CORRECTION TO “THE GHOST OF AN INDEX THEOREM”

ROBIN HARTE

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Abstract. The “ghost of an index theorem” is an isomorphism between products of the kernel spaces and the cokernel spaces of a pair of bounded operators and their product, valid when each operator and also the product is assumed to have a generalized inverse. In this note we correct an error in the original proof, and extend the result to operators with closed range.

Suppose \( T : X \to Y \) and \( S : Y \to Z \) are bounded linear operators between Banach spaces, with generalized inverses \( T' : Y \to X \) and \( S' : Z \to Y \):
\[
T = TT'T, \quad S = SS'S;
\]
suppose also that the product \( ST : X \to Y \) has a generalized inverse \( U : Z \to X \):
\[
ST = STUST.
\]
Under these assumptions we have claimed

1. Theorem. There is an isomorphism
\[
T^{-1}(0) \times S^{-1}(0) \times Z/ST(X) \cong (ST)^{-1}(0) \times Y/T(X) \times Z/S(Y).
\]

In the proof ([1, Theorem 2]) we constructed matrices
\[
U = \begin{pmatrix}
I - T'T & 0 & 0 \\
(I - TUS)T & I - TUS & 0 \\
0 & S(I - TUS) & I - SS'
\end{pmatrix}
\]
and
\[
V = \begin{pmatrix}
I - T'T & T'(I - S'S) & 0 \\
0 & (I - TT')(I - S'S) & (I - TT')S' \\
0 & 0 & I - SS'
\end{pmatrix},
\]
having first normalized the generalized inverses of (0.1) and (0.2) so that
\[
T' = T'T'T, \quad S' = S'SS', \quad U = USTU, \quad U = TVS'.
\]
The argument then proceeds by claiming that \( U \) and \( V \) are mutually generalized inverse, and identifying their ranges with the products of (1.1). Unfortunately \( U \) and \( V \) may fail to be mutually generalized inverse, and their products need not be diagonal as claimed in [1]. To see how to modify their definition we must go back to the source, which is the “one diagram” proof of the index theorem due to Yang [3].

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The argument of Yang ([4, Theorem]) proceeds by the construction of an exact sequence of bounded operators
\[(1.5)\]
\[
\begin{array}{cccccccc}
0 & A_5 & A_4 & A_3 & A_2 & A_1 & W_0 & 0 \\
W_5 & W_4 & W_3 & W_2 & W_1 & W_0 & 0 & \\
\end{array}
\]
where
\[(1.6)\]
\[
W_0 = T^{-1}(0), \quad W_1 = (ST)^{-1}(0), \quad W_2 = S^{-1}(0),
W_3 = Y/T(X), \quad W_4 = Z/(ST)(X), \quad W_5 = Z/S(Y),
\]
and then the mappings \(A_j\) are natural: for example, \(A_1\) is an inclusion map and \(A_5\) is a quotient; \(A_2\) and \(A_4\) are induced by \(T\) and \(S\), respectively, and \(A_3\) comes from the “snake lemma”. Our strategy in [1] is to replace the kernel and cokernel spaces by the originals:
\[(1.7)\]
\[
W_0 = W_1 = X, \quad W_2 = W_3 = Y, \quad W_4 = W_5 = Z.
\]
The mappings \(A_j\) are now more complicated, in that we must define them compatibly on the whole space rather than only on a subspace or a quotient: for example, \(A_1\) and \(A_5\) are the projections induced by the generalized inverses of \(T\) and \(S\). Specifically
\[(1.8)\]
\[
A_1 = I - T'T, \quad A_2 = (I - TUS)T, \quad A_3 = (I - TT')(I - S'S),
A_4 = S(I - TUS), \quad A_5 = I - SS'.
\]
Taking account of the normalization (1.4) we now have an example of a regular chain, in the sense that (with a little abus de notation)
\[(1.9)\]
\[
A_5A_4 = A_4A_3 = A_3A_2 = A_2A_1 = 0,
\]
and each \(A_j\) has a generalized inverse \(A'_j:\)
\[(1.10)\]
\[
A'_1 = I - T'T, \quad A'_2 = T'(I - S'S), \quad A'_3 = I - TUS,
A'_4 = (I - TT')S, \quad A'_5 = I - SS'.
\]
The behaviour of such a regular chain is described ([3 (2.5), (2.6)]) by the matrices
\[(1.11)\]
\[
U = \begin{pmatrix}
A'_1 & 0 & 0 \\
A_2 & A'_3 & 0 \\
0 & A_4 & A'_5
\end{pmatrix}
\quad \text{and} \quad
V = \begin{pmatrix}
A_1 & A'_2 & 0 \\
0 & A_3 & A'_4 \\
0 & 0 & A_5
\end{pmatrix}.
\]
We can now seek to show that \(U\) and \(V\) are mutually generalized inverse, and identify their ranges; but for this to work it is crucial that
\[(1.12)\]
\[
\begin{array}{cccccccc}
0 & A'_1 & A'_2 & A'_3 & A'_4 & A'_5 & 0 & \\
W_0 & W_1 & W_2 & W_3 & W_4 & W_5 & 0 & \\
\end{array}
\]
is another chain; we need
\[(1.13)\]
\[
A'_1A'_2 = A'_2A'_3 = A'_3A'_4 = A'_4A'_5 = 0.
\]
This need not happen; however it is rather easy ([3 page 284]) to replace the generalized inverses \(A'_j\) by generalized inverses \(A''_j\) which form a chain:
2. Lemma. If \( T = TT'T \) and \( S = SS'S \) with \( ST = 0 \), then

