

## THE LINDELÖF PROPERTY AND FRAGMENTABILITY

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**ABSTRACT.** Let  $K$  be a compact Hausdorff space and  $C(K)$  the space of continuous real functions on  $K$ . In this paper we prove that any  $t_p(K)$ -Lindelöf subset of  $C(K)$  which is compact for the topology  $t_p(D)$  of pointwise convergence on a dense subset  $D \subset K$  is norm fragmented; i.e., each non-empty subset of it contains a non-empty  $t_p(D)$ -relatively open subset of small supremum norm diameter. Several applications are given.

### 1. INTRODUCTION

In what follows  $K$  will be a compact Hausdorff space,  $C(K)$  will stand for the space of continuous real functions on  $K$ , and for a given subset  $F \subset K$ ,  $t_p(F)$  will be the topology in  $C(K)$  of pointwise convergence on  $F$ .

The notion of fragmentability as stated below was introduced by Jayne and Rogers [8].

**Definition 1.** Let  $(X, \tau)$  be a topological space and  $\rho$  a metric on  $X$ . We say that  $(X, \tau)$  is *fragmented by  $\rho$*  (or  *$\rho$ -fragmented*) if for each non-empty subset  $A$  of  $X$  and for each  $\varepsilon > 0$  there exists a non-empty  $\tau$ -open subset  $U$  of  $X$  such that  $U \cap A \neq \emptyset$  and  $\rho\text{-diam}(U \cap A) \leq \varepsilon$ .

A result by one of us in [10] implies that every  $t_p(K)$ -compact subset of  $C(K)$  is fragmented by the supremum norm. On the other hand, Bourgin's result in [3, p.98], proved by using a construction by Stegall, states that the Radon-Nikodým property holds for weakly Lindelöf and weak\* compact convex subsets of dual Banach spaces; in other words, weakly Lindelöf and weak\* compact convex subsets of dual Banach spaces are fragmented by the dual norm. The aim of the present paper is to solve affirmatively the problem below (which appears in [4]):

**Problem 1.** Let  $D$  be a dense subset of  $K$  and let  $H$  be a  $t_p(D)$ -compact subset of  $C(K)$ . If  $H$  is  $t_p(K)$ -Lindelöf, is  $H$  fragmented by the norm of  $C(K)$ ?

Thus our main result, Theorem B, states that, if  $H$  satisfies the hypotheses of Problem 1, then the compact space  $(H, t_p(D))$  is fragmented by the norm of  $C(K)$ . Our Theorem B is a common generalization of the results in [10] and [3] cited above. It also extends [5, Proposition 1.1] and the main result in [4]. Our results here are

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very much related to the problem of knowing if  $\ell^\infty = C(\beta\mathbb{N})$  contains a  $t_p(\beta\mathbb{N})$ -Lindelöf subset  $Y$  separating the points of  $\beta\mathbb{N}$ . We prove that this is impossible if  $Y$  is assumed to be  $t_p(\mathbb{N})$ -Čech-analytic.

Compact sets fragmented by a lower-semicontinuous metric are called Radon-Nikodým compact and they are homeomorphic to a weak\*-compact subset of a dual Banach space with the Radon-Nikodým property [11]. One of the consequences of the positive solution of Problem 1 is that the space  $(H, t_p(D))$  appearing there is Radon-Nikodým compact and so, for instance, it is sequentially compact as well [11, Lemma 5.3].

2. PRELIMINARY RESULTS ON  $B_1(H)$

For a topological space  $H$ ,  $C_b(H)$  stands for the space of bounded continuous real functions on  $H$  and  $B_1(H)$  the space of pointwise limits of sequences of continuous functions on  $H$ .  $2^\mathbb{N}$  denotes the compact space of sequences of 0's and 1's endowed with its product topology and  $2^{(\mathbb{N})}$  is the set of finite sequences of 0's and 1's. For a  $t \in 2^{(\mathbb{N})}$ , we let  $|t|$  denote the length of  $t$ . Given  $\sigma \in 2^\mathbb{N}$  and  $n \in \mathbb{N}$ , let  $\sigma|n = (\sigma(1), \dots, \sigma(n)) \in 2^{(\mathbb{N})}$ .

Since the following theorem, which appears in Pol [12, p. 34] with only a sketch of a proof, is essential in our paper, we give a full proof.

**Theorem A.** *Let  $(H, d)$  be a complete metric space,  $D$  a subset of  $C_b(H)$  which is uniformly bounded by 1 and  $K = \overline{D}$  the closure of  $D$  in  $[-1, 1]^H$ . Then the following are equivalent:*

- (a)  $K \not\subset B_1(H)$ ;
- (b) *There exist a homeomorphism  $\varphi : 2^\mathbb{N} \rightarrow \varphi(2^\mathbb{N}) \subset H$ , a sequence  $(f_n)_{n \in \mathbb{N}}$  in  $D$  and numbers  $-1 < s < t < 1$  such that*

$$f_n(\varphi(\sigma)) \in G_{\sigma(n)} \text{ for every } \sigma \in 2^\mathbb{N} \text{ and } n \in \mathbb{N}$$

where  $G_0 := [-1, s)$  and  $G_1 := (t, 1]$ .

