

PURELY INFINITE, SIMPLE C^* -ALGEBRAS ARISING FROM FREE PRODUCT CONSTRUCTIONS. III

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(Communicated by David R. Larson)

ABSTRACT. In the reduced free product of C^* -algebras, $(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$ with respect to faithful states ϕ_1 and ϕ_2 , A is purely infinite and simple if A_1 is a reduced crossed product $B \rtimes_{\alpha, r} G$ for G an infinite group, if ϕ_1 is well behaved with respect to this crossed product decomposition, if $A_2 \neq \mathbf{C}$ and if ϕ is not a trace.

The reduced free product construction for C^* -algebras was invented independently by Voiculescu [11] and, in a more limited sense, Avitzour [1]. (The term “reduced” is to distinguish this construction from the universal or “full” free product of C^* -algebras.) It is a natural construction in Voiculescu’s free probability theory (see [12]). Given unital C^* -algebras A_ι with states ϕ_ι whose GNS representations are faithful ($\iota \in I$), the construction yields

$$(A, \phi) = \ast_{\iota \in I} (A_\iota, \phi_\iota),$$

where A is a unital C^* -algebra containing copies $A_\iota \hookrightarrow A$ and generated by $\bigcup_{\iota \in I} A_\iota$, and where ϕ is a state on A with faithful GNS representation that restricts to give ϕ_ι on A_ι for every $\iota \in I$ and such that $(A_\iota)_{\iota \in I}$ is free with respect to ϕ . Moreover, ϕ is a trace if and only if every ϕ_ι is a trace; by [4], ϕ is faithful on A if and only if ϕ_ι is faithful on A_ι for every $\iota \in I$.

It is a very interesting open question whether every simple, unital C^* -algebra must either have a trace or be purely infinite. Purely infinite C^* -algebras were defined by J. Cuntz [3]. A simple unital C^* -algebra A is purely infinite if and only if for every positive element $x \in A$ there is $y \in A$ with $y^*xy = 1$. An equivalent condition is that every hereditary C^* -subalgebra of A contains an infinite projection.

Let

$$(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$$

be a reduced free product of C^* -algebras. In [8] it was shown that if ϕ_1 or ϕ_2 is nontracial and if A_1 and A_2 are not too small in a specific sense, then A is properly infinite. It is a plausible conjecture that whenever A is simple and at least one of ϕ_1 and ϕ_2 is not a trace, the C^* -algebra A must be purely infinite. The first results in this direction were [7], where in a certain class of examples when ϕ_1 was assumed to be nonfaithful, A was shown to be purely infinite and simple. In [5], assuming ϕ_1

Received by the editors December 17, 1998.

2000 *Mathematics Subject Classification*. Primary 46L09; Secondary 46L54.

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and ϕ_2 faithful, A was shown to be purely infinite and simple in the case when the centralizer of ϕ_1 in A_1 contains a diffuse abelian subalgebra and when A_2 contains a partial isometry that, loosely speaking, scales ϕ_2 by a constant $\lambda \neq 1$. In [9], reduced free products of (countably) infinitely many C^* -algebras that are not too small in a specific sense were shown to be purely infinite.

In this note, we prove a theorem implying that A is purely infinite and simple under somewhat different conditions. For example, if $A_1 = C(\mathbf{T})$ is the algebra of all continuous functions on the circle and if ϕ_1 is given by integration with respect to Haar measure, then A is simple and purely infinite provided only that $A_2 \neq \mathbf{C}$ and ϕ_2 is faithful but not a trace.

Parts of this work were done while the first-named author took part in the program “Quantenergodynamik” organized by Walter Thirring at the Erwin Schrödinger International Institute for Mathematical Physics. She would like to thank the Institute, the Institut für Theoretische Physik at Universität Wien, and Professors Heide Narnhofer and Walter Thirring for their warm hospitality.

Notation. We begin with some notation, which has appeared elsewhere. Given an algebra \mathfrak{A} and subsets $S_\iota \subseteq \mathfrak{A}$ ($\iota \in I$), let $\Lambda^\circ((S_\iota)_{\iota \in I})$ be the set of all words $w = a_1 a_2 \cdots a_n$ where $n \geq 1$, $a_j \in S_{\iota_j}$ and $\iota_1 \neq \iota_2, \iota_2 \neq \iota_3, \dots, \iota_{n-1} \neq \iota_n$. We will refer to the elements a_1, \dots, a_n as the letters of the word w ; we will sometimes regard the word as a product of specific letters, and sometimes as an actual element of the algebra \mathfrak{A} , as it suits the situation.

Moreover, if a C^* -algebra A and a state $\phi : A \rightarrow \mathbf{C}$ are specified, we will denote the kernel of ϕ by A° .

