the corresponding distribution function. In particular

$$\rho_g(\varphi) = \int \varphi(g^{-1})(x)\lambda(dx) \tag{3}$$

for all continuous maps $\varphi : [0,1] \to \mathbb{R}$. It follows that ρ_g and ρ_{g_*} are absolutely continuous with respect to λ (and in fact equivalent to λ). Therefore, by the Radon-Nikodym Theorem, there exist non-negative functions γ_g and γ_{g_*} in $L^1(\lambda)$ such that $\rho_g(dx) = \gamma_g(x)\lambda(dx)$ and $\rho_{g_*}(dx) = \gamma_{g_*}(x)\lambda(dx)$ (equality being considered up to λ-null sets).

For every fixed $g \in G_{a,b}$, we consider the skew product $\Sigma_g = (\Omega \times [0,1], S_g, \mathcal{B}_\Omega \otimes \mathcal{B}_{[0,1]}, \nu \otimes \lambda)$ over $B(\mu)$, where S_g is the transformation defined by

$$S_g(w,x) = (Sw, T_{w_0}(x))$$

with $T_0 = g^{-1}$ and $T_1 = g_*^{-1}$. It follows from (2) that S_g preserves the product measure $\sigma = \nu \otimes \lambda$. Such dynamical systems were considered by Kowalski in [Ko1] and our aim is to show that the set of maps g in $G_{a,b}$ such that Σ_g is totally ergodic, contains a dense G_δ set. We first need the following weaker result:

Proposition 2. *The set of $g \in G_{a,b}$ such that Σ_g is totally ergodic contains a G_δ-set with respect to the uniform topology on $G_{a,b}$.*

The proof will be done in three steps and will use the spectral characterization of total ergodicity. Let $\mathcal{T} = (Y, T, \mathcal{B}, m)$ be a dynamical system and let $f \in L^2(m)$. By the Bochner-Herglotz Theorem, there exists a unique Borel measure m_f (called the spectral measure of f with respect to $\mathcal{T}$) on the torus $\mathbb{R}/\mathbb{Z}$, whose Fourier coefficients are given by

$$\widehat{m}_f(k) = (f \circ T^r, f \circ T^s)$$

where $k = r - s$. It is well known that the ergodicity of T is equivalent to

$$\forall\, f \in L^2(m) :\ (f,1) = 0 \Longrightarrow m_f(\{0\}) = 0. \tag{4}$$

Step 1. We use the above notations with $\mathcal{T} = \Sigma_g$ and g in $G_{a,b}$. Let $\mathcal{M}$ denote the Banach space of signed Borel measures equipped with the total variation norm. The following pair of assertions are well known: the map $f \mapsto \sigma_f$ from $L^2(\sigma)$ to $\mathcal{M}$ is continuous and, for every two functions f_1, f_2 in $L^2(\sigma)$, the spectral measure $\sigma_{f_1+f_2}$ is absolutely continuous with respect to $\sigma_{f_1} + \sigma_{f_2}$ (see [Qu] for example). Therefore, in order to prove property (4) we only have to prove it for a family F of functions which generates a dense subspace of the hyperplane $L^2(\sigma)^o = \{f \in L^2(\sigma)\,;\,(f,1) = 0\}$. For every fixed non-negative integer n, let $\psi_n : \Omega \to \mathcal{R}$ be the map defined by

$$\psi_n(w) = \prod_{k=0}^{\infty} (-1)^{\varepsilon_k(n) w_k} \left(\frac{p\omega_k + (1-p)(1-\omega_k)}{\sqrt{p(1-p)}} \right)^{\varepsilon_k(n)}$$

where $n = \sum_{k\geq 0} \varepsilon_k(n) 2^k$ is the usual expansion of n in base 2. The family $\{\psi_n\,;\, n \in \mathbb{N}\}$ is an orthonormal basis in $L^2(\nu)$ and consequently the family

$$F = \{\psi_n \otimes u\,;\, n \geq 1 \ \&\ u \in L^2(\lambda)\} \cup \{1 \otimes u\,;\, u \in L^2(\lambda)\ \&\ (u,1) = 0\}$$

spans a dense subspace of $L^2(\sigma)^o$.