*Proof.* We note that  $f \in [-1, 1]^H$  is not in  $B_1(H)$  if, and only if, for some non-empty closed set  $F \subset H$  and some pair  $-1 < s < t < 1$  of real numbers,  $\{x \in F : f(x) < s\}$  and  $\{x \in F : f(x) > t\}$  are both dense in  $F$ ; cf. [2, Proposition 1E].

(a)  $\Rightarrow$  (b) Assume that there is  $f \in K \setminus B_1(H)$ . From the former remark we have a closed set  $F \subset H$  and  $s, t$  as above. Let  $G_0 := [-1, s)$ ,  $G_1 := (t, 1]$ . Then,  $\{x \in F : f(x) \in G_0\}$  and  $\{x \in F : f(x) \in G_1\}$  are both dense in  $F$ .

By induction on  $n = |t|$ , we choose a family  $\{U_t : t \in 2^{(\mathbb{N})}\}$  of non-empty relatively open subsets of  $F$  and a sequence  $(f_n)_{n \in \mathbb{N}}$  in  $D$  such that

- (i)  $U_\emptyset = F$ ,
- (ii) for each  $t \in 2^{(\mathbb{N})}$ ,  $\overline{U_{t0}} \cup \overline{U_{t1}} \subset U_t$  and  $\overline{U_{t0}} \cap \overline{U_{t1}} = \emptyset$ ,
- (iii)  $d\text{-diam}(U_t) < \frac{1}{|t|}$  for each  $t \in 2^{(\mathbb{N})}$ , and
- (iv)  $f_n(U_{tj}) \subset G_j$  for  $j = 0$  or  $1$  and  $|t| = n - 1$ .

*Construction.* (i) begins the induction. For  $n \geq 1$ , suppose  $\{U_t : |t| < n\}$ ,  $\{f_i : i < n\}$  have been chosen so as to satisfy (i)–(iv). For each  $t \in 2^{(\mathbb{N})}$  with  $|t| = n - 1$ , choose  $a_t, b_t \in U_t$  such that  $f(a_t) \in G_0$  and  $f(b_t) \in G_1$ . Since  $f \in \overline{D}$ , there is  $f_n \in D$  such that  $f_n(a_t) \in G_0$  and  $f_n(b_t) \in G_1$  for all  $t$  with  $|t| = n - 1$ . Since  $f_n$  is continuous, there are open neighbourhoods  $U_{t0}, U_{t1}$  of  $a_t, b_t$  respectively such that (ii)–(iv) are satisfied. This completes the construction.

Now let

$$\Delta := \bigcap_n \bigcup_{|t|=n} \overline{U}_t = \bigcap_n \bigcup_{|t|=n} U_t.$$

Then  $\Delta \subset H$  is compact and the map  $\varphi : 2^{\mathbb{N}} \rightarrow \Delta$  given by  $\varphi(\sigma) = \bigcap_{n=1}^{\infty} \overline{U_{\sigma|n}}$  is a homeomorphism that together with  $(f_n)$  fulfills condition (b).

(b)  $\Rightarrow$  (a) Let  $g : \overline{G_0} \cup \overline{G_1} \rightarrow \{0, 1\}$  be the obvious map. Let  $f$  be a pointwise cluster point of  $(f_n)$  and let  $\Delta := \varphi(2^{\mathbb{N}})$ . Then  $f(\Delta) \subset \overline{G_0} \cup \overline{G_1}$ , and from (b),  $g \circ f_n \circ \varphi$  is the  $n$ -th projection of  $2^{\mathbb{N}}$  onto  $\{0, 1\}$ . Since  $g \circ f \circ \varphi$  is a cluster point of the sequence  $(g \circ f_n \circ \varphi)$ ,  $g \circ f \circ \varphi$  is not Borel measurable by Sierpinski's theorem; cf. [14]. Hence  $f|_{\Delta}$  is not measurable and therefore  $f \in K \setminus B_1(H)$ .  $\square$

The concept of independent sequence of functions as appears below was introduced by Rosenthal in [13] (see also [2]),

**Definition 2.** A sequence of functions  $(f_n)$  in  $\mathbb{R}^{\Omega}$  is called *independent on*  $A \subset \Omega$  if there are numbers  $s < t$  such that for each pair of finite disjoint subsets  $P, Q \subset \mathbb{N}$  we have

$$(0) \quad \bigcap_{n \in P} \{\omega \in A : f_n(\omega) \leq s\} \cap \bigcap_{n \in Q} \{\omega \in A : f_n(\omega) \geq t\} \neq \emptyset.$$

If  $(f_n)$  is a sequence of *continuous* functions independent on a *compact* set  $A$ , then (0) holds for arbitrary disjoint subsets  $P$  and  $Q$  of  $\mathbb{N}$ . This fact is used in the proof of the next lemma, which links  $\beta\mathbb{N}$  and independent sequences of continuous functions.