Theorem. *Let A_1 be a reduced crossed product C^* -algebra, $A_1 = B \rtimes_{\alpha,r} G$, where G is an infinite discrete group and where B is a unital C^* -algebra. Denote by u_g ($g \in G$) the unitaries in A_1 arising from the reduced crossed product construction and implementing the automorphisms α_g on B . Let ϕ_1 be a faithful state on B that is preserved by all the automorphisms α_g and denote also by ϕ_1 its extension to the state on A_1 that vanishes on the subspace Bu_g for every nontrivial $g \in G$. Let A_2 be a unital C^* -algebra, $A_2 \neq \mathbf{C}$, with a faithful state ϕ_2 ; let*

$$(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$$

be the reduced free product of C^ -algebras. Suppose that at least one of ϕ_1 and ϕ_2 is not a trace.*

Then A is purely infinite and simple.

Proof. Our strategy will be to show that A is itself the reduced crossed product of a C^* -subalgebra D by the group G , where D is (isomorphic to) the reduced free product of infinitely many C^* -algebras; a result from [9] will thereby show that D is purely infinite and simple. We will then show that the action of G on D is properly outer; a result of Kishimoto and Kumjian [10] will thereby imply that A is purely infinite and simple.

Claim 1. The family

$$(B, (u_g^* A_2 u_g)_{g \in G})$$

is free with respect to ϕ .

Proof. We must show that

$$(1) \quad \Lambda^\circ(B^\circ, (u_g^* A_2^\circ u_g)_{g \in G}) \subseteq \ker \phi.$$

Let x be a word belonging to the left-hand side of (1). Splitting off the unitaries u_g^* and u_g from the letters in x , then grouping together any neighbors in the resulting word belonging to A_1 and using that $u_{g_1} B^\circ u_{g_2}^* \subseteq A_1^\circ$ whenever $g_1, g_2 \in G$ and that $u_{g_1} u_{g_2}^* \in A_1^\circ$ if $g_1 \neq g_2$, we see that x is equal to a word $x' \in \Lambda^\circ(A_1^\circ, A_2^\circ)$. Hence $x \in \ker \phi$ by freeness. This finishes the proof of Claim 1.

Let D be the C^* -subalgebra of A generated by $B \cup \bigcup_{g \in G} u_g^* A_2 u_g$.

Claim 2. D is simple and purely infinite.

Proof. Since $A_2 \neq \mathbf{C}$, there is a self-adjoint element $x \in A_2 \setminus \mathbf{C}1$. Let μ be the distribution of x ; namely, μ is the probability measure whose support is the spectrum of x and such that $\phi_2(x^k) = \int_{\mathbf{R}} t^k d\mu(t)$ for all $k \geq 1$. A consequence of Bercovici and Voiculescu's result [2, Prop. 8] is that for some n large enough, the measure arising as the n -fold additive free convolution

$$\mu_n \stackrel{\text{def}}{=} \underbrace{\mu \boxplus \mu \boxplus \cdots \boxplus \mu}_{n \text{ times}}$$

has support equal to an interval $[a, b]$ and is absolutely continuous with respect to Lebesgue measure. If g_1, g_2, \dots, g_n are distinct elements of G , then by Claim 1 the distribution of $y \stackrel{\text{def}}{=} \sum_{j=1}^n u_{g_j}^* x u_{g_j}$ is μ_n ; therefore y generates an abelian subalgebra of

$$D(g_1, \dots, g_n) \stackrel{\text{def}}{=} C^*\left(\bigcup_{j=1}^n u_{g_j}^* A_2 u_{g_j}\right)$$

on which ϕ is given by a measure without atoms; it follows from [6, Prop. 4.1] that $D(g_1, \dots, g_n)$ contains a unitary v satisfying $\phi(v) = 0$ (in fact, this proposition gives $\phi(v^k) = 0$ for all nonzero integers k , but we will not need this). Therefore, partitioning the family $(u_g^* A_2 u_g)_{g \in G}$ into subcollections of cardinality n , and including B in one of these subcollections, we see that D is isomorphic to the free product of infinitely many C^* -algebras with respect to faithful states,

$$(D, \phi) \cong \bigstar_{k=1}^\infty (D_k, \psi_k),$$

where each D_k contains a unitary that evaluates to zero under ψ_k . Moreover, since either ϕ_2 or $\phi_1|_B$ is not a trace, at least one of the ψ_k is not a trace. By [9, Thm. 2.1], D is therefore simple and purely infinite. This finishes the proof of Claim 2.

Claim 3. D has trivial relative commutant in A .

