

## A HIERARCHY OF THIN SETS RELATED TO THE BOUNDEDNESS OF TRIGONOMETRIC SERIES

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(Communicated by Christopher D. Sogge)

**ABSTRACT.** We study the family  $\mathcal{B}_0$  of the sets on which some series of the form  $\sum_{k \in \mathbb{N}} |\sin \pi n_k x|$  is uniformly bounded. We show that the families  $\mathcal{B}_0^c$  of all sets admitting the boundary  $c$  form a hierarchy which is discontinuous with respect to the operations of intersection and union.

Let us recall two kinds of thin sets studied in harmonic analysis. A set  $X \subseteq [0, 1]$  is called an  $N$ -set (in honour of V. V. Nemytskiĭ) if there exists a trigonometric series absolutely converging on  $X$  but not converging absolutely everywhere; it is called an  $N_0$ -set if there exists a series of the form  $\sum_{k \in \mathbb{N}} |\sin \pi n_k x|$  converging on  $X$ . J. Arbault [1] showed that the family  $\mathcal{N}_0$  of all  $N_0$ -sets is a proper subfamily of the family  $\mathcal{N}$  of all  $N$ -sets, and both families share some common properties.

In the paper [3] we examined several modifications of the definitions of the families  $\mathcal{N}$  and  $\mathcal{N}_0$ , and compared the obtained families one to another. We showed that the family  $\mathcal{B}_0$  of all sets on which some series of the form  $\sum_{k \in \mathbb{N}} |\sin \pi n_k x|$  is uniformly bounded differs from previously known “classical” families. In the present paper we continue the study of the family  $\mathcal{B}_0$  considering the hierarchy of the families  $\mathcal{B}_0^c$  of all sets admitting the boundary  $c$ . We show that if  $c < d$ , then  $\mathcal{B}_0^c$  is a proper subfamily of  $\mathcal{B}_0^d$ , and  $\bigcup_{d < c} \mathcal{B}_0^d \subsetneq \mathcal{B}_0^c \subsetneq \bigcap_{d > c} \mathcal{B}_0^d$  for every  $c > 0$ .

For a review of families of trigonometric thin sets, some historical notes, and also for many new results we refer the reader to the paper [2].

We shall deal with the quotient group  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ ; however, we will not distinguish between the elements of  $\mathbb{T}$  and the reals, or between the functions defined on  $\mathbb{T}$  and the corresponding periodic functions on  $\mathbb{R}$ . For a real  $x$ , let  $[x]$  denote the integer part of  $x$  and let  $\|x\|$  denote the distance of  $x$  to the nearest integer, i.e.  $[x] = \max\{k \in \mathbb{Z} : k \leq x\}$  and  $\|x\| = \min\{|x - k| : k \in \mathbb{Z}\}$ . Let us note that  $\|x\| - \|y\| \leq \|x + y\| \leq \|x\| + \|y\|$  and  $\|-x\| = \|x\|$  for all  $x, y \in \mathbb{T}$ . The space  $\mathbb{T}$  equipped with the metric  $\varrho(x, y) = \|x - y\|$  is a compact topological group.

In an accordance with [3], we define  $B_0$ -sets as follows.

**Definition 1.** A set  $X \subseteq \mathbb{T}$  is a  $B_0$ -set if there exists an increasing sequence  $\{n_k\}_{k \in \mathbb{N}}$  of natural numbers such that the series  $\sum_{k \in \mathbb{N}} \|n_k x\|$  is uniformly bounded

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Received by the editors January 12, 1999.

2000 *Mathematics Subject Classification.* Primary 43A46; Secondary 42A05, 42A32.

*Key words and phrases.* Trigonometric thin sets,  $N_0$ -sets,  $B_0$ -sets, uniform boundedness.

This work was supported by grant 2/4034/97 of Slovak Grant Agency VEGA. The research was partly done when the author was visiting the Mathematical Institute of the University in Bonn with financial support by the Graduiertenkolleg.

on  $X$ , i.e. there exists a real  $c$  such that  $\sum_{k \in \mathbb{N}} \|n_k x\| \leq c$  for all  $x \in X$ . The family of all  $B_0$ -sets is denoted by  $\mathcal{B}_0$ .

Since  $2\|x\| \leq |\sin \pi x| \leq \pi\|x\|$  for all  $x \in \mathbb{T}$ , it is clear that the previous definition remains equivalent if we replace  $\sum_{k \in \mathbb{N}} \|n_k x\|$  by  $\sum_{k \in \mathbb{N}} |\sin \pi n_k x|$ .