**Lemma 1.** *Let  $H$  be a compact Hausdorff space and  $(f_n)_{n \in \mathbb{N}}$  a sequence in  $C(H)$  uniformly bounded by 1. If  $(f_n)_{n \in \mathbb{N}}$  is independent on  $H$ , then the mapping  $n \mapsto f_n$  extends to a homeomorphism of  $\beta\mathbb{N}$  onto the closure of  $\{f_n : n \in \mathbb{N}\}$  in  $[-1, 1]^H$ .*

*Proof.* By the definition of  $\beta\mathbb{N}$ , the map  $n \mapsto f_n$  extends to a continuous map  $\delta : \beta\mathbb{N} \rightarrow [-1, 1]^H$ . Clearly  $\delta(\beta\mathbb{N}) = \overline{\{f_n : n \in \mathbb{N}\}}$ . To show that  $\delta$  is a homeomorphism, it is sufficient to prove that  $\delta$  is one-to-one. Suppose  $\alpha, \beta \in \beta\mathbb{N}$  and  $\alpha \neq \beta$ . Then there are disjoint clopen neighbourhoods of  $\alpha$  and  $\beta$  respectively, i.e. for some  $P, Q \subset \mathbb{N}$ ,  $\alpha \in \overline{P}, \beta \in \overline{Q}$  and  $\overline{P} \cap \overline{Q} = \emptyset$ . Since  $H$  is compact and  $\{f_n : n \in \mathbb{N}\}$  is independent on  $H$ , there exist real numbers  $s < t$  and an  $h \in H$  such that  $\delta(n)(h) = f_n(h) \leq s$  for all  $n \in P$  and  $\delta(n)(h) = f_n(h) \geq t$  for all  $n \in Q$ . Since  $\alpha \in \overline{P}$ ,  $\delta(\alpha)(h) \leq s$  by the continuity of  $\delta$ . Similarly  $\delta(\beta)(h) \geq t$ . Hence  $\delta(\alpha) \neq \delta(\beta)$ .  $\square$

**Lemma 2.** *Let  $H$  be a compact metrizable space,  $D$  a uniformly bounded subset of  $C(H)$  that separates the points of  $H$  and let  $K$  be the pointwise closure of  $D$  in  $\mathbb{R}^H$ . If  $H$  is Lindelöf relative to the weak topology  $\sigma_K$  induced by  $K$ , then  $K \subset B_1(H)$ .*

*Proof.* We may and do assume that  $D$  is uniformly bounded by 1 and so  $K \subset [-1, 1]^H$ . We show that  $K \not\subset B_1(H)$  implies that  $(H, \sigma_K)$  is not Lindelöf.

Since  $H$  is a Polish space, according to Theorem A,  $K \not\subset B_1(H)$  implies the existence of a sequence  $(f_n)_{n \in \mathbb{N}}$  in  $D$ , real numbers  $-1 < s < t < 1$  and a homeomorphism of  $2^{\mathbb{N}}$  into  $H$ , denoted as  $\chi_M \mapsto h_M \in H$ , such that for each  $M \subset \mathbb{N}$

- (1)  $f_n(h_M) < s$  for each  $n \in \mathbb{N} \setminus M$ , and
- (2)  $f_n(h_M) > t$  for each  $n \in M$ .

The image  $\Delta$  of this homeomorphism is a compact subset of  $H$  and so  $\sigma_K$ -closed, because, by the assumption,  $D$  separates the points of  $H$  and so the topology of  $H$  is induced by  $D$  and is weaker than  $\sigma_K$ . The inequalities (1) and (2) imply that the sequence  $(f_n)_{n \in \mathbb{N}}$  is independent on  $H$ , and hence by Lemma 1, there exists a homeomorphism  $\delta$  of  $\beta\mathbb{N}$  onto  $C \stackrel{\text{def}}{=} \overline{\{f_n : n \in \mathbb{N}\}} \subset K$  extending the map  $n \mapsto f_n$ . From (1) and (2), it follows that, for each  $M \subset \mathbb{N}$ ,

$$(3) \quad x(h_M) \leq s \text{ if } x \in C \setminus \delta(\overline{M}) \quad \text{and} \quad x(h_M) \geq t \text{ if } x \in \delta(\overline{M}).$$

For each  $x \in C$ , let

$$(4) \quad G_x = \{h_M \in \Delta : x(h_M) \geq t\} = \{h_M \in \Delta : x(h_M) > s\}.$$

Then  $G_x$  is a  $\sigma_K$ -clopen subset of  $\Delta$ .

