SUPERPOSITION OPERATOR IN SOBOLEV SPACES ON DOMAINS

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Abstract. For an arbitrary open set \( \Omega \subset \mathbb{R}^n \) we characterize all functions \( G \) on the real line such that \( G \circ u \in W^{1,p}(\Omega) \) for all \( u \in W^{1,p}(\Omega) \). New element in the proof is based on Maz’ya’s capacitary criterion for the imbedding \( W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega) \).

Let \( \Omega \) be an open set in \( \mathbb{R}^n, n \geq 2 \), and let \( W^{1,p}(\Omega) \) be a Sobolev space with the norm
\[
\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}, \quad 1 \leq p < \infty.
\]

Let \( G : \mathbb{R}^1 \to \mathbb{R}^1 \) be a Borel function. Then the associated superposition operator \( T_G \) is given by
\[
u \mapsto G \circ u.
\]

The purpose of the present paper is to find the necessary and sufficient conditions on \( G \) for
\[
T_G : W^{1,p}(\Omega) \to W^{1,p}(\Omega).
\]

This and more general problems were considered for \( p \neq n \) and domains \( \Omega \) with Lipschitz boundaries in [9], [10], and for \( \Omega = \mathbb{R}^n, 1 \leq p < \infty \), in [3]. Following [3] we say that space \( W^{1,p}(\Omega) \) is supercritical if \( W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega) \), and subcritical if \( W^{1,p}(\Omega) \not\hookrightarrow L^\infty(\Omega) \). By \( |E| \) we denote Lebesgue measure of the set \( E \subset \mathbb{R}^n \). Let \( B(x, r) \) denote an open ball in \( \mathbb{R}^n \) with center \( x \) and radius \( r \).

The core of the works [10], [3] is the criterion for (1) when \( \Omega \) is a cube. If \( \Omega \) is a cube, then (1) is equivalent to \( G' \in L^\infty(\mathbb{R}^1) \) for subcritical \( W^{1,p}(\Omega) \) (that is, for \( 1 \leq p \leq n \)), and to \( G' \in L^\infty_{\text{loc}}(\mathbb{R}^1) \) for supercritical \( W^{1,p}(\Omega) \) (that is, for \( p > n \)). As a direct consequence of this result we obtain that for an arbitrary open set \( \Omega \subset \mathbb{R}^n \)
\[
T_G : W^{1,p}(\Omega) \to W^{1,p}_{\text{loc}}(\Omega)
\]

is equivalent to \( G' \in L^\infty(\mathbb{R}^1) \) if \( 1 \leq p \leq n \), and to \( G' \in L^\infty_{\text{loc}}(\mathbb{R}^1) \) if \( p > n \). Moreover, if for a domain \( \Omega, |\Omega| < \infty \), the exponent \( p = n \), as in the Sobolev imbedding theorem, separates subcritical and supercritical \( W^{1,p}(\Omega) \), then (1) is equivalent to (2). It seems that the broadest class of such domains \( \Omega \) is the class

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of John domains, [10], [2], [5]. For domains with infinite measure one has analogous results provided the condition $G(0) = 0$ is added to the restrictions on $G'$, [10], [3].

For an arbitrary $\Omega$ [11] is not equivalent to [2]. For general open sets $\Omega$ our main result is Theorem 1 below. For an arbitrary $\Omega$ in subcritical case with $p > n$ the condition for (2) is necessary but not sufficient for (1). On the other hand, condition $G' \in L^\infty(\mathbb{R}^1)$ is obviously sufficient for (1), but not necessary for (2) with $p > n$. In this case, in contrast with local constructions in [10], [3], we establish estimates near the boundary.

