CORRECTION TO “THE GHOST OF AN INDEX THEOREM”

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Abstract. The “ghost of an index theorem” is an isomorphism between products of the kernel spaces and the cokernel spaces of a pair of bounded operators and their product, valid when each operator and also the product is assumed to have a generalized inverse. In this note we correct an error in the original proof, and extend the result to operators with closed range.

Suppose $T : X \to Y$ and $S : Y \to Z$ are bounded linear operators between Banach spaces, with generalized inverses $T' : Y \to X$ and $S' : Z \to Y$:

$$\begin{align*}
T &= T'T', \\
S &= SS';
\end{align*}$$

(0.1)

suppose also that the product $ST : X \to Y$ has a generalized inverse $U : Z \to X$:

$$ST = STUST.$$ 

(0.2)

Under these assumptions we have claimed

1. Theorem. There is an isomorphism

$$T^{-1}(0) \times S^{-1}(0) \times Z/ST(X) \cong (ST)^{-1}(0) \times Y/T(X) \times Z/S(Y).$$

(1.1)

In the proof ([1, Theorem 2]) we constructed matrices

$$U = \left( \begin{array}{ccc}
I - T'T & 0 & 0 \\
(I - TUS)T & I - TUS & 0 \\
0 & S(I - TUS) & I - SS'
\end{array} \right)$$

(1.2)

and

$$V = \left( \begin{array}{ccc}
I - T'T & T'(I - S'S) & 0 \\
0 & (I - TT')(I - S'S) & (I - TT'S') \\
0 & 0 & I - SS'
\end{array} \right),$$

(1.3)

having first normalized the generalized inverses of (0.1) and (0.2) so that

$$T' = T'T', \quad S' = S'SS', \quad U = USTU, \quad U = T'VS'.$$

(1.4)

The argument then proceeds by claiming that $U$ and $V$ are mutually generalized inverse, and identifying their ranges with the products of (1.1). Unfortunately $U$ and $V$ may fail to be mutually generalized inverse, and their products need not be diagonal as claimed in [1]. To see how to modify their definition we must go back to the source, which is the “one diagram” proof of the index theorem due to Yang [4].

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The argument of Yang (Theorem) proceeds by the construction of an exact sequence of bounded operators

(1.5) \[ 0 \to W_5 \to W_4 \to W_3 \to W_2 \to W_1 \to W_0 \to 0 \]

where

(1.6) \[
\begin{align*}
W_0 &= T^{-1}(0), & W_1 &= (ST)^{-1}(0), & W_2 &= S^{-1}(0), \\
W_3 &= Y/T(X), & W_4 &= Z/(ST)(X), & W_5 &= Z/S(Y),
\end{align*}
\]

and then the mappings \( A_j \) are natural: for example, \( A_1 \) is an inclusion map and \( A_5 \) is a quotient; \( A_2 \) and \( A_4 \) are induced by \( T \) and \( S \), respectively, and \( A_3 \) comes from the “snake lemma”. Our strategy in [1] is to replace the kernel and cokernel spaces by the originals:

(1.7) \[
\begin{align*}
W_0 &= W_1 = X, & W_2 &= W_3 = Y, & W_4 &= W_5 = Z.
\end{align*}
\]

The mappings \( A_j \) are now more complicated, in that we must define them compatibly on the whole space rather than only on a subspace or a quotient: for example, \( A_1 \) and \( A_5 \) are the projections induced by the generalized inverses of \( T \) and \( S \).

Specifically

(1.8) \[
\begin{align*}
A_1 &= I - T'T, & A_2 &= (I - TUS)T, & A_3 &= (I - TT')(I - S'S), \\
A_4 &= S(I - TUS), & A_5 &= I - SS'.
\end{align*}
\]

Taking account of the normalization (1.4) we now have an example of a regular chain, in the sense that (with a little abus de notation)

(1.9) \[
A_5 A_4 = A_4 A_3 = A_3 A_2 = A_2 A_1 = 0,
\]

and each \( A_j \) has a generalized inverse \( A'_j \):

(1.10) \[
\begin{align*}
A'_1 &= I - T'T, & A'_2 &= T'(I - S'S), & A'_3 &= I - TUS, \\
A'_4 &= (I - TT')S, & A'_5 &= I - SS'.
\end{align*}
\]