We show that, for each  $A \subset C$ ,  $a \in \overline{A}$  if and only if  $G_a \subset \bigcup\{G_x : x \in A\}$ . Suppose  $a \in \overline{A}$ . Then  $h_M \in G_a$  implies  $a(h_M) > s$  and so  $x(h_M) \geq s$  for some  $x \in A$ . Hence  $h_M \in G_x$  for some  $x \in A$  by (4). Conversely, if  $a \notin \overline{A}$ , then there is a closed and open neighbourhood of  $a$  that is disjoint from  $A$ , i.e. for some  $M \subset \mathbb{N}$ ,  $a \in \delta(\overline{M})$  and  $\delta(\overline{M}) \cap A = \emptyset$ . Then by (3),  $a(h_M) \geq t$  and  $x(h_M) \leq s$  for each  $x \in A$ , i.e.  $h_M \in G_a \setminus \bigcup\{G_x : x \in A\}$  by (4).

Finally suppose  $(H, \sigma_K)$  were Lindelöf. Then the  $\sigma_K$ -closed subset  $\Delta$  is also  $\sigma_K$ -Lindelöf. This implies that  $C$  is countably tight, i.e. for each subset  $A$  of  $C$ , each point in  $\overline{A}$  is in the closure of a countable subset of  $A$ . For, suppose  $A \subset C$  and  $a \in \overline{A}$ . Then from above  $G_a \subset \bigcup\{G_x : x \in A\}$ . Since  $G_a, G_x$  are all  $\sigma_K$ -clopen, there is a countable subset  $B$  of  $A$  such that  $G_a \subset \bigcup\{G_x : x \in B\}$  which implies that  $a \in \overline{B}$ . This proves that  $C$  (and hence  $\beta\mathbb{N}$ ) is countably tight. But if this were the case, then each compact separable space, being a continuous image of  $\beta\mathbb{N}$ , is countably tight. But, for instance,  $[0, 1]^{[0, 1]}$  with the product topology is separable and compact without being countably tight. This contradiction shows that  $(H, \sigma_K)$  cannot be Lindelöf.  $\square$

### 3. FRAGMENTABILITY

In this section, we prove the main theorem of this paper. The following lemma is its very special case, but, nevertheless, it contains the crux of the matter.

**Lemma 3.** *Let  $D$  be a uniformly bounded subset of  $C(2^{\mathbb{N}})$  such that, for some  $\varepsilon > 0$ , whenever  $x, x' \in 2^{\mathbb{N}}, x \neq x'$ , then*

$$\rho(x, x') \stackrel{\text{def}}{=} \sup_{f \in D} |f(x) - f(x')| \geq \varepsilon.$$

*If  $K$  is the closure of  $D$  in  $\mathbb{R}^{2^{\mathbb{N}}}$ , then  $2^{\mathbb{N}}$  is not Lindelöf relative to  $\sigma_K$ , the weak topology on  $2^{\mathbb{N}}$  induced by  $K$ .*

*Proof.* Suppose that  $(2^{\mathbb{N}}, \sigma_K)$  is Lindelöf, and we will reach a contradiction. First, by Lemma 2,  $K \subset B_1(2^{\mathbb{N}})$ . Let  $\mu$  denote the normalized Haar measure on  $2^{\mathbb{N}}$ , and let

$$\mathcal{U} = \{U \subset 2^{\mathbb{N}} : U \text{ is Borel, } \sigma_K\text{-open and } \mu(U) = 0\}.$$

Also let  $G = \bigcup \mathcal{U}$  and  $C = 2^{\mathbb{N}} \setminus G$ . Then  $C \neq \emptyset$ , for otherwise, using the Lindelöf property of  $(2^{\mathbb{N}}, \sigma_K)$ ,  $2^{\mathbb{N}}$  can be covered by countably many members of  $\mathcal{U}$  and consequently  $\mu(2^{\mathbb{N}}) = 0$ .

Let  $\varphi : (K, \tau_p) \rightarrow (L^1(2^{\mathbb{N}}, \mu), \text{norm})$  be the map that assigns to each  $f \in K$  the class  $[f]$ . Here  $\tau_p$  is the topology of pointwise convergence in  $\mathbb{R}^{2^{\mathbb{N}}}$ . Since  $(K, \tau_p)$  is angelic by [2, Theorem 3F], one can show that  $\varphi$  is continuous. In fact, if  $A \subset K$  and  $f \in A^{-\tau_p}$ , then there is a sequence in  $A$  converging pointwise to  $f$ . Hence by the bounded convergence theorem,  $\varphi(f) \in \varphi(A)^-$ . It follows that  $\varphi(K)$  is compact and metrizable and  $\varphi : (K, \tau_p) \rightarrow \varphi(K)$  is a quotient map. We claim that if  $f, g \in K$  and  $\varphi(f) = \varphi(g)$ , then  $f(x) = g(x)$  for each  $x \in C$ . For, let  $V = \{t \in 2^{\mathbb{N}} : f(t) \neq g(t)\}$ ; then  $V \in \mathcal{U}$  and so  $V$  and  $C$  are disjoint. It follows that each member  $x$  of  $C$  defines a continuous function  $\hat{x}$  on  $\varphi(K)$  satisfying  $\hat{x}(\varphi(f)) = f(x)$ . Hence  $\hat{C}$  is a subset of  $C(\varphi(K))$  and the latter is norm separable. Note that if  $x, x' \in C$ , then

