INTEGRAL REPRESENTATION OF LINEAR FUNCTIONALS
ON SPACES OF UNBOUNDED FUNCTIONS

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Abstract. Let $L$ be a vector lattice of real functions on a set $\Omega$ with $1 \in L$, and let $P$ be a linear positive functional on $L$. Conditions are given which imply the representation
$$P(f) = \int f d\pi, \quad f \in L,$$
for some bounded charge $\pi$. As an application, for any bounded charge $\pi$ on a field $\mathcal{F}$, the dual of $L^1(\pi)$ is shown to be isometrically isomorphic to a suitable space of bounded charges on $\mathcal{F}$. In addition, it is proved that, under one more assumption on $L$, $P$ is the integral with respect to a $\sigma$-additive bounded charge.

1. Introduction

Throughout, $L$ is a class of real functions on a set $\Omega$ and $B$ is the class of all real bounded functions on $\Omega$. It is assumed that $L$ is a vector lattice including the constants; that is, $1 \in L$ and $af + bg, f \vee g, f \wedge g$ are in $L$ whenever $f, g \in L$ and $a, b \in \mathbb{R}$. Moreover, $P: L \to \mathbb{R}$ is a linear positive functional.

To get integral representations for $P$ is a classical task. Indeed, in addition to the non-recent celebrated results (of Riesz, Daniell, Stone, Radon and many others), there is also recent work in this area; for instance, to prove versions of the Riesz theorem for an arbitrary Hausdorff space, or else to characterize some dual spaces. See [4], [8], [10], [11] and [12]. However, most results concern situations where $L \subseteq B$, or where the main goal is a finitely additive representation, are almost neglected (an exception is [10]).

In this paper, instead, $L$ is not necessarily a subset of $B$. In such a framework, conditions are given under which
$$P(f) = \int f d\pi \quad \text{for all } f \in L$$
for some positive bounded charge $\pi$. By a bounded charge, it is meant a real, bounded, finitely additive measure defined on some field $\mathcal{F}$ of subsets of $\Omega$. A positive bounded charge (p.b.c.) is a bounded charge $\pi$ such that $\pi(A) \geq 0$ for all $A \in \mathcal{F}$. Further, all integrals in this paper are intended in the sense of Dunford and Schwartz; cf. [3] and [6].

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By the Daniell-Stone theorem, if
\[(f_n) \subset L \text{ and } f_n(\omega) \downarrow 0 \text{ for all } \omega \in \Omega \Rightarrow P(f_n) \downarrow 0,\]
then (1) holds for a unique \(\sigma\)-additive p.b.c. \(\pi\) defined on the \(\sigma\)-field generated by the elements of \(L\). Plainly, condition (2) is a form of \(\sigma\)-additivity for the functional \(P\). Thus, one can conjecture that, when (2) is dropped, (1) still holds for some not necessarily \(\sigma\)-additive \(\pi\). This is not the case, however, as the following simple example shows.

**Example 1.** Let \(\Omega = [0, \infty), f = \sum_{n=1}^{\infty} n I_{[n-1,n)}\) and \(L\) be the linear span of \(f\) and all the simple functions. After some algebra, it can be checked that \(g^+ \in L\) whenever \(g \in L\) and thus \(L\) is a vector lattice including the constants. Let \(\nu\) be any p.b.c. on the power set of \(\Omega\) such that \(f\) is \(\nu\)-integrable, and let \(a \in \mathbb{R}\) be such that \(a > \int f d\nu\). If \(g \in L\), then \(g = h + bf\) for some unique simple function \(h\) and \(b \in \mathbb{R}\). Accordingly, for each \(g = h + bf \in L\), define \(P(g) = \int h d\nu + ab\). Then, \(P\) is linear, positive and (1) fails for any p.b.c. \(\pi\). To check positivity, fix \(g = h + bf \in L\) with \(g \geq 0\). Since \(g \geq 0\), one has \(b \geq 0\), and hence
\[P(g) \geq \int h d\nu + b \int f d\nu = \int g d\nu \geq 0.\]