**Proposition 2.** *A set  $X \subseteq \mathbb{T}$  is a  $B_0$ -set if and only if there exist an increasing sequence  $\{n_k\}_{k \in \mathbb{N}}$  of natural numbers and a sequence  $\{a_k\}_{k \in \mathbb{N}}$  of elements of  $\mathbb{T}$  such that the series  $\sum_{k \in \mathbb{N}} \|n_k x + a_k\|$  is uniformly bounded on  $X$ .*

*Proof.* It follows immediately from Lemma 3 below.  $\square$

**Lemma 3.** *Let  $\{n_k\}_{k \in \mathbb{N}}$  be an increasing sequence of natural numbers and let  $\{a_k\}_{k \in \mathbb{N}}$  be a sequence of elements of  $\mathbb{T}$ . Then for every  $\varepsilon > 0$  there exists an increasing sequence  $\{m_j\}_{j \in \mathbb{N}}$  of natural numbers such that*

$$\sum_{j \in \mathbb{N}} \|m_j x\| \leq \sum_{k \in \mathbb{N}} \|n_k x + a_k\| + \varepsilon$$

for every  $x \in \mathbb{T}$ .

*Proof.* We use a classical argument. Since  $\mathbb{T}$  is compact, there exists an increasing function  $h : \mathbb{N} \rightarrow \mathbb{N}$  such that the sequence  $\{n_{h(2j+1)} - n_{h(2j)}\}_{j \in \mathbb{N}}$  is increasing and  $\sum_{j \in \mathbb{N}} \|a_{h(2j+1)} - a_{h(2j)}\| \leq \varepsilon$ . It suffices to put  $m_j = n_{h(2j+1)} - n_{h(2j)}$ .  $\square$

However, we do not know the answer to the following question.

**Problem 4.** Let  $\{n_k\}_{k \in \mathbb{N}}$  be an increasing sequence of natural numbers and let  $\{a_k\}_{k \in \mathbb{N}}$  be a sequence of elements of  $\mathbb{T}$ . Does there exist a sequence  $\{m_j\}_{j \in \mathbb{N}}$  such that  $\sum_{j \in \mathbb{N}} \|m_j x\| \leq \sum_{k \in \mathbb{N}} \|n_k x + a_k\|$  for all  $x \in \mathbb{T}$ ?

Let us now define the families  $\mathcal{B}_0^c$ . Here the use of the series  $\sum_{k \in \mathbb{N}} \|n_k x\|$  and  $\sum_{k \in \mathbb{N}} |\sin \pi n_k x|$  is not equivalent; for simplicity we will consider the first one.

**Definition 5.** Let  $c$  be a positive real. A set  $X \subseteq \mathbb{T}$  is called a  $B_0^c$ -set if there exists an increasing sequence  $\{n_k\}_{k \in \mathbb{N}}$  of natural numbers such that  $\sum_{k \in \mathbb{N}} \|n_k x\| \leq c$  for all  $x \in X$ . The family of all  $B_0^c$ -sets is denoted by  $\mathcal{B}_0^c$ .

We show that in the previous definition ‘ $\leq$ ’ can be replaced by ‘ $<$ ’.

**Lemma 6.** *Let  $c > 0$ ,  $X \subseteq \mathbb{T}$ , and let  $\{n_k\}_{k \in \mathbb{N}}$  be such that  $\sum_{k \in \mathbb{N}} \|n_k x\| \leq c$  for all  $x \in X$ . Then there exists  $j \in \mathbb{N}$  such that  $\sum_{k > j} \|n_k x\| < c$  for all  $x \in X$ .*

*Proof.* Suppose for a contradiction that for every  $j$  there exists  $x_j \in X$  such that  $\sum_{k > j} \|n_k x_j\| \geq c$ . Thus  $\|n_k x_j\| = 0$  for all  $k \leq j$  and  $\|n_{k_j} x_j\| > 0$  for some  $k_j > j$ . Hence for every  $j$  and  $k$  such that  $k \geq k_j$  we have  $\|n_{k_j} x_j\| > 0$  and  $\|n_{k_j} x_k\| = 0$ , and thus  $x_j \neq x_k$ . It follows that the set  $\{x_j : j \in \mathbb{N}\}$  is infinite. However, this is impossible since for every  $k$  there are only finitely many  $x \in \mathbb{T}$  such that  $\|n_k x\| = 0$ .  $\square$

Let us recall that a set  $X \subseteq \mathbb{T}$  is called a *Dirichlet set* (a *D-set*) if there exists an increasing sequence  $\{n_k\}_{k \in \mathbb{N}}$  of natural numbers such that the sequence  $\{\|n_k x\|\}_{k \in \mathbb{N}}$  is uniformly converging to 0 on  $X$ . The family of all D-sets is denoted by  $\mathcal{D}$ . From the definition we can immediately see that  $\bigcup_{c > 0} \mathcal{B}_0^c = \mathcal{B}_0$  and  $\bigcap_{c > 0} \mathcal{B}_0^c = \mathcal{D}$ .

We do not know whether an analogue of Proposition 2 holds true in the case of  $B_0^c$ -sets.

**Problem 7.** Let  $c > 0$ , let  $\{n_k\}_{k \in \mathbb{N}}$  be an increasing sequence of natural numbers and let  $\{a_k\}_{k \in \mathbb{N}}$  be a sequence of elements of  $\mathbb{T}$ . Does there exist a sequence  $\{m_j\}_{j \in \mathbb{N}}$  such that for all  $x \in \mathbb{T}$ ,  $\sum_{j \in \mathbb{N}} \|m_j x\| \leq c$  whenever  $\sum_{k \in \mathbb{N}} \|n_k x + a_k\| \leq c$ ?