For every $f = \psi_n \otimes u$ in F with $n \geq 1$ and for every integer $k > \log_2 n$, a straightforward computation shows that

$$\begin{aligned}\widehat{\sigma}_f(k) &= (\int_\Omega \psi_n d\nu) \sum_{w_0 \ldots w_{k-1}} p_{w_0} \ldots p_{w_{k-1}} \psi_n(w_0, \ldots, w_{k-1}) \int_0^1 u(T_{w_{k-1}} \ldots T_{w_0} y) \overline{u(y)} \lambda(dy) \\ &= 0\end{aligned}$$

where the sum runs over all binary strings of length k and $p_0 = p$, $p_1 = 1 - p$. This means that σ_f is absolutely continuous with respect to the Haar measure of $\mathbb{R}/\mathbb{Z}$. In particular

$$\forall\, n \geq 1\,,\quad \sigma_{\psi_n \otimes u}(\{0\}) = 0\,. \tag{5}$$

Now we claim that

Lemma 1. *For all $g \in G_{a,b}$ the transformation Σ_g is ergodic if and only if $\sigma_{1\otimes u}(\{0\}) = 0$ for all $u \in L^2(\lambda)$ such that $(u, 1) = 0$.*

In fact, from (5), property (4) holds if $\sigma_{1\otimes u}(\{0\}) = 0$ for every $u \in L^2(\lambda)$ with $(u, 1) = 0$; the ergodicity of Σ_g follows. The converse is obvious.

As a by-product we obtain

Proposition 3. *Assume $g \in G_{a,b}$. Then $f \in L^2(\sigma)$ is invariant under S_g if and only if there exists $u \in L^2(\lambda)$ such that $f = 1 \otimes u$ σ-a.e. and $u \circ g^{-1} = u \circ g_*^{-1} = u$ λ-a.e.*

Proof. For every fixed $f \in L^2(\sigma)$ there exists a sequence $(u_n)_n$ in $L^2(\lambda)$ such that $f = \sum_{n=0}^{\infty} \psi_n \otimes u_n$, the series being convergent in L^2. Assume that f is invariant under S_g. Then $\sigma_f = ||f||_2\delta_0$ where δ_0 stands for the Dirac probability on $\mathbb{R}/\mathbb{Z}$ at the origin. But $\sigma_f(\{0\}) = \sigma_{1\otimes u_0}(\{0\})$ from (5). Consequently $\sigma_f = \sigma_{1\otimes u_0}$ and $u_n = 0$ for all $n \geq 1$. Hence $f = 1 \otimes u_0$ with $u_0 \circ g^{-1} = u_0 \circ g_*^{-1} = u_0$ λ-a.e. The converse is obvious. □

Step 2. Given a Borel measure τ on $\mathbb{R}/\mathbb{Z}$ we introduce the property

$$(E) \qquad \forall n \in \mathbb{N} \setminus \{0\}\ ,\ \exists\, L_n\ ,\ \max_{L_n \leq k < L_n + n} |\widehat{\tau}(k)| < \frac{1}{n}\,.$$

This definition has the following application:

Lemma 2. *Let $\mathcal{T} = (Y, T, \mathcal{B}, m)$ be any dynamical system and assume that for every function f in $L^2(m)$ such that $(f, 1) = 0$ the property (E) holds with $\tau = m_f$. Then $\mathcal{T}$ is totally ergodic.*

Proof. We use notations in (E) with $\tau = m_f$. An immediate application of the Lebesgue's dominated convergence theorem gives

$$\lim_{n\to\infty} \int_{\mathbb{R}/\mathbb{Z}} \Big(\frac{1}{n} \sum_{0\leq k<n} e^{2i\pi(L_n+k)t}\Big) m_f(dt) = m_f(\{0\}).$$