In general, the integral representation of \(P\) can fail only on \(L - B\). In fact, by the Hahn-Banach theorem, there is a linear positive functional \(T\) on \(B\) agreeing with \(P\) on \(L \cap B\). Let \(\pi(A) = T(I_A)\) for \(A \subset \Omega\). Then, \(\pi\) is a p.b.c. on the power set of \(\Omega\) and \(T(f) = \int f d\pi\) whenever \(f\) is a simple function. Since simple functions are dense in \(B\) in the sup-norm, \(T(f) = \int f d\pi\) for all \(f \in B\). Hence, if \(\mathcal{F}\) is any field which makes every \(f \in L \cap B\) \(\pi\)-integrable, where \(\pi\) is the restriction of \(\pi\) to \(\mathcal{F}\), then \(P(f) = T(f) = \int f d\pi\) for all \(f \in L \cap B\).

Thus, to get an integral representation on all of \(L\), the key point is the possibility of approximating \(P\) on \(L - B\) through its restriction to \(L \cap B\).

The main result of this paper is an equivalent condition for such an approximation, and thus for (1). A sufficient condition for (1), suggested by the proof of the main result, is also obtained. As an application, for any bounded charge \(\pi\) on a field \(\mathcal{F}\), the dual of \(L^1(\pi)\) is shown to be isometrically isomorphic to \(\{\nu: \nu\text{ is a bounded charge on } \mathcal{F} \text{ and } |\nu| \leq c|\pi| \text{ for some } c > 0\}\). Moreover, a condition on \(L\) is given which implies that (1) holds for a \(\sigma\)-additive p.b.c. \(\pi\). The condition is just a strengthening of the above-mentioned sufficient condition for (1) (which does not grant \(\sigma\)-additivity of \(\pi\)).

A last note is that the content of this paper is connected with a notion of coherence, for expectations of random variables with values in a Banach space, given in [2]. Other related references are [3] and [4]. Indeed, if \(L\) is regarded as a class of real random variables, an expectation on \(L\) can be defined as any functional on \(L\) which meets some suitable coherence condition. This happens in some approaches to the foundations of probability theory, and in particular in de Finetti’s approach. Then, one question is whether a given expectation is an integral with respect to some finitely additive probability measure.

2. Basic definitions and notation

We briefly recall some definitions (see [3] for more information). Given a bounded charge \(\pi\) on a field \(\mathcal{F}\) of subsets of \(\Omega\), let \(\pi^+(A) = \sup\{\pi(A \cap F): F \in \mathcal{F}\}\) for \(A \in \mathcal{F}\), \(\pi^- = (-\pi)^+\) and \(|\pi| = \pi^+-\pi^-\). Then, \(\pi^+, \pi^-\) and \(|\pi|\) are p.b.c.’s and \(\pi = \pi^+-\pi^-\).
For $A \subset \Omega$, let $|\pi|^*(A) = \inf\{|||F|||: A \subset F \in \mathcal{F}\}$. Given an $\mathcal{F}$-simple function $f = \sum_{i=1}^n a_i 1_{A_i}$, where $a_1,\ldots,a_n \in \mathbb{R}$ and $\{A_1,\ldots,A_n\}$ is a partition of $\Omega$ in $\mathcal{F}$, $\int f d\pi$ is defined as $\int f d\pi := \sum_{i=1}^n a_i \pi(A_i)$. An arbitrary function $f: \Omega \rightarrow \mathbb{R}$ is $\pi$-integrable if there is a sequence $(f_n)$ of $\mathcal{F}$-simple functions such that

$$\lim_{n,m} \int |f_n - f_m| d\pi = 0 \quad \text{and} \quad \lim_n |\pi|^*(|f_n - f| > \varepsilon) = 0 \quad \text{for each} \quad \varepsilon > 0.$$  

If $f$ is $\pi$-integrable and $(f_n)$ is as above, $\int f d\pi$ is defined as $\int f d\pi := \lim_n \int f_n d\pi$. Moreover, if (3) holds for some sequence $(f_n)$ of $\pi$-integrable (but not necessarily $\mathcal{F}$-simple) functions, then $f$ is $\pi$-integrable and $\int |f_n - f| d\pi | \rightarrow 0$.