$$\rho(x, x') = \sup_{f \in D} |f(x) - f(x')| = \sup_{f \in K} |f(x) - f(x')| = \| \hat{x} - \hat{x}' \| .$$

Here  $\| \cdot \|$  is the supremum norm of  $C(\varphi(K))$ . Hence  $(\hat{C}, \text{norm})$  and  $(C, \rho)$  are isometric. Since the former is separable and the latter discrete, we see that  $C$  is countable. It follows that  $C$  and  $G = 2^{\mathbb{N}} \setminus C$  are Borel sets and  $\mu(C) = 0$ . We reach our contradiction by observing that  $\mu(G) = 0$ . To see this, let  $L$  be a compact subset of  $G$ . Then  $L$  is  $\sigma_K$  closed in  $2^{\mathbb{N}}$ , because the assumption that  $D$  separates points of  $2^{\mathbb{N}}$  implies that  $\sigma_K$  is finer than the topology of  $2^{\mathbb{N}}$ . It follows that  $L$  is  $\sigma_K$ -Lindelöf and hence it is covered by countably many members from  $\mathcal{U}$ . Therefore  $\mu(L) = 0$ . By the regularity of  $\mu$ ,  $\mu(G) = 0$ . □

From now onwards, except in the last corollary in this section,  $D$  will be a dense subset of  $K$ . Given a  $t_p(D)$ -compact subset  $H$  of  $C(K)$  we will look at the elements of  $K$  as functions on  $H$ : for each point  $k$  in  $K$  we will denote by  $\hat{k}$  the restriction to  $H$  of the “point mass” at  $k$ , that is  $\hat{k}(f) := f(k)$ . It is clear that  $\hat{D} = \{\hat{d} : d \in D\}$  is a pointwise bounded set of continuous functions on the compact space  $(H, t_p(D))$ , and  $\hat{K} = \{\hat{k} : k \in K\}$  is a pointwise compact set of continuous functions on  $(H, t_p(K))$ . Obviously, the closure of  $\hat{D}$  in  $\mathbb{R}^H$  is  $\hat{K}$ .

In the proof of the following theorem, we use the simple fact that, for  $(X, \tau)$  in Definition 1 to be  $\rho$ -fragmented, it is sufficient that each  $\tau$ -closed non-empty subset of  $X$  has non-empty relatively  $\tau$ -open subsets of arbitrarily small  $\rho$ -diameter. Also in the proof  $\| \cdot \|$  will denote the supremum norm of  $C(K)$ .

**Theorem B.** *Let  $K$  be a compact Hausdorff space and let  $D$  be a dense subset of  $K$ . Then, every  $t_p(D)$ -compact subset of  $C(K)$  which is  $t_p(K)$ -Lindelöf is fragmented by the supremum norm, and so, it is a Radon-Nikodým compact space.*

*Proof.* Let  $H$  be a  $t_p(D)$  compact subset of  $C(K)$  and let  $B$  denote the unit ball of  $C(K)$ . Then  $B$  is  $t_p(D)$ -closed. If  $A$  is a non-empty  $t_p(D)$ -closed subset of  $H$ , then by the Baire category theorem, there exists an  $n \in \mathbb{N}$  such that  $(nB) \cap A$  has non-empty relative  $t_p(D)$ -interior. Hence in order to prove that  $(H, t_p(D))$  is fragmented by the norm it is sufficient to prove each  $(nB) \cap A$  is fragmented by the norm. So we may and do assume that  $H$  is uniformly bounded.

Suppose that  $(H, t_p(D))$  is not fragmented by the norm. Then, for some non-empty  $t_p(D)$ -compact subset  $C$  of  $H$  and  $\varepsilon > 0$ , each non-empty  $t_p(D)$ -open subset of  $C$  has norm diameter greater than  $\varepsilon$ . By induction on  $n = |t|$ ,  $t \in 2^{(\mathbb{N})}$ , we construct a family  $\{U_t : t \in 2^{(\mathbb{N})}\}$  of non-empty relatively  $t_p(D)$ -open subsets of  $C$  and a family  $\{x_t : t \in 2^{(\mathbb{N})}\}$  of points of  $D$ , satisfying the following conditions, where the closures are relative to  $t_p(D)$ :

- (i)  $U_\emptyset = C$ ,
- (ii) for each  $t$ ,  $\overline{U_{t0}} \cup \overline{U_{t1}} \subset U_t$ ,
- (iii)  $|(f - g)(x_t)| > \varepsilon$  for each  $f \in U_{t0}$  and  $g \in U_{t1}$ , and
- (iv) whenever  $s, t \in 2^{(\mathbb{N})}$  and  $|s| < |t|$ ,  $\text{diam } \hat{x}_s(\overline{U_{tj}}) < |t|^{-1}$  for  $j = 0, 1$ .

