SUPERPOSITION OPERATOR IN SOBOLEV SPACES ON DOMAINS

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ABSTRACT. For an arbitrary open set \( \Omega \subseteq \mathbb{R}^n \) we characterize all functions \( G \) on the real line such that \( G \circ u \in W^{1,p}(\Omega) \) for all \( u \in W^{1,p}(\Omega) \). New element in the proof is based on Maz’ya’s capacitary criterion for the imbedding \( W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega) \).

Let \( \Omega \) be an open set in \( \mathbb{R}^n, n \geq 2 \), and let \( W^{1,p}(\Omega) \) be a Sobolev space with the norm
\[
\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}, \quad 1 \leq p < \infty.
\]
Let \( G : \mathbb{R}^1 \to \mathbb{R}^1 \) be a Borel function. Then the associated superposition operator \( T_G \) is given by
\[
u \mapsto G \circ \nu.
\]
The purpose of the present paper is to find the necessary and sufficient conditions on \( G \) for
\[
T_G : W^{1,p}(\Omega) \to W^{1,p}(\Omega).
\]
This and more general problems were considered for \( p \neq n \) and domains \( \Omega \) with Lipschitz boundaries in [9], [10], and for \( \Omega = \mathbb{R}^n, 1 \leq p < \infty \), in [3]. Following [2] we say that space \( W^{1,p}(\Omega) \) is supercritical if \( W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega) \), and subcritical if \( W^{1,p}(\Omega) \not\hookrightarrow L^\infty(\Omega) \). By \( |E| \) we denote Lebesgue measure of the set \( E \subseteq \mathbb{R}^n \). Let \( B(x, r) \) denote an open ball in \( \mathbb{R}^n \) with center \( x \) and radius \( r \).

The core of the works [10], [3] is the criterion for (1) when \( \Omega \) is a cube. If \( \Omega \) is a cube, then (1) is equivalent to \( G' \in L^\infty(\mathbb{R}^1) \) for subcritical \( W^{1,p}(\Omega) \) (that is, for \( 1 \leq p \leq n \)), and to \( G' \in L^\infty_{\text{loc}}(\mathbb{R}^1) \) for supercritical \( W^{1,p}(\Omega) \) (that is, for \( p > n \)). As a direct consequence of this result we obtain that for an arbitrary open set \( \Omega \subseteq \mathbb{R}^n \)
\[
T_G : W^{1,p}_{\text{loc}}(\Omega) \to W^{1,p}_{\text{loc}}(\Omega)
\]
is equivalent to \( G' \in L^\infty(\mathbb{R}^1) \) if \( 1 \leq p \leq n \), and to \( G' \in L^\infty_{\text{loc}}(\mathbb{R}^1) \) if \( p > n \). Moreover, if for a domain \( \Omega, |\Omega| < \infty \), the exponent \( p = n \), as in the Sobolev imbedding theorem, separates subcritical and supercritical \( W^{1,p}(\Omega) \), then (1) is equivalent to (2). It seems that the broadest class of such domains \( \Omega \) is the class

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of John domains, [3], [2], [5]. For domains with infinite measure one has analogous results provided the condition \( G(0) = 0 \) is added to the restrictions on \( G' \), [10], [3].

For an arbitrary \( \Omega \) [4] is not equivalent to [2]. For general open sets \( \Omega \) our main result is Theorem 1 below. For an arbitrary \( \Omega \) in subcritical case with \( p > n \) the condition for (2) is necessary but not sufficient for (1). On the other hand, condition \( G' \in L^\infty(\mathbb{R}^1) \) is obviously sufficient for (1), but not necessary for (2) with \( p > n \). In this case, in contrast with local constructions in [10], [3], we establish estimates near the boundary.