The behaviour of such a regular chain is described ([3] (2.5), (2.6)) by the matrices

(1.11) \[
\begin{bmatrix}
A'_1 & 0 & 0 \\
A_2 & A'_3 & 0 \\
0 & A_4 & A'_5
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
A_1 & A'_2 & 0 \\
0 & A_3 & A'_4 \\
0 & 0 & A_5
\end{bmatrix}.
\]

We can now seek to show that \( U \) and \( V \) are mutually generalized inverse, and identify their ranges; but for this to work it is crucial that

(1.12) \[
\begin{align*}
0 \to W_0 \to W_1 \to W_2 \to W_3 \to W_4 \to W_5 \to 0
\end{align*}
\]

is another chain; we need

(1.13) \[
A'_1 A'_2 = A'_2 A'_3 = A'_3 A'_4 = A'_4 A'_5 = 0.
\]

This need not happen; however it is rather easy ([3] page 284) to replace the generalized inverses \( A'_j \) by generalized inverses \( A''_j \) which form a chain:
2. Lemma. If $T = TT'T$ and $S = SS'S$ with $ST = 0$, then

\begin{equation}
T = TT'T, \quad S = SS'S, \quad T''S'' = 0
\end{equation}

where

\begin{equation}
(T'', S'') = (T', (I - TT')S') \text{ or } (T'', S'') = (T'(I - S'T), S').
\end{equation}

Proof. Clear. 

The extension of this procedure from regular chains of length 2 to regular chains of length $n$ is clear: for the chain $A$ of (1.5) we may take

\begin{equation}
A_0 = A_1; A_2 = (I - A_1A_1')A_2', \quad A_3 = (I - A_1A_2')A_3',
\end{equation}

\begin{align*}
A_4' &= (I - A_1A_3'')A_4', \quad A_5'' = (I - A_4A_4')A_5'.
\end{align*}

If we now write

\begin{equation}
U' = \begin{pmatrix} A_1' & 0 & 0 \\ A_2 & A_2'' & 0 \\ 0 & A_4 & A_5'' \end{pmatrix} \quad \text{and} \quad V' = \begin{pmatrix} A_1 & A_2' & 0 \\ 0 & A_3 & A_4' \\ 0 & 0 & A_5 \end{pmatrix},
\end{equation}

then we find (as claimed spuriously in [1]) that $U'V'$ and $V'U'$ are each diagonal, with $U'$ and $V'$ mutually generalized inverse. Thus to salvage our proof of the ghost of an index theorem [1] we must replace $U$ and $V$ of (1.2) and (1.3) by the appropriate $U'$ and $V'$ specialized to the $A_j$ and $A_j'$ of (1.7) and (1.9); we leave it to the reader to work this out explicitly.

If we relax the assumptions on the operators $S$, $T$ and $ST$, asking only that all three operators have closed range, then we still have a ghost; we must however work with the operators of Yang as in (1.5):

3. Theorem. If $T : X \to Y$, $S : Y \to Z$ and $ST : X \to Z$ have closed range, then there exist isomorphisms

\begin{equation}
Z/SY \cong (Z/STX)/(SY/STX),
\end{equation}

\begin{equation}
SY/STX \cong (Y/STX)/((SY + T^{-1}(0))/TX),
\end{equation}

\begin{equation}
(SY + T^{-1}(0))/TX \cong S^{-1}(0)/(S^{-1}(0) \cap TX),
\end{equation}

\begin{equation}
S^{-1}(0) \cap TX \cong (ST)^{-1}(0)/T^{-1}(0).
\end{equation}

Proof. The sequence (1.5) is exact at each point $W_j$, and hence for each $j$ there is an isomorphism

\begin{equation}
A_{j+1}(W_j) \cong W_j/A_j(W_{j-1}).
\end{equation}

If we abandon all assumptions about the operators $S$ and $T$ we might expect to replace ranges by their closure, and hence salvage something; this [2] is however a forlorn hope. The reader is invited to write down the sequence (1.5) when ([2 Example 1]) $T : X \to Y$ is one-one with dense range and $S : Y \to Z$ is of rank one with $S^{-1}(0) \cap TX = \{0\}$. 

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REFERENCES


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