*Construction.* (i) starts the induction from  $n = 0$ . Next, for some  $n > 0$ , assume that  $\{U_t : |t| < n\}$  and  $\{x_s : |s| < n - 1\}$  have been constructed. Fix a  $t \in 2^{(\mathbb{N})}$  with  $|t| = n - 1$ . By assumption, for some  $f_0, f_1 \in U_t$ ,  $\|f_0 - f_1\| > \varepsilon$ , which means that  $|(f_0 - f_1)(x_t)| = |\hat{x}_t(f_0) - \hat{x}_t(f_1)| > \varepsilon$  for some  $x_t \in D$ . Since  $\hat{x}_t$  and  $\hat{x}_s$ ,  $|s| < |t|$ , are all continuous on  $(H, t_p(D))$ , one can select  $t_p(D)$ -open neighbourhoods  $U_{t0}$  and  $U_{t1}$  of  $f_0$  and  $f_1$ , respectively, so that (ii), (iii) and (iv) are satisfied. This completes the construction. Note that (iii) implies that  $\overline{U_{t0}} \cap \overline{U_{t1}} = \emptyset$  for each  $t \in 2^{(\mathbb{N})}$ .

Let  $F := \bigcap_n \bigcup_{|t|=n} \overline{U_t}$ . Then  $F$  is a compact subset of  $(H, t_p(D))$  and it is partitioned as  $F = \bigcup_{\sigma \in 2^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} \overline{U_{\sigma|n}}$ . Define  $\varphi : F \rightarrow 2^{\mathbb{N}}$  by  $\varphi^{-1}(\sigma) = \bigcap_{n \in \mathbb{N}} \overline{U_{\sigma|n}}$ . Then clearly  $\varphi$  is a continuous and onto map. For each  $t \in 2^{(\mathbb{N})}$  and  $\sigma \in 2^{\mathbb{N}}$ ,  $\hat{x}_t(\varphi^{-1}(\sigma))$  is a singleton by (iv). Hence  $\hat{x}_t$  ‘lifts’ to a continuous function  $x_t^*$  on  $2^{\mathbb{N}}$  such that

$$(5) \quad f(x_t) = \hat{x}_t(f) = x_t^*(\varphi(f))$$

for every  $f \in F$ . If  $\sigma, \sigma' \in 2^{\mathbb{N}}$  and  $\sigma \neq \sigma'$ , then for some  $n \in \{0\} \cup \mathbb{N}$ ,  $\sigma|n = \sigma'|n$  and  $\sigma(n+1) \neq \sigma'(n+1)$ . If we let  $t = \sigma|n$ , then by (iii),  $|x_t^*(\sigma) - x_t^*(\sigma')| > \varepsilon$ . This means that the hypothesis for  $D$  in Lemma 3 is satisfied by  $\{x_t^* : t \in 2^{(\mathbb{N})}\}$ . Let  $L$  be the closure of  $\{x_t^* : t \in 2^{(\mathbb{N})}\}$  in  $\mathbb{R}^{2^{\mathbb{N}}}$ . Then by Lemma 3,  $2^{\mathbb{N}}$  is not Lindelöf for the weak topology  $\sigma_L$ . However, we show below that our assumptions imply that  $(2^{\mathbb{N}}, \sigma_L)$  is Lindelöf and this contradiction proves the theorem. To see that  $(2^{\mathbb{N}}, \sigma_L)$  is Lindelöf, it is sufficient to prove that  $\varphi$  is  $(t_p(K) - \sigma_L)$ -continuous, since  $(F, t_p(K))$  is Lindelöf by hypothesis. For the continuity of  $\varphi$ , we must prove that the map  $f \mapsto \xi(\varphi(f))$  is  $t_p(K)$ -continuous on  $F$  for each  $\xi \in L$ . Fix  $\xi \in L$ . Then there is a net  $\{x_{t_\alpha}^*\}$  that converges to  $\xi$  pointwise. By the compactness of  $K$ , we may assume that  $x_{t_\alpha}$  converges to  $x \in K$ . Then by (5) we have that

$$\xi(\varphi(f)) = \lim_{\alpha} x_{t_\alpha}^*(\varphi(f)) = \lim_{\alpha} f(x_{t_\alpha}) = f(x)$$

for each  $f \in F$ , and  $f \mapsto f(x)$  is clearly  $t_p(K)$ -continuous. □

Properties of  $t_p(D)$ -compact norm bounded sets  $H$  which are fragmented by the supremum norm can be found in [5]. For instance, it is proved there that for such an  $H$  the closed convex hull of it,  $\overline{\text{co}(H)}^{t_p(D)}$ , is again  $t_p(D)$ -compact, satisfies  $\overline{\text{co}(H)}^{t_p(D)} = \overline{\text{co}(H)}^{\|\cdot\|}$  and this closed convex hull has the usual Radon-Nikodým property.