Let $L^1(\pi)$ be the set of $\pi$-integrable functions quotiented by the equivalence relation $f \sim g$ if and only if $|\pi|^*|f - g| > \varepsilon) = 0$ for each $\varepsilon > 0$. Setting $||f||_1 = \int |f| d\pi|$, $L^1(\pi)$ becomes a normed space. As usual, when confusion does not arise, $L^1(\pi)$ is also regarded as a space of functions (and not of equivalence classes). For instance, we will write $f \in L^1(\pi)$ or $L \subset L^1(\pi)$ to mean that the function $f$ or every element of $L$, respectively, is $\pi$-integrable.

Finally, when $\Omega$ is a topological space, $C(\Omega)$ denotes the class of all real continuous functions on $\Omega$.

3. FINITELY ADDITIVE INTEGRAL REPRESENTATION OF FUNCTIONALS

By the discussion in Section 1 there exist a field $\mathcal{F}$ of subsets of $\Omega$ and a p.b.c. $\pi$ on $\mathcal{F}$ such that $L \cap B \subset L^1(\pi)$ and $P(f) = \int f d\pi$ for $f \in L \cap B$. The next result is based on this fact.

**Theorem 2.** Let $L$ be a vector lattice of real functions on $\Omega$ with $1 \in L$, and let $P$ be a linear positive functional on $L$. Fix a field $\mathcal{F}$ and a p.b.c. $\pi$ on $\mathcal{F}$ such that $L \cap B \subset L^1(\pi)$ and $P(f) = \int f d\pi$ for $f \in L \cap B$. Then, $L \subset L^1(\pi)$. Moreover, $P(f) = \int f d\pi$ for all $f \in L$ if and only if

$$\sup\{P(\phi) : 0 \leq \phi \leq g, \phi \in L\} < \infty \quad \text{for every} \quad g \in \overline{L} \quad \text{with} \quad g \geq 0,$$

where $\overline{L}$ is the closure of $L$ in the $L^1(\pi)$-norm.

**Proof.** Let $f \in L$. Since $f = f^+ - f^-$, for proving $f \in L^1(\pi)$ it can be assumed $f \geq 0$. Then, since $f \cap n \in L \cap B$, one has

$$\sup_n \int f \cap n d\pi = \sup_n P(f \cap n) \leq P(f),$$

and this implies $\lim_{n,m} \int |f \cap n - f \cap m| d\pi = 0$. For each $n$, there is a p.b.c. $\pi_n$ on the power set of $\Omega$ such that $\pi_n = \pi$ on $\mathcal{F}$ and $\pi_n(f > n) = \pi^*(f > n)$; cf. [3], Theorem 3.3.3, p. 73. Since $\pi_n$ extends $\pi$, $\int (f \cap n)d\pi_n = \int (f \cap n)d\pi$. Hence, given $\varepsilon > 0$,

$$\pi^*(|f \cap n - f| > \varepsilon) \leq \pi_n(f > n) \leq \frac{1}{n} \int f \cap n d\pi_n \leq \frac{1}{n} P(f) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$  

It follows that $f \in L^1(\pi)$ and $\int f d\pi = \sup_n \int (f \cap n)d\pi \leq P(f)$. Next, if $P(f) = \int f d\pi$ for all $f \in L$, then $P(\phi) \leq \int g d\pi < \infty \quad \text{whenever} \quad \phi \in L, \quad g \in \overline{L} \quad \text{and} \quad 0 < \phi \leq g$. Conversely, suppose that $P(f) \neq \int f d\pi$ for some $f \in L$. We prove that, under this assumption, (4) fails. Since $P(f^+) > P(f^-) \neq \int f^+ d\pi - \int f^- d\pi$, it can be assumed
that \( f \geq 0 \). Then, since \( P(f) \geq \int fd\pi \), it must be \( P(f) > \int fd\pi \). Define a function \( g \in \mathcal{T} \) as follows. Since
\[
\sum_{j=1}^{\infty} \int [f \land j - f \land (j-1)]d\pi = \int fd\pi < \infty,
\]
there is a sequence \((a_j)\) such that \( a_j > 0 \) for all \( j \), \( a_j \uparrow \infty \), and
\[
(5) \quad \sum_{j=1}^{\infty} a_j \int [f \land j - f \land (j-1)]d\pi < \infty.
\]
Let \( g_n = \sum_{j=1}^{n} a_j [f \land j - f \land (j-1)] \) and \( g = \lim_n g_n \). By (5), \( \lim_{n,m} \int |g_n - g_m|d\pi = 0 \). Moreover, \( (g_n) \subseteq L \) and \( \pi^*(|g_n - g| > \varepsilon) \leq \pi^*(f > n) \rightarrow 0 \) for each \( \varepsilon > 0 \), so that \( g \in \mathcal{T} \). Observe now that, if \( \phi_j := a_{j+1}(f - f \land j) \), then \( \phi_j \in L, 0 \leq \phi_j \leq g \), and
\[
P(\phi_j) = a_{j+1} \left( P(f) - \int f \land jd\pi \right) \geq a_{j+1} \left( P(f) - \int f d\pi \right).
\]
Hence, since \( P(f) > \int fd\pi \) and \( a_j \uparrow \infty \), (4) fails.