Proof. Let

$$D_0 = C^*\left(\bigcup_{g \in G} u_g^* A_2 u_g\right) \subseteq D;$$

we will show that D_0 has trivial relative commutant in A , which will imply the same for D . Suppose that $x \in A$ and x commutes with D_0 ; our goal is to show that x must belong to $\mathbf{C}1$. Let $x_0 = x - \phi(x)1$ and suppose, to obtain a contradiction, that $x_0 \neq 0$. Since ϕ is faithful, $\|x_0\|_2 = \phi(x_0^* x_0)^{1/2} > 0$. Choose ϵ so that $0 < \epsilon < \frac{\|x_0\|_2}{3}$. Since

$$(2) \quad \mathbf{C}1 + \text{span } \Lambda^\circ\left(B^\circ \cup \bigcup_{g \in G \setminus \{e\}} B u_g, A_2^\circ\right)$$

is a dense $*$ -subalgebra of A , and since $\Lambda^\circ(B^\circ \cup \bigcup_{g \in G \setminus \{e\}} Bu_g, A_2^\circ) \subseteq \ker \phi$, there is a sum of finitely many words, $y = w_1 + w_2 + \cdots + w_m$ with $w_1, w_2, \dots, w_m \in \Lambda^\circ(B^\circ \cup \bigcup_{g \in G \setminus \{e\}} Bu_g, A_2^\circ)$, such that $\|x_0 - y\| < \epsilon$. Let F be the finite subset of G whose elements are the identity element and all nontrivial elements $g \in G$ for which some w_j has a letter coming from Bu_g . From the proof of Claim 2, there is $n \in \mathbf{N}$ such that for any n distinct elements, g_1, g_2, \dots, g_n of G , there is a unitary

$$v \in D(g_1, g_2, \dots, g_n) = C^*\left(\bigcup_{j=1}^n u_{g_j}^* A_2 u_{g_j}\right)$$

with $\phi(v) = 0$. We take this unitary v having ensured that the n distinct elements satisfy $g_j \notin F$ and $g_j^{-1} \notin F$ for every $j \in \{1, \dots, n\}$.

Let us show that vy and yv are orthogonal with respect to the inner product on A induced by ϕ , i.e. that $\langle yv, vy \rangle_\phi = \phi(v^* y^* v y) = 0$. Since

$$C1 + \text{span } \Lambda^\circ(u_{g_1}^* A_2^\circ u_{g_1}, u_{g_2}^* A_2^\circ u_{g_2}, \dots, u_{g_n}^* A_2^\circ u_{g_n})$$

is a dense $*$ -subalgebra of $D(g_1, \dots, g_n)$ and since (as can be seen using Claim 1)

$$\Lambda^\circ(u_{g_1}^* A_2^\circ u_{g_1}, u_{g_2}^* A_2^\circ u_{g_2}, \dots, u_{g_n}^* A_2^\circ u_{g_n}) \subseteq \ker \phi,$$

for every $\eta > 0$ there is a sum of finitely many words $z = w'_1 + w'_2 + \cdots + w'_p$ with

$$w'_1, \dots, w'_p \in \Lambda^\circ(u_{g_1}^* A_2^\circ u_{g_1}, u_{g_2}^* A_2^\circ u_{g_2}, \dots, u_{g_n}^* A_2^\circ u_{g_n}),$$

such that $\|v - z\| < \eta$. But we see that each w'_j is equal to a word

$$w''_j \in \Lambda^\circ(\{u_g \mid g \in G \setminus \{e\}\}, A_2^\circ)$$

where w''_j begins with $u_{g_j^{-1}}$ and ends with u_{g_k} for some $j, k \in \{1, \dots, n\}$, and where w''_j has length at least three. Since

$$w_1, \dots, w_m \in \Lambda^\circ\left(B^\circ \cup \bigcup_{g \in F \setminus \{e\}} Bu_g, A_2^\circ\right),$$

when we consider a product $(w''_{i_1})^* w_{j_1}^* w''_{i_2} w_{j_2}$ for arbitrary $i_1, i_2 \in \{1, \dots, p\}$ and $j_1, j_2 \in \{1, \dots, m\}$, the choice of the elements g_1, \dots, g_n ensures that there is not too much cancellation and we are left with a reduced word

$$(w''_{i_1})^* w_{j_1}^* w''_{i_2} w_{j_2} = w \in \Lambda^\circ\left(B^\circ \cup \bigcup_{g \in G \setminus \{e\}} Bu_g, A_2^\circ\right);$$

hence $\phi((w''_{i_1})^* w_{j_1}^* w''_{i_2} w_{j_2}) = 0$. This implies that $\phi(z^* y^* z y) = 0$. Since $\eta > 0$ was arbitrary and $|\phi(v^* y^* v y) - \phi(z^* y^* z y)| \leq \eta(2 + \eta)\|y\|^2$, we have $\phi(v^* y^* v y) = 0$, i.e. vy and vy are orthogonal.