Our consideration is based on the following criterion for $W^{1,p}(\Omega)$, which was proved by Maz’ya. For $x \in \Omega$ we define

$$V_\Omega(x) = \{ f \in C^\infty_\text{loc}(\Omega) : 0 \leq f \leq 1, f(x) = 1 \}.$$ 

Relative $p$-capacity in $\Omega$ of a point $x \in \Omega$ with respect to the ball $B(x, r)$ is defined (see [2]) as

$$\text{cap}_p(x, B(x, r); \Omega) = \inf \left\{ \int_\Omega |\nabla f|^p : f \in V_\Omega(x), f|_{\Omega \setminus B(x, r)} \equiv 0 \right\}.$$ 

Maz’ya [7] proved that $W^{1,p}(\Omega) \not\rightarrow L^\infty(\Omega)$ if and only if

$$\inf \{ \text{cap}_p(x, B(x, r); \Omega) : x \in \Omega \} = 0 \quad \text{for all } r > 0.$$ 

Note that for $B(x, r) \subset \subset \Omega$ and $p > n$

$$\text{cap}_p(x, B(x, r); \Omega) = c(n, p)r^{n-p} \to \infty \quad \text{as } r \to 0$$

(see [7]).

It should be mentioned that there is a vast literature concerning characterisation of $G$ for $T_G$ to act in Besov and Lizorkin-Triebel spaces on $\mathbb{R}^n$. We refer to surveys [4], [12], [13], and to the book [11]. Results for superposition operators in other spaces of real functions, such as Lebesgue spaces, $BV$, and ideal spaces, can be found in [4].

**Theorem 1.** Let $\Omega \subset \mathbb{R}^n$ be an open set, $|\Omega| < \infty$, $1 \leq p < \infty$. The following conditions on the function $G : \mathbb{R}^1 \to \mathbb{R}^1$ are necessary and sufficient for $T_G : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$:

(i) $G' \in L^\infty(\mathbb{R}^1)$, if $W^{1,p}(\Omega)$ is subcritical.

(ii) $G' \in L^\infty_\text{loc}(\mathbb{R}^1)$, if $W^{1,p}(\Omega)$ is supercritical.

The proof of the theorem is given at the end of the paper. First we make some remarks.

**Remark 2.** Condition (3) is equivalent to $W^{1,p}(\Omega) \not\rightarrow L^\infty(\Omega)$ also in the case $|\Omega| = \infty$. For open sets $\Omega$ of infinite measure Theorem 1 is valid provided we add the obvious necessary condition $G(0) = 0$ in both (i) and (ii). The proof is the same as below.

**Remark 3.** Let $\Omega$ be a connected open set and let $\omega$ be an open nonempty set with compact closure $\overline{\omega} \subset \Omega$. Following [2] we introduce the Banach space $L^{1,p}(\Omega)$ with the norm

$$\|u\|_{L^{1,p}(\Omega)} = \|u\|_{L^p(\omega)} + \|\nabla u\|_{L^p(\Omega)}.$$
It is known that norms corresponding to different choices of $\omega$ are equivalent (see [7]). If $|\Omega| < \infty$ then condition (3) is equivalent to $L^{1,p}(\Omega) \not\subset L^\infty(\Omega)$ (see [7]). Theorem [1] holds for the space $L^{1,p}(\Omega)$ whenever $|\Omega| < \infty$. The proof is even easier than for $W^{1,p}(\Omega)$, as we do not need to control $\|u\|_{L^p(\Omega)}$

\textbf{Remark 4.} From [3] and [10] it follows that the nonlinear operator $T_G$ in Theorem [1] is bounded and continuous in $W^{1,p}(\Omega)$ as soon as [11] is valid. The same is true for $T_G$ in $L^{1,p}(\Omega)$ provided $|\Omega| < \infty$, and in $W^{1,p}(\Omega)$ if $|\Omega| = \infty$.

\textbf{Proof of Theorem [7].} The only part of the theorem not covered by conditions from [3] for (2) is the necessity for $p > n$ in (i).

Thus we need to prove that if $T_G : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$, $p > n$, and $W^{1,p}(\Omega) \not\subset L^\infty(\Omega)$, then $G' \in L^\infty(\mathbb{R}^1)$. From [3] and [11] it follows that $G$ is absolutely continuous on $\mathbb{R}^1$ and $G' \in L^\infty_{\text{loc}}(\mathbb{R}^1)$. Moreover, for all $u \in W^{1,p}(\Omega)$ one has $\nabla(G \circ u) = G'(u) \nabla u$ a.e. in $\Omega$.