Generally, Theorem 2 helps to decide whether \( P \) admits an integral representation; see the next section. Moreover, the proof of Theorem 2 suggests the following criterion.

**Theorem 3.** Let \( L, P \) and \( \pi \) be as in Theorem 2 and let \( \mathcal{T} \) denote the closure of \( L \) in the \( L^1(\pi) \)-norm (recall that, by Theorem 2, \( L \subseteq L^1(\pi) \)). In order that \( P(f) = \int fd\pi \) for all \( f \in L \), it is sufficient that
\[
(i) \quad g \in \mathcal{T}, g \geq 0 \text{ and } g \land n \in L \text{ for each } n \in \mathbb{N} \Rightarrow g \in L.
\]

**Proof.** Suppose that \( P(f) > \int fd\pi \) for some \( f \in L \) with \( f \geq 0 \). Define \((a_j), (g_j), (\phi_j)\) and \( g \) as in the second part of the proof of Theorem 2. Then, \( g \geq 0 \) and \( g \in \mathcal{T} \). Given \( n \), take \( j \) with \( a_j > n \), and note that \( g = g_j \) on \( \{f \leq j\} \) and \( g \geq g_j \geq a_j > n \) on \( \{f > j\} \). Hence, \( g \land n = g_j \land n \in L \). However, since \( P(\phi_j) \rightarrow \infty \) and \( g \geq \phi_j \) for all \( j \), one has \( g \not\in L \). Thus, (i) fails.

Condition (i) trivially holds when \( L = \mathcal{T} \). Hence, Theorem 2 yields
\[
P(f) = \int fd\pi \quad \text{for all } f \in L \text{ whenever } L \text{ is closed in the } L^1(\pi) \text{-norm}.
\]
In particular, one consequence of the above fact is the following. Fix a p.b.c. \( \nu \) on a field \( \mathcal{F} \). If \( L = L^1(\nu) \) and \( P \) satisfies \( P(I_A) = \nu(A) \) for \( A \in \mathcal{F} \), then \( P \) is the integral with respect to \( \nu \); see [2], Theorem (2.13). We now give two more applications of Theorem 3.

**Example 4.** Let \( \Omega \) be a topological space. Fix a p.b.c. \( \nu \) on the field generated by the open sets and define \( T(f) = \int fd\nu \) for \( f \in C(\Omega) \cap B \). Then, the only extension of \( T \) to \( L^1(\nu) \cap C(\Omega) \), as a linear positive functional, is the integral with respect to \( \nu \). In fact, for any \( g : \Omega \rightarrow \mathbb{R} \) one has \( g \in C(\Omega) \) whenever \( g \land n \in C(\Omega) \) for all \( n \), and thus it suffices to apply Theorem 3 with \( L = L^1(\nu) \cap C(\Omega) \) and \( P \) such that \( P = T \) on \( C(\Omega) \cap B \).
Example 5. Let $\Omega = \mathbb{R}$ and let $\mathcal{D}$ be the class of those real functions which are almost everywhere differentiable (with respect to Lebesgue measure). Given $g: \mathbb{R} \to \mathbb{R}$, it can be checked that $|g| \in \mathcal{D}$ whenever $g \in \mathcal{D}$ (and thus $\mathcal{D}$ is a vector lattice), and that $g \in \mathcal{D}$ whenever $g \wedge n \in \mathcal{D}$ for all $n$. Fix a p.b.c. $\nu$ on the field $\mathcal{F}$ generated by the intervals and define $T(I_A) = \nu(A)$ for $A \in \mathcal{F}$. Then, the only extension of $T$ to $L^1(\nu) \cap \mathcal{D}$, as a linear positive functional, is the integral with respect to $\nu$. This follows by Theorem 3 by setting $L = L^1(\nu) \cap \mathcal{D}$ and by taking $P$ such that $P(I_A) = \nu(A)$ for $A \in \mathcal{F}$.

