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ON THE NUMBER OF GENERATORS OF COHEN-MACAULAY IDEALS

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ABSTRACT. Several bounds on the number of generators of Cohen-Macaulay ideals known in the literature follow from a simple inequality which bounds the number of generators of such ideals in terms of mixed multiplicities. Results of Cohen and Akizuki, Abhyankar, Sally, Rees and Boratynski-Eisenbud-Rees are deduced very easily from this inequality.

1. INTRODUCTION

The objective of this note is to present a novel approach to several results for the number of generators of Cohen-Macaulay ideals in Cohen-Macaulay local rings. Let (R, m) be a Cohen-Macaulay local ring of dimension d. An ideal I of R is called a *Cohen-Macaulay ideal* if R/I is a Cohen-Macaulay ring.

To state our main result we need to recall the basic notation for mixed multiplicities of ideals. Let (R, m) be a local ring. Let I be an ideal of positive height. Consider the function $C(r, s) = \ell(m^r I^s / m^{r+1} I^s)$. This function is given by a polynomial $Q(r, s)$ in two variables r and s for all large values of r and s [B]. This polynomial can be written as

$$Q(r,s) = \sum_{i+j \leq d-1} e_{ij} \binom{r+i}{i} \binom{s+j}{j}$$

where e_{ij} are integers for all $i, j = 0, 1, 2, \ldots, d-1$. When $i + j = d - 1$, we write $e_{i+1j} = e_j(m|I)$. These integers which appear with the monomials of degree $d-1$ in $Q(r, s)$ are nonnegative and they are called the *mixed multiplicities* of m and I. Let $\mu(I)$ denote the minimum number of generators for I. The principal result in this paper is the following:

Theorem 1.1. *Let (R, m) be a Cohen-Macaulay local ring of dimension d. Let I be a Cohen-Macaulay ideal of R of positive height h. Then for $q = 0, 1, 2, \ldots, h$,*

$$\mu(I) \leq h - q + (q-1)e(R/I) + e_{h-q}(m|I).$$

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We shall recover and generalize several results known in the literature which give upper bounds for the minimum number of generators of Cohen-Macaulay ideals. This will be done quite easily by invoking the above inequality and then applying standard results about mixed multiplicities.

2. Bound for $\mu(I)$ for m-primary ideals

The proof of Theorem 1.1 is by induction on $s = dimR/I$. When $s = 0$, the ideal I is m-primary. Therefore we treat this case first in this section. We begin by recalling a few facts about mixed multiplicities of ideals.

(1) Let I and J be m-primary ideals in a d-dimensional local ring (R, m). The function $B(r, s) = \ell(R/I^r J^s)$ is called the *Bhattacharya function* of I and J. Bhattacharya [B] proved that for large values of r and s, the Bhattacharya function is given by a polynomial $P(r, s)$ of total degree d in r and s with rational coefficients. Moreover it can be written as

$$P(r,s) = \sum_{i+j\leq d} e_{ij}\binom{r+i}{i}\binom{s+j}{j}.$$

The coefficients e_{ij} are integers and the ones for which $i+j = d$ are positive and they are called mixed multiplicities of I and J. We will use the notation $e_j(I|J) = e_{ij}$ for the mixed multiplicities of I and J.

(2) Rees [R1] showed that $e_0(I|J) = e(I)$ and $e_d(I|J) = e(J)$. Here $e(.)$ denotes the *Hilbert-Samuel multiplicity*.

(3) Risler and Teissier [T1] provided an interpretation of the other mixed multiplicities. They showed that the jth mixed multiplicity $e_j(I|J)$ is the multiplicity of an ideal generated by $d - j$ elements of I and j elements of J chosen sufficiently generally.