On the other hand, Property (E) implies

$$\Big| \int_{\mathbb{R}/\mathbb{Z}} \Big(\frac{1}{n} \sum_{0\leq k<n} e^{2i\pi(L_n+k)t}\Big) m_f(dt)\Big| \leq \frac{1}{n}$$

and passing to the limit, we get $m_f(\{0\}) = 0$. Therefore, T is ergodic. Due to the obvious fact that the Fourier coefficient at k, $k \in \mathbb{Z}$, of the spectral measure of f with respect to T^n is $m_f(nk)$, a similar argument shows that T^n is also ergodic. This proves Lemma 2. □

Step 3. In this last step, we assume that g belongs to $G_{a,b}$ and we denote by $\sigma_q^{(g)}$ the spectral measure of the function $(w, y) \mapsto e^{2i\pi qy}$ $(q \in \mathbb{Z})$ with respect to S_g. For $n \geq 1$ given, let us introduce the subset

$$\Gamma(q, n, L) = \{g \in G_{a,b}\,;\ |\widehat{\sigma}_q^{(g)}(L + k)| < \frac{1}{n} \text{ for } 0 \leq k < n\}\,.$$

The next lemma will be useful to prove that the set $\Gamma(q, n, L)$ is open in $G_{a,b}$.

Lemma 3. *Let g and h be elements of $G_{a,b}$ and put $g_0 = g$, $h_0 = h$, $g_1 = g_*$, $h_1 = h_*$. Then, for every binary string $w_0 \dots w_{k-1}$ of length k, one has*

$$||g^{-1}_{w_{k-1}} \dots g^{-1}_{w_0} - h^{-1}_{w_{k-1}} \dots h^{-1}_{w_0}||_\infty \le d\Big(\frac{d^k-1}{d-1}\Big)||g-h||_\infty \tag{6}$$

where $d = \max\{\frac{1}{a}, \frac{bp}{b-p}, \frac{b(1-p)}{b-p}\}$.

Proof. We prove the assertion by induction on k. The case $k = 1$ follows immediately from just properties (iii) or (iii$_*$) and the fact that $p(g-h)+(1-p)(g_*-h_*) = 0$. Now assume that inequality (6) holds for a given $k \ge 1$, so we have successively

$$\begin{aligned} ||g^{-1}_{w_k} \dots g^{-1}_{w_0} - h^{-1}_{w_k} \dots h^{-1}_{w_0}||_\infty &\le ||g^{-1}_{w_k} \circ (g^{-1}_{w_{k-1}} \dots g^{-1}_{w_0}) - h^{-1}_{w_k} \circ (g^{-1}_{w_{k-1}} \dots g^{-1}_{w_0})||_\infty \\ &\quad + ||h^{-1}_{w_k} \circ (g^{-1}_{w_{k-1}} \dots g^{-1}_{w_0}) - h^{-1}_{w_k} \circ (h^{-1}_{w_{k-1}} \dots h^{-1}_{w_0})||_\infty \\ &\le d||g^{-1}_{w_k} - h^{-1}_{w_k}||_\infty + d||g^{-1}_{w_{k-1}} \dots g^{-1}_{w_0} - h^{-1}_{w_{k-1}} \dots h^{-1}_{w_0}||_\infty \\ &\le d\Big(1 + d\big(\frac{d^k-1}{d-1}\big)\Big)||g-h||_\infty \\ &= d\Big(\frac{d^{k+1}-1}{d-1}\Big)||g-h||_\infty. \end{aligned}$$

Hence (6) holds for $k+1$, and the proof is complete. □

The inequality (6) leads to

$$\begin{aligned} |\widehat{\sigma}^{(g)}_q(k) - \widehat{\sigma}^{(h)}_q(k)| &\le \sum_{w_0 \dots w_{k-1}} \int_0^1 |e^{2i\pi q g^{-1}_{w_{k-1}} \dots g^{-1}_{w_0}(y)} - e^{2i\pi q h^{-1}_{w_{k-1}} \dots h^{-1}_{w_0}(y)}|\lambda(dy) \\ &\le 2\pi q d\Big(\frac{d^k-1}{d-1}\Big)||g-h||_\infty \end{aligned}$$

and by means of a standard argument, $\Gamma(q,n,L)$ is open in $G_{a,b}$.