Our consideration is based on the following criterion for \( W^{1,p}(\Omega) \), which was proved by Maz’ya. For \( x \in \Omega \) we define

\[
V_\Omega(x) = \{ f \in C_0^\infty(\Omega) : 0 \leq f \leq 1, f(x) = 1 \}.
\]

Relative \( p \)-capacity in \( \Omega \) of a point \( x \in \Omega \) with respect to the ball \( B(x, r) \) is defined (see [2]) as

\[
\text{cap}_p(x, B(x, r); \Omega) = \inf \left\{ \int_\Omega |\nabla f|^p : f \in V_\Omega(x), f|_{\Omega \setminus B(x, r)} \equiv 0 \right\}.
\]

Maz’ya [7] proved that \( W^{1,p}(\Omega) \not\hookrightarrow L^\infty(\Omega) \) if and only if

\[
(3) \quad \inf \{ \text{cap}_p(x, B(x, r); \Omega) : x \in \Omega \} = 0 \quad \text{for all } r > 0.
\]

Note that for \( B(x, r) \subset \subset \Omega \) and \( p > n \)

\[
(4) \quad \text{cap}_p(x, B(x, r); \Omega) = c(n, p)r^{n-p} \rightarrow \infty \quad \text{as } r \rightarrow 0
\]

(see [2]).

It should be mentioned that there is a vast literature concerning characterisation of \( G \) for \( T_G \) to act in Besov and Lizorkin-Triebel spaces on \( \mathbb{R}^n \). We refer to surveys [4], [12], [13], and to the book [11]. Results for superposition operators in other spaces of real functions, such as Lebesgue spaces, \( BV \), and ideal spaces, can be found in [4].

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( |\Omega| < \infty \), \( 1 \leq p < \infty \). The following conditions on the function \( G : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \) are necessary and sufficient for \( T_G : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega) \):

(i) \( G' \in L^\infty(\mathbb{R}^1) \), if \( W^{1,p}(\Omega) \) is subcritical.

(ii) \( G' \in L^\infty_{\text{loc}}(\mathbb{R}^1) \), if \( W^{1,p}(\Omega) \) is supercritical.

The proof of the theorem is given at the end of the paper. First we make some remarks.

**Remark 2.** Condition (3) is equivalent to \( W^{1,p}(\Omega) \not\hookrightarrow L^\infty(\Omega) \) also in the case \( |\Omega| = \infty \). For open sets \( \Omega \) of infinite measure Theorem 1 is valid provided we add the obvious necessary condition \( G(0) = 0 \) in both (i) and (ii). The proof is the same as below.

**Remark 3.** Let \( \Omega \) be a connected open set and let \( \omega \) be an open nonempty set with compact closure \( \omega \subset \Omega \). Following [2] we introduce the Banach space \( L^{1,p}(\Omega) \) with the norm

\[
\|u\|_{L^{1,p}(\Omega)} = \|u\|_{L^p(\omega)} + \|\nabla u\|_{L^p(\Omega)}.
\]
It is known that norms corresponding to different choices of \( \omega \) are equivalent (see [7]). If \(|\Omega| < \infty\) then condition (3) is equivalent to \( L^{1,p}(\Omega) \hookrightarrow L^{\infty}(\Omega) \) (see [7]). Theorem 1 holds for the space \( L^{1,p}(\Omega) \) whenever \(|\Omega| < \infty\). The proof is even easier than for \( W^{1,p}(\Omega) \), as we do not need to control \( \|u\|_{L^p(\Omega)} \).

Remark 4. From [9] and [10] it follows that the nonlinear operator \( T_G \) in Theorem 1 is bounded and continuous in \( W^{1,p}(\Omega) \) as soon as (1) is valid. The same is true for \( T_G \) in \( L^{1,p}(\Omega) \) provided \(|\Omega| < \infty\), and in \( W^{1,p}(\Omega) \) if \(|\Omega| = \infty\).

Proof of Theorem 2. The only part of the theorem not covered by conditions from [3] for (2) is the necessity for \( p > n \) in (i).

Thus we need to prove that if \( T_G : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega), p > n, \) and \( W^{1,p}(\Omega) \hookrightarrow L^{\infty}(\Omega) \), then \( G' \in L^{\infty}(\mathbb{R}^1) \). From [11] and [12] it follows that \( G \) is absolutely continuous on \( \mathbb{R}^1 \) and \( G' \in L^{\infty}_{\text{loc}}(\mathbb{R}^1) \). Moreover, for all \( u \in W^{1,p}(\Omega) \) one has \( \nabla(G \circ u) = G'(u)\nabla u \) a.e. in \( \Omega \).