Seeking a contradiction suppose that $G' \not\in L^\infty(\mathbb{R}^1)$. Then we shall construct a function $U \in W^{1,p}(\Omega)$ such that $\|\nabla(G \circ U)\|_{L^p(\Omega)} = \infty$. To do this we first construct a special sequence of functions $\{u_j\}$, $u_j \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \cap C_{\text{loc}}(\Omega)$ for $j = 1, 2, \ldots$, and a sequence of pairwise disjoint balls $\{B(y_j, r_j)\} = \{B_j\}$, $B_j \subset \subset \Omega$, with the properties

\begin{equation}
\|u_j\|_{L^p(\Omega)} + \left(\int_{\Omega \setminus \bigcup_{m \leq j} B_m} |\nabla u_j|^p\right)^{1/p} \leq \left(1 + \cdots + \frac{1}{2^j}\right) \quad \text{for all } j \geq 1,
\end{equation}

\begin{equation}
u_m|_{B_j} = u_j|_{B_j} \quad \text{for } m \geq j.
\end{equation}

The following inequalities are valid for all $j \geq 1$:

\begin{equation}1/2^j}
\end{equation}

\begin{equation}
\int_{B_j} |\nabla u_j|^p \leq 1/j^2,
\end{equation}

\begin{equation}
\int_{B_j} |G'(u_j)|^p |\nabla u_j|^p \geq C(n),
\end{equation}

with some $C(n) > 0$ independent of $j$.

By (5)--(7) the sequence $\|u_j\|_{W^{1,p}(\Omega)}$ is bounded, and we can extract a subsequence converging weakly to some function $U \in W^{1,p}(\Omega)$. Now (4) and (8) imply that $\nabla(G \circ U) \not\in L^p(\Omega)$.

We construct the sequences $\{u_j\}$ and $\{B_j\}$ by induction. Using (4) we choose a constant $R = R(n, p)$ such that for $B(z, R) \subset \subset \Omega$

\begin{equation}
\text{cap}_p(z, B(z, R); \Omega) \geq 2.
\end{equation}

Without loss of generality we can assume that $G' \not\in L^\infty(\mathbb{R}^1_+)$, where $\mathbb{R}^1_+ = (0, +\infty)$.

To construct $u_1$ we take the number $t_1$ such that $t_1 > 1$, $|G'(t_1)| \geq 1$ and $t_1$ is a Lebesgue point of $G'$. The latter implies that

\begin{equation}\left|\left\{s : |t_1 - s| \leq \delta, |G'(s)| \geq 1\right\}\right| \geq \delta
\end{equation}

for all sufficiently small $\delta$. Next we choose $0 < r_1 < R$ so small that $t_1 |B(0, r_1)|^{1/p} < 1/4$. For these $t_1$ and $r_1$ using (3) we find a ball $B(y_1, r_1)$, $y_1 \in \Omega$, and a function $v_1 \in V_\Omega(y_1)$ supported in $B(y_1, r_1) \cap \Omega$ such that $\int_{\Omega} |\nabla v_1|^p \leq 1/(4t_1)^p$, or
equivalently
\[
\left( \int_\Omega |\nabla (2t_1 v_1)|^p \right)^{1/p} \leq 1/2.
\]

Now we put \( w_1 = \min\{t_1, 2t_1 v_1 \} \). Note that truncating does not increase norm in \( W^{1,p}(\Omega) \). Thus because of our choice of \( r_1 \) we have
\[
\|w_1\|_{W^{1,p}(\Omega)} \leq \|2t_1 v_1\|_{W^{1,p}(\Omega)} \leq 1.
\]

Note that \( v_1(y_1) = 1 \). Consequently there exists a ball \( B_1 = B(y_1, \rho_1) \subset \subset \Omega \) such that \( w_1|_{B_1} \equiv t_1 \). To finish construction of \( u_1 \) we use the function \( \phi(x) = \max\{1 - |x|, 0\}, x \in \mathbb{R}^n \). We denote \( \phi_* (x) = \phi(x/\varepsilon) \) for \( \varepsilon > 0 \). We can choose \( a > 0 \) so small that \( u_1 \) is satisfied for \( \delta = a, \varepsilon < \rho_1 \) such that
\[
\|a\phi_*\|_{L^p(\mathbb{R}^n)} \leq C_1(n)\varepsilon^{n/p} \leq 1/2,
\]
\[
\int_{\mathbb{R}^n} |\nabla (a\phi_*)|^p = C_2(n)a^p/\varepsilon^{n-p} = 1/2.
\]