To complete the proof of Proposition 2, let us introduce the set

$$\mathcal{E} = \bigcap_{q \in \mathbb{Z}\setminus\{0\}} \bigcap_{n \ge 1} \bigcap_{\ell \ge 1} \bigcup_{L \ge \ell} \Gamma(q,n,L)$$

which is clearly a G_δ-set. Let χ_q denote the exponential map $y \mapsto e^{2i\pi q y}$ defined on $[0,1]$, choose $g \in \mathcal{E}$ and consider the family

$$F = \{\psi_n \otimes \chi_q\,;(n,q) \in \mathbb{N} \times \mathbb{Z} \setminus \{(0,0)\}\}.$$

For every $f \in F$, the spectral measure $\sigma^{(g)}_f$ satisfies (E). This is clear by the choice of g for $f = 1 \otimes \chi_q$. Otherwise, $f = \psi_n \otimes \chi_q$ for suitable n and q, but in that case $\lim_{k\to\infty} \widehat{\sigma}^{(g)}_f(k) = 0$; hence $\sigma^{(g)}_f$ still satisfies (E). Now notice that $F \cup \{\mathbf{1}\}$ is an orthonormal basis in $L^2(\sigma)$ and, if (f_n) is a sequence in $L^2(\sigma)$ which converges to f such that all the spectral measures $\sigma^{(g)}_{f_n}$ satisfy property (E), then $\sigma^{(g)}_f$ also satisfies property (E). In fact, $\limsup_{n\to\infty} \sup_{k\in\mathbb{Z}} |\widehat{\sigma}^{(g)}_{f_n}(k) - \widehat{\sigma}^{(g)}_f(k)| \le \lim_{n\to\infty} |||\sigma^{(g)}_{f_n} - \sigma^{(g)}_f||| = 0$. Here, $|||\cdot|||$ denotes the total variation norm. Lemma 2 finishes the proof of Proposition 2. □

Remark 3. There exist many functions g in $G_{a,b}$ such that S_g is not ergodic. For example, if g has a fixed point in $]0,1[$, then S_g cannot be ergodic.

3. Genericity of the K-property

Let $\Phi_g : L^1(\sigma) \to L^1(\sigma)$ be the Perron-Frobenius operator associated to Σ_g. For every $F \in L^1(\sigma)$, the following formula

$$\Phi_g(F)(w,y) = p\gamma_g(y)F(0w, g(y)) + (1-p)\gamma_{g_*}(y)F(1w, g_*(y))$$

is classical. We denote by T_g the operator defined on the space $L^1([0,1], \mathcal{B}_{[0,1]}, \lambda)$ by

$$T_g(f) = \Phi_g(1 \otimes f).$$

Notice that T_g is doubly stochastic and the proof of the next proposition shows that the mixing property of Φ_g depends on that of T_g.

Proposition 4. *For any $g \in G_{a,b}$, ergodicity of Σ_g implies strong mixing.*

Proof. Let $A : L^1(\lambda) \to L^1(\lambda)$ be the linear positive operator defined by

$$A(h) = p\, h \circ g + (1-p) h \circ g_*.$$

If $h \in \mathcal{C}^1_{\mathbb{R}}([0,1])$, then

$$\frac{d}{dx} Ah = T_g\Big(\frac{d}{dx} h\Big) \quad \lambda - \text{a.e.}$$

Therefore, we can apply the argument of Theorem 1 in [Ko2] to deduce the strong mixing property of Σ_g. □

Using Theorem 1 in [Ko3] we obtain from Proposition 4 a similar result to the classification of D. Rudolph [Ru]:

Proposition 5. *For every $g \in G_{a,b}$, the ergodicity of Σ_g implies the K-property of Σ_g.*

We are ready to study the genericity of the K-property from a topological point of view.