\[
T = TT'T, \quad S = SS'S, \quad T''S'' = 0
\]

where

\[
(T'', S'') = (T', (I - TT')S') \text{ or } (T'', S'') = (T'(I - S'S), S').
\]

Proof. Clear. 

The extension of this procedure from regular chains of length 2 to regular chains of length \( n \) is clear: for the chain of (1.5) we may take

\[
A^{00}_{j+1} = A^0_j, \quad A^{00}_{j+2} = (I - A^0_j A^0_{j+1}) A^2_{j+1}, \quad A^{00}_{j+4} = (I - A^0_{j+2} A^0_{j+3}) A^{00}_{j+3}, \quad A^{00}_{j+5} = (I - A^0_{j+4} A^0_{j+5}) A^0_{j+5}.
\]

If we now write

\[
U' = \begin{pmatrix} A^{00}_1 & 0 & 0 \\ A^1_2 & A^{00}_2 & 0 \\ 0 & A^1_4 & A^{00}_5 \end{pmatrix} \quad \text{and} \quad V' = \begin{pmatrix} A_1 & A^{00}_2 & 0 \\ 0 & A_3 & A^0_4 \\ 0 & 0 & A_5 \end{pmatrix},
\]

then we find (as claimed spuriously in [1]) that \( U'V' \) and \( V'U' \) are each diagonal, with \( U' \) and \( V' \) mutually generalized inverse. Thus to salvage our proof of the ghost of an index theorem [1] we must replace \( U \) and \( V \) of (1.2) and (1.3) by the appropriate \( U' \) and \( V' \) specialized to the \( A_j \) and \( A'_j \) of (1.7) and (1.9); we leave it to the reader to work this out explicitly.

If we relax the assumptions on the operators \( S, T \) and \( ST \), asking only that all three operators have closed range, then we still have a ghost; we must however work with the operators of Yang as in (1.5):

3. Theorem. If \( T : X \to Y, \quad S : Y \to Z \) and \( ST : X \to Z \) have closed range, then there exist isomorphisms

\[
Z/SY \cong (Z/STX)/(SY/STX),
\]

\[
SY/STX \cong (Y/STX)/((SY + T^{-1}(0))/TX),
\]

\[
(SY + T^{-1}(0))/TX \cong S^{-1}(0)/(S^{-1}(0) \cap TX),
\]

\[
S^{-1}(0) \cap TX \cong (ST)^{-1}(0)/T^{-1}(0).
\]

Proof. The sequence (1.5) is exact at each point \( W_j \), and hence for each \( j \) there is an isomorphism

\[
A_{j+1}(W_j) \cong W_j/A_j(W_{j-1}).
\]

If we abandon all assumptions about the operators \( S \) and \( T \) we might expect to replace ranges by their closure, and hence salvage something; this [2] is however a forlorn hope. The reader is invited to write down the sequence (1.5) when ([2, Example 1]) \( T : X \to Y \) is one-one with dense range and \( S : Y \to Z \) is of rank one with \( S^{-1}(0) \cap TX = \{0\} \).
REFERENCES


School of Mathematics, Trinity College, University of Dublin, Dublin 2, Ireland

E-mail address: rharte@maths.tcd.ie