Clearly, Theorems 2 and 3 also apply to those functionals $T: L \to \mathbb{R}$ which admit the representation $T = T_1 - T_2$ with $T_1$ and $T_2$ linear and positive. For later purposes, we recall standard conditions on $T$ which are equivalent to such a representation.

Lemma 6. Let $T: L \to \mathbb{R}$. Then, $T$ has the representation $T = T_1 - T_2$, where $T_1$ and $T_2$ are linear and positive, if and only if $T$ is linear and

$$\alpha(f) := \sup \{ T(\phi) : 0 \leq \phi \leq f, \phi \in L \} < \infty \text{ for every } f \in L \text{ with } f \geq 0.$$ 

Moreover, $T_1$ can be taken as $T_1(f) = \alpha(f^+) - \alpha(f^-)$ for every $f \in L$.

4. Two related results

Let $\pi$ be a bounded charge on a field $\mathcal{F}$ of subsets of $\Omega$, let

$$\mathcal{B}(\pi) = \{ \nu : \nu \text{ is a bounded charge on } \mathcal{F} \text{ and } |\nu| \leq c|\pi| \text{ for some } c > 0 \},$$

and let us define a norm on $\mathcal{B}(\pi)$ as

$$||\nu|| = \inf \{ c > 0 : |\nu| \leq c|\pi| \} \text{ for } \nu \in \mathcal{B}(\pi).$$

Moreover, let $T(\pi)$ be the dual of $L^1(\pi)$, with the usual norm $||T|| = \sup \{|T(f)| : f \in L^1(\pi), ||f||_1 = 1\}$, $T \in T(\pi)$. As an application of the material in Section 3 we now prove that $\mathcal{B}(\pi)$ is isometrically isomorphic to $T(\pi)$.

Theorem 7. Let $I(\nu)(f) = \int f d\nu$, where $\nu \in \mathcal{B}(\pi)$ and $f \in L^1(\pi)$. Then, $I: \mathcal{B}(\pi) \to T(\pi)$ is an isometric isomorphism.

Proof. Let $\nu \in \mathcal{B}(\pi)$. Since $L^1(\pi) \subset L^1(\nu)$, $I(\nu)(f)$ is well defined and $|I(\nu)(f)| \leq ||\nu|| ||f||_1$ for each $f \in L^1(\pi)$. Thus, $I(\nu) \in T(\pi)$ and $||I(\nu)|| \leq ||\nu||$. To prove $||I(\nu)|| \geq ||\nu||$, it can be assumed $\nu \neq 0$. Then, given $\varepsilon > 0$, there is $A \in \mathcal{F}$ with

$$\nu(A) > (||\nu|| - \varepsilon)|\pi|(A) \text{ and } |\pi|(A) > 0.$$ 

Moreover, by 3, Theorem 2.6.2, p. 56, there is $E \in \mathcal{F}$ with

$$\nu(E \cap F) > -\varepsilon |\pi|(A) \text{ and } \nu(E^c \cap F) < \varepsilon |\pi|(A) \text{ for each } F \in \mathcal{F}.$$

Let $f = (I_A I_E - I_A I_{E^c})/|\pi|(A)$. Then, $||f||_1 = 1$ and

$$I(\nu)(f)|\pi|(A) = \nu(A \cap E) - \nu(A \cap E^c) = |\pi|(A) - 2(\nu^+(A \cap E^c) + \nu^-(A \cap E)) + (||\nu|| - \varepsilon)|\pi|(A) - 4\varepsilon |\pi|(A).$$