If a topological space  $(X, \tau)$  has a countable base (or more generally, is hereditarily Lindelöf) and if it is fragmented by a metric  $\rho$ , then a simple argument with points of condensation shows that  $(X, \rho)$  is separable. Combining this with Theorem B, we obtain the following.

**Corollary C.** *Let  $K$  be a compact and  $D$  a dense and countable subset of  $K$ . Then, every  $t_p(D)$ -compact subset of  $C(K)$  which is  $t_p(K)$ -Lindelöf is separable for the supremum norm.*

In the next corollary, the rôles of  $K, D$  and  $H$  will be as in Lemma 2.

**Corollary D.** *Let  $H$  be a compact Hausdorff space,  $D$  a pointwise bounded subset of  $C(H)$  that separates points of  $H$  and let  $K$  be the pointwise closure of  $D$  in  $\mathbb{R}^H$ . If  $H$  is Lindelöf relative to the weak topology  $\sigma_K$  induced by  $K$ , then, for each closed subset  $F$  of  $H$ , there is a dense  $G_\delta$ -subset  $Z_F$  of  $F$  such that  $\{f|_F : f \in K\}$  is equicontinuous at each point of  $Z_F$ .*

*Proof.* With the pointwise topology  $K$  is a compact Hausdorff space with the dense subset  $D$ . As usual, each  $h \in H$  gives rise to  $\hat{h} \in C(K)$  such that  $\hat{h}(f) = f(h)$  for each  $f \in K$ . Then  $h \mapsto \hat{h}$  is a homeomorphism  $H \rightarrow (\hat{H}, t_p(D))$  and by hypothesis  $\hat{H}$  is  $t_p(K)$ -Lindelöf. Hence by Theorem B,  $(\hat{H}, t_p(D))$  is fragmented by the norm, or equivalently  $H$  is  $\rho$ -fragmented, where  $\rho$  is a metric on  $H$  given by

$$\rho(h, h') = \sup_{f \in K} |f(h) - f(h')| \quad \text{for } h, h' \in H.$$

Consequently, using the category argument (cf. [11]), one sees that, given a closed subset  $F$  of  $H$ , there is a dense  $G_\delta$ -subset  $Z_F$  of  $F$  such that the identity map  $H \rightarrow (H, \rho)$  is continuous at each point of  $Z_F$ , which is equivalent to the conclusion of the corollary.  $\square$

*Remark.* The corollary above is a generalization of Théorème 4.1 in [17] where the space  $(H, \sigma_K)$  is assumed to be  $K$ -analytic. However, Talagrand remarks that one can prove the theorem with only assuming that  $(H, \sigma_K)$  is Lindelöf and that “*la démonstration serait alors beaucoup plus longue*”. The proof of the stronger theorem has never been published.

#### 4. APPLICATIONS

Given a Banach space  $(X, \|\cdot\|)$ , a subset  $B$  of the dual unit ball  $B_{X^*}$  is said to be norming if  $\|x\| = \sup\{|x^*(x)| : x^* \in B\}$  for every  $x \in X$ . As a consequence of the Hahn-Banach separation theorem, if  $B$  is norming, then its absolutely convex hull  $D$  is weak\* dense in  $B_{X^*}$ . In this way,  $X$  endowed with the topology  $\sigma(X, B)$  of pointwise convergence on  $B$  (or equivalently, on  $D$ ) appears as a subspace of  $(C(B_{X^*}), t_p(D))$  and Theorem B can be used to state the corollary below.

**Corollary E.** *Let  $(X, \|\cdot\|)$  be a Banach space and  $B \subset B_{X^*}$  a norming subset. Then, every weakly Lindelöf  $\sigma(X, B)$ -compact subset of  $X$  is fragmented by the norm and so it is a Radon-Nikodým compact space under  $\sigma(X, B)$ .*

This corollary implies the result in [3] mentioned in the Introduction without the convexity assumption: a weakly Lindelöf weak\*-compact subset of a dual Banach space is fragmented by the dual norm. In [15], Srivatsa proves that, if  $f$  is a continuous map from a metric space  $T$  into a Banach space  $X$  with its weak topology, then  $f$  is the pointwise (norm)-limit of a sequence of continuous functions  $T \rightarrow (X, \|\cdot\|)$ , i.e.  $f \in B_1(T, X)$ . The following corollary shows that, if the weak topology in the above is replaced by  $\sigma(X, B)$  for some norming subset  $B$  of  $B_{X^*}$ , one still reaches the same conclusion provided that the image of  $f$  is contained in a weakly Lindelöf subset of  $X$  and  $T$  is complete.