We now obtain the contradiction. Since x_0 belongs to the commutant of D_0 , we must have $vx_0 - x_0v = 0$. But by orthogonality of vy and yv ,

$$\|vy - yv\| \geq \|vy - yv\|_2 > \|vy\|_2 = \|y\|_2$$

and hence

$$\|vx_0 - x_0v\| \geq \|vy - yv\| - 2\epsilon > \|y\|_2 - 2\epsilon \geq \|x_0\|_2 - 3\epsilon > 0,$$

which is a contradiction. This finishes the proof of Claim 3.

Claim 4. For every nontrivial $g \in G$, $\beta_g \stackrel{\text{def}}{=} \text{Ad}(u_g)$ is an outer automorphism of D , $g \mapsto \beta_g$ is a group homomorphism and A is isomorphic to the reduced crossed product $D \rtimes_{\beta, r} G$.

Proof. Clearly, $\text{Ad}(u_g)$ is an automorphism of D , for every $g \in G$, and $g \mapsto \beta_g$ is a group homomorphism. From the density of (2) in A and the fact that $u_g B = B u_g$, we see that $\text{span} \bigcup_{g \in G} D u_g$ is dense in A . Moreover, whenever $g' \in G$ is nontrivial, $D u_{g'} \subseteq \ker \phi$; this can be seen by approximating an arbitrary element of $D u_{g'}$ by sums of words each belonging to $\{u_{g'}\} \cup \Lambda^\circ(B^\circ, (u_g^* A_2^\circ u_g)_{g \in G}) u_{g'}$. As the GNS representation of ϕ is faithful on A , one sees that A is isomorphic to the reduced crossed product $D \rtimes_{\beta, r} G$.

We will now show that β_g is an outer automorphism of D whenever $g \neq e$. Indeed, if it were inner then letting $v_g \in D$ be such that $\beta_g = \text{Ad}(v_g)$, we would have $u_g^* v_g$ commuting with D . By Claim 3, this would imply that u_g is a scalar multiple of v_g , hence belongs to D , which contradicts that $D u_g \subseteq \ker \phi$. This finishes the proof of Claim 4.

Now that A is seen to be the crossed product of a simple, purely infinite C^* -algebra by an infinite discrete group acting by outer automorphisms, Kishimoto and Kumjian's result [10, Lemma 10] shows that A is simple and purely infinite. \square

REFERENCES

1. D. Avitzour, *Free products of C^* -algebras*, Trans. Amer. Math. Soc. **271** (1982), 423-465. MR **83h**:46070
2. H. Bercovici, D. Voiculescu, *Superconvergence to the free central limit theorem and failure of Cramér theorem for free random variables*, Prob. Theory Relat. Fields **102** (1995), 215-222. MR **96k**:46115
3. J. Cuntz, *K -theory for certain C^* -algebras*, Ann. of Math. **113** (1981), 181-197. MR **84c**:46058
4. K.J. Dykema, *Faithfulness of free product states*, J. Funct. Anal. **154** (1998), 223-229. MR **99e**:46066
5. ———, *Purely infinite simple C^* -algebras arising from free product constructions, II*, Math. Scand. (to appear).
6. K.J. Dykema, U. Haagerup, M. Rørdam, *The stable rank of some free product C^* -algebras*, Duke Math. J. **90** (1997), 95-121; *correction*, vol. 94, 1998, p. 213. MR **99g**:46077a
7. K.J. Dykema, M. Rørdam, *Purely infinite simple C^* -algebras arising from free product constructions*, Can. J. Math. **50** (1998), 323-341. MR **99d**:46074
8. ———, *Projections in free product C^* -algebras*, Geom. Funct. Anal. **8** (1998), 1-16. MR **99d**:46075
9. ———, *Projections in free product C^* -algebras, II*, Math. Z. (to appear).
10. A. Kishimoto, A. Kumjian, *Crossed products of Cuntz algebras by quasi-free automorphisms*, Fields Inst. Commun. **13** (1997), 173-192. MR **98h**:46076
11. D. Voiculescu, *Symmetries of some reduced free product C^* -algebras*, Operator Algebras and Their Connections with Topology and Ergodic Theory, Lecture Notes in Mathematics, vol. 1132, Springer-Verlag, 1985, pp. 556-588. MR **87d**:46075
12. D. Voiculescu, K.J. Dykema, A. Nica, *Free Random Variables*, CRM Monograph Series vol. 1, American Mathematical Society, 1992. MR **94c**:46133

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