(4) Rees [R2] introduced the important concept of joint reductions of ideals which helps in calculation of mixed multiplicities. An ideal $K \subset J$ is called a reduction of J if there exists an $n \in \mathbb{N}$ such that $KJ^n = J^{n+1}$ [NR]. Let $I_1, I_2, \ldots, I_d$ be m-primary ideals of R. A set of elements $(x_1, x_2, \ldots, x_d)$ where $x_i \in I_i$, $i = 1, 2, \ldots, d$, is called a *joint reduction* of the set of ideals $(I_1, I_2, \ldots, I_d)$ if the ideal $\sum_{i=1}^{i=d} x_i I_1 I_2 \ldots I_{i-1} I_{i+1} \ldots I_d$ is a reduction of $I_1 I_2 \ldots I_d$. Rees showed [R3] that if R/m is infinite, then joint reductions exist. The following is a crucial result in the theory of mixed multiplicities:

Theorem 2.1 (Rees's Mixed Multiplicity Theorem [R3])**.** *Let $(x_1, x_2, \ldots, x_d)$ be a joint reduction of the set of ideals $(I, I, \ldots, I, J, J, \ldots, J)$ where I is repeated $d - q$ times and J is repeated q times. Then $e_q(I|J) = e((x_1, x_2, \ldots, x_d))$.*

Now we prove our main theorem for m-primary ideals.

Theorem 2.2. *Let (R, m) be a d-dimensional Cohen-Macaulay local ring and I be an m-primary ideal of R. Then for $q = 0, 1, 2, \ldots, d$,*

$$\mu(I) \leq d - q + (q-1)\,\ell(R/I) + e_{d-q}(m|I).$$

Proof. We may assume that R/m is infinite. Let $(x_1, x_2, \ldots, x_q, a_1, a_2, \ldots, a_{d-q})$ be a joint reduction of $(m, m, \ldots, m, I, I, \ldots, I)$ where m is repeated q times and I is repeated $d - q$ times. Let $\underline{x}$ and $\underline{a}$ denote the ideals $(x_1, x_2, \ldots, x_q)$ and $(a_1, a_2, \ldots, a_{d-q})$ respectively. Consider the R-module homomorphism

$$\phi : (R/m)^{d-q} \oplus (R/I)^q \longrightarrow \frac{(\underline{x}, \underline{a})}{\underline{x}I + \underline{a}m},$$

given by

$$\phi(y_1',\ldots,y_{d-q}',b_1',\ldots,b_q') = (y_1a_1+\ldots+y_{d-q}a_{d-q}+b_1x_1+\ldots+b_qx_q)',$$

where primes denote the residue classes. Hence

$$\ell\left(\frac{(\underline{x},\underline{a})}{\underline{x}I+\underline{a}m}\right) \le d-q+q\,\ell(R/I).$$

But

$$\begin{aligned}\ell\left(\frac{(\underline{x},\underline{a})}{\underline{x}I+\underline{a}m}\right) &= \ell(R/\underline{x}I+\underline{a}m)-\ell\left(R/(\underline{x}+\underline{a})\right).\\ &= \ell(R/I)+\ell(I/Im)+\ell(Im/\underline{x}I+\underline{a}m)-e_{d-q}(m|I).\end{aligned}$$

Hence $\mu(I)\le d-q+(q-1)\,\ell(R/I)+e_{d-q}(m|I)$.

Corollary 2.3 (Akizuki [Ak], Cohen [C])**.** *Let (R,m) be a one-dimensional Cohen-Macaulay local ring. Then for any m-primary ideal I of R, $\mu(I)\le e(m)$.*

Proof. Put $d=q=1$ to get $\mu(I)\le e_0(m|I)=e(m)$.

The next result was proved by Abhyankar [A] for the maximal ideal.

Corollary 2.4. *Let (R,m) be a d-dimensional Cohen-Macaulay local ring. Let I be an m-primary ideal. Then*

$$\mu(I)\le e(I)-\ell(R/I)+d.$$

Proof. Put $q=0$ to get

$$\mu(I)\le d-\ell(R/I)+e_d(m|I) = d-\ell(R/I)+e(I).$$

Recall that the nilpotency degree of a nilpotent ideal I is the smallest integer t for which $I^t=0$. The next result was proved by Sally [S] for $q=1$.

