

CORRECTION TO “THE GHOST OF AN INDEX THEOREM”

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ABSTRACT. The “ghost of an index theorem” is an isomorphism between products of the kernel spaces and the cokernel spaces of a pair of bounded operators and their product, valid when each operator and also the product is assumed to have a generalized inverse. In this note we correct an error in the original proof, and extend the result to operators with closed range.

Suppose $T : X \rightarrow Y$ and $S : Y \rightarrow Z$ are bounded linear operators between Banach spaces, with *generalized inverses* $T' : Y \rightarrow X$ and $S' : Z \rightarrow Y$:

$$(0.1) \quad T = TT'T, \quad S = SS'S;$$

suppose also that the product $ST : X \rightarrow Z$ has a generalized inverse $U : Z \rightarrow X$:

$$(0.2) \quad ST = STUST.$$

Under these assumptions we have claimed

1. Theorem. *There is an isomorphism*

$$(1.1) \quad T^{-1}(0) \times S^{-1}(0) \times Z/ST(X) \cong (ST)^{-1}(0) \times Y/T(X) \times Z/S(Y).$$

In the proof ([1, Theorem 2]) we constructed matrices

$$(1.2) \quad \mathbf{U} = \begin{pmatrix} I - T'T & 0 & 0 \\ (I - TUS)T & I - TUS & 0 \\ 0 & S(I - TUS) & I - SS' \end{pmatrix}$$

and

$$(1.3) \quad \mathbf{V} = \begin{pmatrix} I - T'T & T'(I - S'S) & 0 \\ 0 & (I - TT')(I - S'S) & (I - TT')S' \\ 0 & 0 & I - SS' \end{pmatrix},$$

having first normalized the generalized inverses of (0.1) and (0.2) so that

$$(1.4) \quad T' = T'TT', \quad S' = S'SS', \quad U = USTU, \quad U = T'VS'.$$

The argument then proceeds by claiming that \mathbf{U} and \mathbf{V} are mutually generalized inverse, and identifying their ranges with the products of (1.1). Unfortunately \mathbf{U} and \mathbf{V} may fail to be mutually generalized inverse, and their products need not be diagonal as claimed in [1]. To see how to modify their definition we must go back to the source, which is the “one diagram” proof of the index theorem due to Yang [4].

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The argument of Yang ([4, Theorem]) proceeds by the construction of an exact sequence of bounded operators

$$(1.5) \quad \begin{array}{ccccccccccc} & 0 & & A_5 & & A_4 & & A_3 & & A_2 & & A_1 & & 0 \\ 0 & \longleftarrow & W_5 & \longleftarrow & W_4 & \longleftarrow & W_3 & \longleftarrow & W_2 & \longleftarrow & W_1 & \longleftarrow & W_0 & \longleftarrow & 0 \end{array}$$

where

$$(1.6) \quad \begin{aligned} W_0 &= T^{-1}(0), & W_1 &= (ST)^{-1}(0), & W_2 &= S^{-1}(0), \\ W_3 &= Y/T(X), & W_4 &= Z/(ST)(X), & W_5 &= Z/S(Y), \end{aligned}$$

and then the mappings A_j are natural: for example, A_1 is an inclusion map and A_5 is a quotient; A_2 and A_4 are induced by T and S , respectively, and A_3 comes from the “snake lemma”. Our strategy in [1] is to replace the kernel and cokernel spaces by the originals:

$$(1.7) \quad W_0 = W_1 = X, \quad W_2 = W_3 = Y, \quad W_4 = W_5 = Z .$$

The mappings A_j are now more complicated, in that we must define them compatibly on the whole space rather than only on a subspace or a quotient: for example, A_1 and A_5 are the projections induced by the generalized inverses of T and S . Specifically

$$(1.8) \quad \begin{aligned} A_1 &= I - T'T, & A_2 &= (I - TUS)T, & A_3 &= (I - TT')(I - S'S), \\ A_4 &= S(I - TUS), & A_5 &= I - SS'. \end{aligned}$$

Taking account of the normalization (1.4) we now have an example of a *regular chain*, in the sense that (with a little *abus de notation*)

$$(1.9) \quad A_5A_4 = A_4A_3 = A_3A_2 = A_2A_1 = 0,$$

and each A_j has a generalized inverse A'_j :

$$(1.10) \quad \begin{aligned} A'_1 &= I - T'T, & A'_2 &= T'(I - S'S), & A'_3 &= I - TUS, \\ A'_4 &= (I - TT')S, & A'_5 &= I - SS'. \end{aligned}$$