Note that (10) and the linear radial nature of \( \phi \) imply that for such \( a \)
\[
\left\{|z \in \mathbb{R}^n : |z| < \varepsilon, |G' \circ (t_1 + a\phi_*)(z)| \geq 1\right\} \geq C_3(n)\varepsilon^n.
\]

Therefore one also has
\[
(11) \quad \int_{\mathbb{R}^n} |\nabla (G \circ (t_1 + a\phi_*)|^p \geq C_3(n)a^p/\varepsilon^{n-p} \geq C_3(n)/2C_2(n).
\]

Now taking \( u_1(x) = w_1(x) + a\phi_*(x - y_1), x \in \Omega \), we see that \( u_1 \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \cap C_{\text{loc}}(\Omega) \) and (10), (7), (8) are valid for \( j = 1 \).

Now we construct \( u_k \) assuming that the functions \( u_1, \ldots, u_{k-1} \) and the balls \( B_1, \ldots, B_{k-1} \) with all required properties have already been constructed. We can find \( t_k \) such that
\[
(12) \quad t_k \geq 4\|u_{k-1}\|_{L^\infty(\Omega)},
\]
\[
|G'(t_k)| \geq k, \text{ and } t_k \text{ is a Lebesgue point of } G'. \text{ Next we choose } 0 < r_k < R \text{ so small that}
\]
\[
(13) \quad 4r_k \leq \min\{\text{dist}(B_j, \partial \Omega), j = 1, \ldots, k - 1\},
\]
and \( t_k|B(0, r_k)|^{1/p} < 1/2^{k+2} \). For \( r_k, t_k \) we find, using (10), a ball \( B(y_k, r_k), y_k \in \Omega \), and a function \( v_k \in V_{\Omega}(y_k) \) supported in \( B(y_k, r_k) \cap \Omega \) such that
\[
(14) \quad \left( \int_\Omega |\nabla (2t_k v_k)|^p \right)^{1/p} \leq 1/2^{k+2}.
\]

We note that (10) and (14) imply that \( \partial B(y_k, r_k) \cap \partial \Omega \neq \emptyset \). Thus \( B(y_k, r_k) \) does not intersect any of \( B_1, \ldots, B_{k-1} \) because of (13). We define \( u_k \) in a similar way as \( w_1 \). Let \( w_k = \min\{t_k, u_{k-1} + 2t_k v_k\} \). Because of (12) \( w_k \) coincides with \( u_{k-1} \) in the balls \( B_1, \ldots, B_{k-1} \). From (12) one has \( u_{k-1}(y_k) + 2t_k v_k(y_k) > t_k \). Therefore there is a ball \( B_k = B(y_k, \rho_k) \subset \subset \Omega, B_k \subset B(y_k, r_k) \), not intersecting \( B_1, \ldots, B_{k-1} \), such that \( w_k|_{B_k} \equiv t_k \). By induction we have the estimate
\[
\|w_k\|_{W^{1,p}(\Omega)} \leq \left( 1 + \ldots + \frac{1}{2^{k-1}} + \frac{1}{2^{k+1}} \right).
\]
To construct $u_k$ we modify $w_k$ in the ball $B_k$. We use the function $a\phi_\varepsilon$ defined as above. We choose $a$ and $\varepsilon$ such that $\text{supp}(a\phi_\varepsilon) \subset B_k$.

\begin{equation}
\{s : |t_k - s| \leq a, |G'(s)| \geq k\} \geq a,
\end{equation}
\begin{equation}
\|a\phi_\varepsilon\|_{L^p(\mathbb{R}^n)} \leq 1/2^{k+1},
\end{equation}
\begin{equation}
\int_{\mathbb{R}^n} |\nabla (a\phi_\varepsilon)|^p = 1/k^2.
\end{equation}

By analogy with (11) from (15) and (16) we have
\begin{equation}
\int_{\mathbb{R}^n} |\nabla (G \circ (t_k + a\phi_\varepsilon))|^p \geq C_4(n)k^p/k^2 \geq C_5(n).
\end{equation}

Now we define $u_k(x) = w_k(x) + a\phi_\varepsilon(x - y_k)$, and (5)--(8) hold. This completes the proof.

REFERENCES