Corollary 2.5. *Let (R,m) be a d-dimensional Cohen-Macaulay local ring. Let I be an m-primary ideal. Let the nilpotency degree of m/I be t. Then*

$$\mu(I)\le d-q+(q-1)\,\ell(R/I)+t^{d-q}\,e(m).$$

Proof. It is easy to prove the following: (1) $e_i(m^p|I^q)=q^ip^{d-i}e_i(m|I)$ (2) $e_i(I|I)=e(I)$ and (3) for an ideal $K\subset I, e_i(m|I)\le e_i(m|K)$. These imply that

$$\begin{aligned}\mu(I) &\le d-q+(q-1)\,\ell(R/I)+e_{d-q}(m|m^t)\\ &= d-q+(q-1)\,\ell(R/I)+t^{d-q}\,e_{d-q}(m|m)\\ &= d-q+(q-1)\,\ell(R/I)+t^{d-q}\,e(m).\end{aligned}$$

The next result generalizes a bound due to Boratynski, Eisenbud and Rees. This follows from Theorem 2.2 and by the following Minkowski type inequality for mixed multiplicities due to Rees and Sharp [RS] and Teissier [T2].

Theorem 2.6. *Let I and J be m-primary ideals of a local ring (R,m) of dimension d. Then for $i=0,1,\ldots,d$,*

$$e_i(I|J)\le\sqrt[d]{e(I)^{d-i}e(J)^i}.$$

Corollary 2.7. *Let I be an m-primary ideal in a d-dimensional Cohen-Macaulay local ring. Then for $q=0,1,\ldots,d$,*

$$\mu(I)\le d-q+(q-1)\,\ell(R/I)+\sqrt[d]{e(m)^qe(I)^{d-q}}.$$

3. Bound for $\mu(I)$ for Cohen-Macaulay ideals

The proof of Theorem 1.1 is by induction on the dimension of R/I. The following lemma of Rees provides us with a tool to pass to one lower dimension.

Lemma 3.1 (Rees's Lemma [R3])**.** *Let (R, m) be a local ring with infinite residue field R/m. Let $(I_1, I_2, \dots, I_g)$ be a set of ideals of R. Let $\mathcal{P}$ be a finite set of prime ideals so that none of the primes in $\mathcal{P}$ contain any of the ideals $I_1, I_2, \dots, I_g$. Then there exist integers $s_i \geq 0$ and elements $x_i \in I_i \setminus \bigcup\{p : p \in \mathcal{P}\}$ where $i = 1, 2, \dots, g$ so that for all $r_i \geq s_i$ and for all $r_j \geq 0$, $j \neq i$,*

$$x_i R \cap I_1^{r_1} I_2^{r_2} \dots I_g^{r_g} = x_i I_1^{r_1} I_2^{r_2} \dots I_i^{r_i - 1} \dots I_g^{r_g}.$$

Definition 3.2. The element $x_i \in I_i$ in Rees's lemma is called superficial for the set of ideals $I_1, I_2, \dots, I_g$.

Lemma 3.3 ([KV])**.** *Let (R, m) be a local ring. Let I be an ideal of positive height h. If $x \in m$ is superficial for I and m, then for $i = 0, \dots, h$,*

$$e_i(m|I) = e_i\left(\frac{m}{xR} \middle| \frac{I + xR}{xR}\right).$$

Proof of Theorem 1.1. Apply induction on $s = \dim R/I$. If $s = 0$, then I is m-primary. Thus $\ell(R/I) = e(R/I)$. Therefore the theorem follows from Theorem 2.2. Suppose that $s \geq 1$. Then we can choose a nonzerodivisor $x \in m$ which is superficial for m and I and its image is a nonzerodivisor in and superficial for m/I. Put $\overline{R} = R/xR$ and $\overline{I} = I/xR$. Then

$$\mu(\overline{I}) = \dim(\overline{I}/\overline{m}\overline{I}) = \dim\left(\frac{I + xR}{mI + xR}\right) = \dim\left(\frac{I}{mI + (xR \cap I)}\right) = \mu(I).$$

It is easy to see that $ht(\overline{I}) = ht(I)$. Since $\overline{x}$ is superficial for R/I, $e(R/I) = e(\overline{R}/\overline{I})$. By Lemma 3.3, $e_{h-q}(\overline{m}|\overline{I}) = e_{h-q}(m|I)$. The theorem follows by induction.