Theorem 1. *Assume that $a, b, p \in]0,1[, p < b$. Then the set of g in $G_{a,b}$ such that Σ_g has the K-property contains a dense G_δ-set with respect to the uniform topology.*

Proof. By Propositions 1 and 5, it is enough to find a dense subset of elements in $G_{a,b}$ such that Σ_g is ergodic. Let D be the set of functions $g \in G_{a,b}$ such that there exists $x_0 \in]0,1[$ with the following properties:

(j) The restriction of g to the interval $[g(x_0), g_*(x_0)]$ is twice continuously differentiable;
(jj) $g'(x_0) = 1$, $g'(x) < 1$ for all x in $[g(x_0), x_0[$ and $g'(x) > 1$ for all $x \in]x_0, g_*(x_0)]$;
(jjj) $g(x) < x$ for all $x \in]0,1[$.

It is easily checked that D is dense in $G_{a,b}$ with respect to the uniform topology. Moreover, for every $g \in D$ the skew product Σ_g is ergodic. This fact is a direct consequence of the proof of Theorem 3 in [Ko1]. □

Open problem: In connection with the example of an exact transformation without a one-sided generator with finite entropy given in [KaKoLi], show that the exactness property is generic in $G_{a,b}$.

4. A family of uniformly distributed sequences

Let $g_0 = g$ be a homeomorphism of $[0,1]$ such that $g_1 = g_*$ (the map defined by (1), section 2) is also a homeomorphism of $[0,1]$ and the sequence $n \mapsto g_{\omega_n}^{-1} \dots g_{\omega_0}^{-1} y$ is well defined for each $\omega \in \Omega$.

Theorem 2. *If Σ_g is ergodic, then, for almost every point $\omega \in \Omega$, the sequence*

$$n \mapsto g_{\omega_n}^{-1} \dots g_{\omega_0}^{-1} y \tag{7}$$

is uniformly distributed in $[0,1]$ for every $y \in]0,1[$.

Proof. Let $s(\omega, y)$ denote the sequence defined by (7). Assume that for $\omega \in \Omega$ there exist u and v in $]0,1[$, $u < v$, such that both sequences $s(\omega,u)$ and $s(\omega,v)$ are uniformly distributed. Now, the monotonicity of g and g_* implies that, for every $t \in [0,1]$ and every $y \in [u,v]$, one has

$$\mathbf{1}_{[0,t[}(s(\omega,v)_n) \le \mathbf{1}_{[0,t[}(s(\omega,y)_n) \le \mathbf{1}_{[0,t[}(s(\omega,u)_n).$$

By our assumption on u and v we obtain

$$\forall\, t \in [0,1]\,,\ \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_{[0,t[}(s(\omega,y)_n) = t.$$

This means that the sequence $s(\omega,y)$ is uniformly distributed in $[0,1]$. But, because S_g is ergodic, for almost all ω, the set $I(\omega)$ of elements x in $[0,1]$ such that $s(\omega,x)$ is uniformly distributed in $[0,1]$, has measure 1. Therefore, the above result shows that in fact $I(\omega) =]0,1[$. □

Remark 4. The uniformly distributed sequence $s(\omega,y)$ in Theorem 2 is not completely uniformly distributed (see [KuNi] for the definition). In fact, for every given continuous function $f : [0,1] \times [0,1] \to \mathbb{C}$, a straightforward computation shows that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(s(\omega,y)_n, s(\omega,y)_{n+1}) = p \int_0^1 f(x, g^{-1}x)\lambda(dx) + (1-p) \int_0^1 f(x, g_*^{-1}x)\lambda(dx)$$

for all $y \in]0,1[$ and almost all ω. In particular, for such ω the sequence

$$n \mapsto (s(\omega,y)_n, s(\omega,y)_{n+1})$$

is distributed according to a measure carried by the union of the graphs of g^{-1} and g_*^{-1}.

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