Seeking a contradiction suppose that \( G' \not\in L^{\infty}(\mathbb{R}^1) \). Then we shall construct a function \( U \in W^{1,p}(\Omega) \) such that \( \|\nabla(G \circ U)\|_{L^p(\Omega)} = \infty \). To do this we first construct a special sequence of functions \( \{u_j\} \), \( u_j \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega) \cap C_{\text{loc}}(\Omega) \) for \( j = 1, 2, \ldots \), and a sequence of pairwise disjoint balls \( \{B(y_j, \rho_j)\} = \{B_j\}, B_j \subset \subset \Omega, \) with the properties

\[
\|u_j\|_{L^p(\Omega)} + \left( \int_{\Omega \setminus \bigcup_{m \leq j} B_m} |\nabla u_j|^p \right)^{1/p} \leq \left( 1 + \cdots + \frac{1}{2^j} \right) \quad \text{for all } j \geq 1,
\]

\[
u_m|_{B_j} = u_j|_{B_j} \quad \text{for } m \geq j.
\]

The following inequalities are valid for all \( j \geq 1 \):

\[
\int_{B_j} |\nabla u_j|^p \leq 1/j^2,
\]

\[
\int_{B_j} |G'(u_j)|^p |\nabla u_j|^p \geq C(n),
\]

with some \( C(n) > 0 \) independent of \( j \).

By (5)–(7) the sequence \( \|u_j\|_{W^{1,p}(\Omega)} \) is bounded, and we can extract a subsequence converging weakly to some function \( U \in W^{1,p}(\Omega) \). Now (1) and (8) imply that \( \nabla(G \circ U) \not\in L^p(\Omega) \).

We construct the sequences \( \{u_j\} \) and \( \{B_j\} \) by induction. Using (4) we choose a constant \( R = R(n, p) \) such that for \( B(z, R) \subset \subset \Omega \)

\[
cap_p(z, B(z, R); \Omega) \geq 2.
\]

Without loss of generality we can assume that \( G' \not\in L^\infty(\mathbb{R}_+^1), \) where \( \mathbb{R}_+^1 = (0, +\infty) \).

To construct \( u_1 \) we take the number \( t_1 \) such that \( t_1 > 1, |G'(t_1)| \geq 1 \) and \( t_1 \) is a Lebesgue point of \( G' \). The latter implies that

\[
\left| \{ s : |t_1 - s| \leq \delta, |G'(s)| \geq 1 \} \right| \geq \delta
\]

for all sufficiently small \( \delta \). Next we choose \( 0 < r_1 < R \) so small that \( t_1 |B(0, r_1)|^{1/p} < 1/4 \). For these \( t_1 \) and \( r_1 \) using (3) we find a ball \( B(y_1, r_1), y_1 \in \Omega, \) and a function \( v_1 \in V_\Omega(y_1) \) supported in \( B(y_1, r_1) \cap \Omega \) such that \( \int_{\Omega} |\nabla v_1|^p \leq 1/(4t_1)^p \), or
equivalently
\[
\left( \int_{\Omega} |\nabla (2t_1v_1)|^p \right)^{1/p} \leq 1/2.
\]
Now we put \(w_1 = \min\{t_1, 2t_1v_1\}\). Note that truncating does not increase norm in \(W^{1,p}(\Omega)\). Thus because of our choice of \(r_1\) we have
\[
\|w_1\|_{W^{1,p}(\Omega)} \leq \|2t_1v_1\|_{W^{1,p}(\Omega)} \leq 1.
\]
Note that \(v_1(y_1) = 1\). Consequently there exists a ball \(B_1 = B(y_1, \rho_1) \subset \Omega\) such that \(w_1|_{B_1} \equiv t_1\). To finish construction of \(u_1\) we use the function \(\phi(x) = \max\{1 - |x|, 0\}, x \in \mathbb{R}^n\). We denote \(\phi_\varepsilon(x) = \phi(x/\varepsilon)\) for \(\varepsilon > 0\). We can choose \(a > 0\) so small that (10) is satisfied for \(\delta = a\), and \(\varepsilon < \rho_1\) such that
\[
\|a\phi_\varepsilon\|_{L^p(\mathbb{R}^n)} \leq C_1(n)a\varepsilon^{n/p} \leq 1/2,
\]
\[
\int_{\mathbb{R}^n} |\nabla (a\phi_\varepsilon)|^p = C_2(n)a^p/\varepsilon^{p-n} = 1/2.
\]
Note that (10) and the linear radial nature of \(\phi\) imply that for such \(a\)
\[
\left| \{z \in \mathbb{R}^n : |z| < \varepsilon, |G' \circ (t_1 + a\phi_\varepsilon(z))| \geq 1\} \right| \geq C_3(n)\varepsilon^n.
\]
Therefore one also has
\[
(11) \quad \int_{\mathbb{R}^n} |\nabla (G \circ (t_1 + a\phi_\varepsilon))|^p \geq C_3(n)a^p/\varepsilon^{p-n} \geq C_3(n)/2C_2(n).
\]
Now taking \(u_1(x) = w_1(x) + a\phi_\varepsilon(x - y_1), x \in \Omega\), we see that \(u_1 \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \cap C_{\text{loc}}(\Omega)\) and (4), (7), (8) are valid for \(j = 1\).