The behaviour of such a regular chain is described ([3, (2.5), (2.6)]) by the matrices

$$(1.11) \quad \mathbf{U} = \begin{pmatrix} A'_1 & 0 & 0 \\ A_2 & A'_3 & 0 \\ 0 & A_4 & A'_5 \end{pmatrix} \quad \text{and} \quad \mathbf{V} = \begin{pmatrix} A_1 & A'_2 & 0 \\ 0 & A_3 & A'_4 \\ 0 & 0 & A_5 \end{pmatrix} .$$

We can now seek to show that \mathbf{U} and \mathbf{V} are mutually generalized inverse, and identify their ranges; but for this to work it is crucial that

$$(1.12) \quad \begin{array}{ccccccccccc} & 0 & & A'_1 & & A'_2 & & A'_3 & & A'_4 & & A'_5 & & 0 \\ 0 & \longleftarrow & W_0 & \longleftarrow & W_1 & \longleftarrow & W_2 & \longleftarrow & W_3 & \longleftarrow & W_4 & \longleftarrow & W_5 & \longleftarrow & 0 \end{array}$$

is another chain; we need

$$(1.13) \quad A'_1A'_2 = A'_2A'_3 = A'_3A'_4 = A'_4A'_5 = 0.$$

This need not happen; however it is rather easy ([3, page 284]) to replace the generalized inverses A'_j by generalized inverses A''_j which form a chain:

2. Lemma. *If $T = TT'T$ and $S = SS'S$ with $ST = 0$, then*

$$(2.1) \quad T = TT''T, \quad S = SS''S, \quad T''S'' = 0$$

where

$$(2.2) \quad (T'', S'') = (T', (I - TT')S') \text{ or } (T'', S'') = (T'(I - S'S), S').$$

Proof. Clear. •

The extension of this procedure from regular chains of length 2 to regular chains of length n is clear: for the chain A of (1.5) we may take

$$(2.3) \quad A_1'' = A_1', \quad A_2'' = (I - A_1A_1')A_2', \quad A_3'' = (I - A_2A_2'')A_3', \\ A_4'' = (I - A_3A_3'')A_4', \quad A_5'' = (I - A_4A_4'')A_5'.$$

If we now write

$$(2.4) \quad \mathbf{U}' = \begin{pmatrix} A_1'' & 0 & 0 \\ A_2 & A_3'' & 0 \\ 0 & A_4 & A_5'' \end{pmatrix} \quad \text{and} \quad \mathbf{V}' = \begin{pmatrix} A_1 & A_2'' & 0 \\ 0 & A_3 & A_4'' \\ 0 & 0 & A_5 \end{pmatrix},$$

then we find (as claimed spuriously in [1]) that $\mathbf{U}'\mathbf{V}'$ and $\mathbf{V}'\mathbf{U}'$ are each diagonal, with \mathbf{U}' and \mathbf{V}' mutually generalized inverse. Thus to salvage our proof of the ghost of an index theorem [1] we must replace \mathbf{U} and \mathbf{V} of (1.2) and (1.3) by the appropriate \mathbf{U}' and \mathbf{V}' specialized to the A_j and A_j' of (1.7) and (1.9); we leave it to the reader to work this out explicitly.

If we relax the assumptions on the operators S , T and ST , asking only that all three operators have closed range, then we still have a ghost; we must however work with the operators of Yang as in (1.5):

3. Theorem. *If $T : X \rightarrow Y$, $S : Y \rightarrow Z$ and $ST : X \rightarrow Z$ have closed range, then there exist isomorphisms*

$$(3.1) \quad Z/SY \cong (Z/STX)/(SY/STX),$$

$$(3.2) \quad SY/STX \cong (Y/STX)/((SY + T^{-1}(0))/TX),$$

$$(3.3) \quad (SY + T^{-1}(0))/TX \cong S^{-1}(0)/(S^{-1}(0) \cap TX),$$

$$(3.4) \quad S^{-1}(0) \cap TX \cong (ST)^{-1}(0)/T^{-1}(0).$$

Proof. The sequence (1.5) is exact at each point W_j , and hence for each j there is an isomorphism

$$(3.5) \quad A_{j+1}(W_j) \cong W_j/A_j(W_{j-1}). \bullet$$

If we abandon all assumptions about the operators S and T we might expect to replace ranges by their closure, and hence salvage something; this [2] is however a forlorn hope. The reader is invited to write down the sequence (1.5) when ([2, Example 1]) $T : X \rightarrow Y$ is one-one with dense range and $S : Y \rightarrow Z$ is of rank one with $S^{-1}(0) \cap TX = \{0\}$.

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