It follows that $||I(\nu)|| > ||\nu|| - 5\varepsilon$, and thus $I$ is an isometry. Since $I$ is clearly linear, it remains only to prove that $I$ is surjective. Fix $T \in T(\pi)$. By using Lemma 6 it is easily seen that $T = T_1 - T_2$, where $T_1$ and $T_2$ are linear, positive and continuous in the $L^1(\pi)$-norm. Hence, it can be assumed that $T$ is positive. Let $\lambda$ be a p.b.c.
on the power set of $\Omega$ such that $T(f) = \int f \, d\lambda$ for $f \in L^1(\pi) \cap B$, and let $\nu$ be the restriction of $\lambda$ to $\mathcal{F}$. Then,

$$\nu(F) = T(I_F) \leq \|T\|_{L^1(\pi)} \cdot \|\lambda\|_{L^1(\pi)}$$

for all $F \in \mathcal{F}$.

Thus, $\nu \in \mathcal{B}(\pi)$, and this implies $L^1(\pi) \subset L^1(\nu)$ and $T(f) = \int f \, d\nu$ for $f \in L^1(\pi) \cap B$. Let $0 \leq \phi \leq g$, where $\phi \in L^1(\pi)$ and $g$ is in the closure of $L^1(\pi)$ in the $L^1(\nu)$-norm. Then

$$T(\phi) = \lim_n T(\phi \wedge n) = \lim_n \int \phi \wedge n \, d\nu \leq \int g \, d\nu$$

where the first equality depends on continuity of $T$ in the $L^1(\pi)$-norm. By Theorem 2 (applied with $L = L^1(\pi)$ and $P = T$), it follows that $T = I(\nu)$.

It is worth noting that, given $\nu \in \mathcal{B}(\pi)$, $I(\nu)$ need not be of the form $I(\nu)(f) = \int fg \, d\pi$ for each $f \in L^1(\pi)$ and some $\pi$-essentially bounded $g \in \Omega \to \mathbb{R}$. In fact, $\nu$ can fail to have a density with respect to $\pi$. In general, to decide whether $I(\nu)$ has the above form, one needs some finitely additive version of the Radon-Nikodym theorem. See, for instance, [1].

Let us turn now to the second result in this section. Next, Theorem 8 states that, for $P$ to admit a $\sigma$-additive integral representation, it is enough that $L$ meets a suitable strengthening of condition (i) in Theorem 4.

**Theorem 8.** Let $L$ be a vector lattice with $1 \in L$, and let $P$ be a linear positive functional on $L$. If

**(ii)**

$$g \geq 0 \text{ and } g \wedge n \in L \text{ for each } n \in \mathbb{N} \Rightarrow g \in L,$$

then there is a unique $\sigma$-additive p.b.c. $\pi$, defined on the $\sigma$-field generated by the elements of $L$, such that $L \subset L^1(\pi)$ and $P(f) = \int f \, d\pi$ for all $f \in L$.

Theorem 8 includes some known results as particular cases. For instance, when $\Omega$ is a topological space and $L = C(\Omega)$, Theorem 8 reduces to a result of Hewitt; cf. [7]. Theorems 14 and 15. Likewise, when $L$ is the class of all real functions on $\Omega$ which are measurable with respect to a given $\sigma$-field, Theorem 8 has been obtained by Dubins; cf. [4], Lemmas 1–4 and Theorem 1. On the other hand, in addition to unifying the previous results, Theorem 8 has the merit of giving a general criterion for proving a $\sigma$-additive integral representation. As an example, when $\Omega = \mathbb{R}$, Theorem 8 works for $L = D$ or $L = D \cap C(\mathbb{R})$, where $D$ is the class of real functions which are almost everywhere differentiable (with respect to Lebesgue measure); cf. Example 5.