Now we construct \(u_k\) assuming that the functions \(u_1, \ldots, u_{k-1}\) and the balls \(B_1, \ldots, B_{k-1}\) with all required properties have already been constructed. We can find \(t_k\) such that
\[
(12) \quad t_k \geq 4\|u_{k-1}\|_{L^\infty(\Omega)},
\]
\(|G'(t_k)| \geq k\), and \(t_k\) is a Lebesgue point of \(G'\). Next we choose \(0 < r_k < R\) so small that
\[
(13) \quad 4r_k \leq \min\{\text{dist}(B_j, \partial\Omega), j = 1, \ldots, k-1\},
\]
and \(t_k|B(0, r_k)|^{1/p} < 1/2^{k+2}\). For \(r_k, t_k\) we find, using (4), a ball \(B(y_k, r_k), y_k \in \Omega\), and a function \(v_k \in V_{\Omega}(y_k)\) supported in \(B(y_k, r_k) \cap \Omega\) such that
\[
(14) \quad \left( \int_{\Omega} |\nabla (2t_kv_k)|^p \right)^{1/p} \leq 1/2^{k+2}.
\]
We note that (4) and (14) imply that \(\partial B(y_k, r_k) \cap \partial\Omega \neq \emptyset\). Thus \(B(y_k, r_k)\) does not intersect any of \(B_1, \ldots, B_{k-1}\) because of (13). We define \(w_k\) in a similar way as \(w_1\). Let \(w_k = \min\{t_k, u_{k-1} + 2t_kv_k\}\). Because of (12) \(w_k\) coincides with \(u_{k-1}\) in the balls \(B_1, \ldots, B_{k-1}\). From (12) one has \(u_{k-1}(y_k) + 2t_kv_k(y_k) > t_k\). Therefore there is a ball \(B_k = B(y_k, \rho_k) \subset \Omega, B_k \subset B(y_k, r_k)\), not intersecting \(B_1, \ldots, B_{k-1}\), such that \(w_k|_{B_k} \equiv t_k\). By induction we have the estimate
\[
\|w_k\|_{W^{1,p}(\Omega)} \leq \left( 1 + \ldots + \frac{1}{2^{k-1}} + \frac{1}{2^{k+1}} \right).
\]
To construct \( u_k \) we modify \( w_k \) in the ball \( B_k \). We use the function \( a\phi_x \) defined as above. We choose \( a \) and \( \varepsilon \) such that \( \text{supp}(a\phi_x) \subset B_k \),

\[
\left| \left\{ s : |t_k - s| \leq a, |G'(s)| \geq k \right\} \right| \geq a,
\]

\[
\|a\phi_x\|_{L^p(\mathbb{R}^n)} \leq 1/2^{k+1},
\]

\[
\int_{\mathbb{R}^n} |\nabla (a\phi_x)|^p = 1/k^2.
\]

By analogy with (11) from (15) and (16) we have

\[
\int_{\mathbb{R}^n} |\nabla (G \circ (t_k + a\phi_x))|^p \geq C_4(n)k^p/k^2 \geq C_5(n).
\]

Now we define \( u_k(x) = w_k(x) + a\phi_x(x - y_k) \), and (5)-(8) hold. This completes the proof.

REFERENCES


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