**Corollary F.** *Let  $K$  be a compact Hausdorff space,  $D \subset K$  a dense subset,  $T$  a complete metric space and  $f : T \rightarrow C(K)$  a  $t_p(D)$ -continuous function. If there is a  $t_p(K)$ -Lindelöf subset  $Y \subset C(K)$  such that  $f(T) \subset Y$ , then  $f \in B_1(T, C(K))$ .*

*Proof.* To prove that  $f \in B_1(T, C(K))$  it is enough to prove that for every compact subset  $W$  of  $T$  the restriction  $f|_W$  has a point of norm continuity, [16]. Given a compact  $W \subset T$ , the image  $f(W)$  is  $t_p(D)$ -compact and so norm fragmented after Theorem B. According to [11, Lemma 1.1] the identity map

$$id : (f(W), t_p(D)) \rightarrow (f(W), \|\cdot\|)$$

has a point of continuity and thus we get that  $f|_W$  has a point of norm continuity and the proof is done.  $\square$

It is a usual exercise in elementary measure theory that, if  $f$  is a real-function on  $\mathbb{R} \times \mathbb{R}$  such that  $f_t \stackrel{\text{def}}{=} f(t, \cdot)$  is continuous for each  $t \in \mathbb{R}$  and such that  $f^s \stackrel{\text{def}}{=} f(\cdot, s)$  is continuous for each  $s$  belonging to a dense subset of  $\mathbb{R}$ , then  $f \in B_1(\mathbb{R} \times \mathbb{R})$ . The following corollary, which is a straightforward consequence of the previous one, is a far reaching generalization of this. For related results concerning measurability of separately continuous functions, see [9], [18] and references cited therein.

**Corollary G.** *Let  $K$  be a compact Hausdorff space and  $T$  a complete metric space. Let  $f : T \times K \rightarrow [-1, 1]$  be a function verifying*

- (i) *There is a  $t_p(K)$ -Lindelöf subset  $Y \subset C(K)$  such that  $\{f_t : t \in T\} \subset Y$ .*
- (ii) *The set  $\{x \in K : f^x \in C(T)\}$  is dense in  $K$ .*

*Then,  $f \in B_1(T \times K)$ .*

In [1, p. 610], Arkhangelskii raises the following question: Suppose  $K$  is a compact Hausdorff space. If there exists a Lindelöf subset  $Y$  of  $(C(K), t_p(K))$  that separates points of  $K$ , must  $K$  be countably tight? As far as we know, this question, in the usual set theory, is still open. When  $K = \beta\mathbb{N}$ , which is not countably tight as seen in Section 2, the question above is the same as Problem 2 in [4]. The following corollary is a partial answer to this problem.

**Corollary H.**  *$\ell^\infty = C(\beta\mathbb{N})$  cannot contain a  $t_p(\mathbb{N})$ -Čech-analytic and  $t_p(\beta\mathbb{N})$ -Lindelöf subset  $Y$  separating the points of  $\beta\mathbb{N}$ .*

*Proof.* Assume that there is such a  $Y$ . Since  $(C(\beta\mathbb{N}), t_p(\mathbb{N})) = (\ell^\infty, t_p(\mathbb{N}))$  is a metric analytic space, if  $Y$  is  $t_p(\mathbb{N})$ -Čech analytic, then  $Y$  is metric analytic too. The last implies the existence of a continuous map from a Polish space  $P$  onto  $(Y, t_p(N))$ , say,

$$\varphi : P \rightarrow (Y, t_p(N)) \hookrightarrow (C(\beta\mathbb{N}), t_p(\mathbb{N})).$$

Now, Corollary F can be applied to deduce that  $\varphi \in B_1(P, C(\beta\mathbb{N}))$ . This implies that its range, which contains  $Y$ , is norm separable and so  $\beta\mathbb{N}$  must be metrizable which is impossible.  $\square$

The next result is more general than [4, Corollary F] and it is in the same vein as results [7, Theorem 1] and [6, Corollary 8]. The ideas of [4, Corollary F] can be used to provide a proof of this corollary.

**Corollary I.** *Let  $K$  be a compact Hausdorff space,  $D \subset K$  a dense subset,  $T$  a topological space that contains a dense Čech-complete subspace and  $f : T \rightarrow C(K)$  a  $t_p(D)$ -continuous function. If there is a  $t_p(K)$ -Lindelöf subset  $Y \subset C(K)$  such that  $f(T) \subset Y$ , then  $f$  is norm-continuous at each point of a dense  $G_\delta$  subset of  $T$ .*

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