Corollary 3.4. *Let I be a Cohen-Macaulay ideal of positive height in a d-dimensional Cohen-Macaulay local ring (R, m). Then for all $q = 0, 1, \dots, h$,*

$$\mu(I) \leq h - q + (q - 1)\, e(R/I) + e(R/I)^{h-q} e(R).$$

Proof. Put $s = \dim(R/I)$. Suppose $s = 0$. Then I is m-primary. Let the nilpotency degree of m/I be t. Hence

$$\begin{aligned} e_{d-q}(m|I) \leq e_{d-q}(m|m^t) &= t^{d-q} e(R) \\ &\leq \ell(R/I)^{d-q} e(R) \\ &= e(R/I)^{d-q} e(R). \end{aligned}$$

Now let $s \geq 1$. Pick $x \in m \setminus I$ so that it is superficial for m and I and $\overline{x}$ is a nonzerodivisor in R/I and it is superficial for m/I. Then

$$\begin{aligned} \mu(I) = \mu(\overline{I}) &\leq h - q + (q - 1)\, e(\overline{R}/\overline{I}) + e(\overline{R}/\overline{I})^{h-q} e(\overline{R}) \\ &= h - q + (q - 1)\, e(R/I) + e(R/I)^{h-q} e(R). \end{aligned}$$

Corollary 3.5 (Sally [S])**.** $\mu(I) \leq h - 1 + e(R/I)^{h-1} e(R)$.

Corollary 3.6 (Rees [R4])**.** *Suppose that $ht(I) = 2$. Then $\mu(I) \leq e(R) + e(R/I)$.*

Proof. Put $h = q = 2$.

Corollary 3.7 (Rees [R4]). *Suppose that $ht(I) = 1$. Then $\mu(I) \leq e(R)$.*

Proof. Put $h = q = 1$.

4. Comparison with other bounds

In this section we present some examples to show that our bounds can sometimes give better results than the previously known bounds.

First we consider a bound found by Valla in [V].

Theorem 4.1. *Let (R, m) be a CM local ring of dimension d and multiplicity e. Let I be a CM ideal of height h. Suppose that $e(R/I) = \epsilon$. Put $r = min(e, \epsilon)$. Then*

a) *If $h = 0$, then $\mu(I) \leq e - \epsilon$.*

b) *If $h > 0$, then $\mu(I) \leq e + \epsilon(h-1)^2/h + r(h-1)/h$.*

c) *If $h \geq 2$, and $I \subseteq m^2$, then $\mu(I) \leq e + \epsilon(h-1)^2/h + min(r+h, \epsilon)(h-1)/h - \binom{h}{2}$.*

Let (R, m) be the three-dimensional regular local ring $k[[x, y, z]]$ where k is any field. Consider the ideal $I = p^{(2)}$ where p is generated by the defining equations of the monomial space curve (t^3, t^4, t^5). Then $e_1(m|I) = 3$. To calculate Valla's bound notice that $e = 1, \epsilon = e(R/p^{(2)}) = 9$, by the associativity formula. Thus Valla's bound gives $\mu(I) \leq 6$. Our bound in Theorem 1.1 gives $\mu(I) \leq 4$. In fact I is four generated. Next we consider a very appealing bound found in [DGV].

Theorem 4.2. *Let (R, m) be a CM local ring of dimension $d \geq 1$. Let I be an m-primary ideal such that $m^s \subseteq I$. Then*

$$\mu(I) \leq e(R)\binom{s+d-2}{d-1} + \binom{s+d-2}{d-2}.$$

Consider the ideal $I = (x^2, xy, y^n)$, $n \geq 2$, of the power series ring $k[[x, y]]$ over a field k. Then $e_1(m|I) = 2$, hence the bound in Theorem 1.1 tells us that I is generated by at most 3 elements. The bound in [DGV] tells us that I is generated by at most $n + 1$ elements. On the other hand the bound in [DGV] is often better for large powers of ideals. If (R, m) is a regular local ring of dimension d, then Theorem 1.1 implies that m^n is generated by at most $d - 1 + n^{d-1}$ elements. The bound in [DGV] gives the exact number of generators. Hence our bound is inferior to the bound in [DGV] in this case.

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