It is possible to give a proof of Theorem 8 based on Theorem 2 only. However, since $\pi$ is to be $\sigma$-additive, it is more convenient to use the result of Daniell and Stone mentioned in Section 4. In fact, if we can prove condition (2), then Theorem 8 automatically follows. Let us fix a p.b.c. $\nu$ on the power set of $\Omega$ such that $P(f) = \int f \, d\nu$ for $f \in L \cap B$. In order to prove (2) a useful fact is that, under (ii), Theorem 8 implies

$$L \subset L^1(\nu) \quad \text{and} \quad P(f) = \int f \, d\nu \quad \text{for all } f \in L.$$  

We also need two lemmas. In both, $L_\nu$ denotes the class of those $f \in \Omega \to \mathbb{R}$ such that, for some $(f_n) \subset L$, $f_n \to f$ uniformly.
Lemma 9. Let $E = \{ \phi \leq a \}$ and $F = \{ \phi \leq b \}$, where $\phi \in L_u$ and $a < b$. Given $f \in L \cap B$ with $f \geq 0$ and a scalar $c \geq \sup f$, there is $g \in L \cap B$ such that
\[ g \geq f, \quad g = f \quad \text{on } E, \quad g = c \quad \text{on } F_c. \]

Proof. Take $\phi_0 \in L$ with $|\phi(\omega) - \phi_0(\omega)| < (b - a)/3$ for all $\omega \in \Omega$, and define
\[ \psi = \left( \frac{3\phi_0 - 2a - b}{b - a} \right)^+ \land 1, \]
and $g = f \lor (\psi)$. \hfill \boxed{}

Lemma 10. Let $A_n = \{ \phi \leq a_n \}$, $n \in \mathbb{N}$, where $\phi \in L_u$ and $a_1 < a_2 < \cdots$. If $L$ meets (ii) and $A_n \uparrow \Omega$, then $\nu(A_n^c) = 0$ for some $n$.

Proof. It can be assumed that $A_1 \neq \emptyset$ and $A_n \neq A_{n-1}$ for $n > 1$. Towards a contradiction, suppose that $\nu(A_n^c) > 0$ for all $n$. By Lemma 9, there is $(g_n) \subset L \cap B$ such that $g_n = 1$ and, for $n > 1$
\[ g_n \geq g_{n-1}, \quad g_n = g_{n-1} \quad \text{on } A_{n-1}, \quad g_n = c_n \quad \text{on } A_n^c, \]
where $c_n = n\nu(\Omega)(1 + \sup g_{n-1})/\nu(A_n^c)$. Let $F_1 = A_1, F_n = A_n - A_{n-1}$ for $n > 1$, and $g = \sum_{n=1}^{\infty} I_{F_n} g_n$. Then, for each $n > 1$, $g = g_n$ on $A_n$ and $g \geq g_n = c_n > n$ on $A_n^c$, and thus $g \land n = g_n \land n \in L$. By (ii), $g \in L$ and thus $g \in L^1(\nu)$. This leads to a contradiction. In fact,
\[ c_n \nu(g \geq c_n) \geq c_n \nu(A_n^c) > n\nu(\Omega) \quad \text{for all } n > 1, \]
and thus $g \not\in L^1(\nu)$. \hfill \boxed{}

Proof of Theorem 8. It is enough to prove condition (2). Fix $(f_n) \subset L$ such that $f_n \downarrow 0$. Given $\varepsilon > 0$, it suffices to show that $\nu(f_j \geq \varepsilon) = 0$ for some $j$. In this case, in fact,
\[ 0 \leq P(f_n) \leq P(f_j) = \int_{f_j} d\nu \leq \varepsilon \nu(\Omega) \quad \text{for all } n \geq j. \]
Define $\phi_n = (f_n/\varepsilon) \land 1, \phi = \sum_{n=1}^{\infty} 2^{-n} \phi_n$ and $A_n = \{ \phi \leq (n-1)/n \}$. Then, $\phi \in L_u$. Since $f_n \downarrow 0$, $\phi(\omega) < 1$ for all $\omega \in \Omega$, and thus $A_n \uparrow \Omega$. By Lemma 11, $\nu(A_n^c) = 0$ for some $n$. To conclude the proof it suffices to note that, for some $j$, $(f_j \geq \varepsilon) \subset A_n^c$. \hfill \boxed{}

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References


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