Fix a positive real  $c$ . Let us denote by  $\mathcal{B}_0^{*c}$  the family of all sets  $X \subseteq \mathbb{T}$  for which there exist sequences  $\{n_k\}_{k \in \mathbb{N}}$  and  $\{a_k\}_{k \in \mathbb{N}}$  such that  $\sum_{k \in \mathbb{N}} \|n_k x + a_k\| \leq c$  for all  $x \in X$ . Clearly  $\mathcal{B}_0^c \subseteq \mathcal{B}_0^{*c}$ , and Problem 7 asks whether the equality holds true. From Lemma 3 it follows that  $\mathcal{B}_0^{*c} \subseteq \bigcap_{d > c} \mathcal{B}_0^d$ . Thus for every  $c > 0$  we have

$$\bigcup_{d < c} \mathcal{B}_0^d = \bigcup_{d < c} \mathcal{B}_0^{*d} \subseteq \mathcal{B}_0^c \subseteq \mathcal{B}_0^{*c} \subseteq \bigcap_{d > c} \mathcal{B}_0^d = \bigcap_{d > c} \mathcal{B}_0^{*d}.$$

In Propositions 10 and 11 we will show that the first and the last inclusions are proper.

Let  $\{m_j\}_{j \in \mathbb{N}}$  be an increasing sequence of natural numbers and let  $\{a_j\}_{j \in \mathbb{N}}$  be a sequence of elements of  $\mathbb{T}$ . For every positive integer  $n$ , let  $\alpha(n), \beta(n) \in \mathbb{N}$  be such that  $n = 2^{\alpha(n)}\beta(n)$  and  $\beta(n)$  is odd.

**Lemma 8.** *Let  $p, r \in \mathbb{N}$ ,  $p \geq 1$ . Assume that the set  $\{\alpha(m_j) : j \in \mathbb{N}\}$  is infinite. There exist  $j, q \in \mathbb{N}$  and  $s \in \{-1, 1\}$  such that  $q \geq r$  and  $\|sm_j 2^{-q2^p} + a_j\| \geq 2^{-2^p}$ .*

*Proof.* Put  $J = \{j \in \mathbb{N} : \|a_j\| \geq 2^{-2^p}\}$ . Clearly either  $\{\alpha(m_j) : j \in J\}$  or  $\{\alpha(m_j) : j \notin J\}$  is infinite.

In the first case find  $j \in J$  such that  $\alpha(m_j) \geq r2^p$ , and put  $q = r$  and  $s = 1$ . Then  $sm_j 2^{-q2^p}$  is an integer and thus  $\|sm_j 2^{-q2^p} + a_j\| = \|a_j\| \geq 2^{-2^p}$ .

In the latter case find  $j \notin J$  such that  $\alpha(m_j) \geq (r - 1)2^p$  and put  $q = \lceil \alpha(m_j) 2^{-p} \rceil + 1$ . Then clearly  $q \geq r$ . We also have  $0 < q2^p - \alpha(m_j) \leq 2^p$  and thus  $\|m_j 2^{-q2^p}\| = \|\beta(m_j) 2^{\alpha(m_j) - q2^p}\| \geq 2^{-2^p}$ . Since  $\|a_j\| \leq 2^{-2^p} \leq 2^{-2}$ , we can conclude that either  $\|m_j 2^{-q2^p} + a_j\| \geq 2^{-2^p}$  or  $\|m_j 2^{-q2^p} - a_j\| \geq 2^{-2^p}$ .  $\square$

**Lemma 9.** *Let  $\theta$  be a real such that  $0 < \theta \leq 1/2$ , let  $\{p_k\}_{k \in \mathbb{N}}$  be a sequence of positive integers, and let  $\{\varepsilon_k\}_{k \in \mathbb{N}}$  be a sequence of positive reals. Assume that  $\lim_{j \rightarrow \infty} \alpha(m_j) = \infty$ . There exists  $x \in \mathbb{T}$  such that*

- (1)  $\|x\| \leq \theta$ ;
- (2) *there exists an increasing sequence  $\{n_k\}_{k \in \mathbb{N}}$  of natural numbers such that  $\|2^{2^k} x\| \leq 2^{-2^{p_k}} + 2^{-2^{n_k}}$  for all  $k \in \mathbb{N}$ , and  $\|2^{2^n} x\| \leq 2^{-2^n}$  for all  $n \notin \{n_k : k \in \mathbb{N}\}$ ;*
- (3) *there exists an increasing sequence  $\{j_k\}_{k \in \mathbb{N}}$  of natural numbers such that  $\|m_{j_k} x + a_{j_k}\| \geq 2^{-2^{p_k}} - \varepsilon_k$  for all  $k \in \mathbb{N}$ .*

*Proof.* We will define sequences  $\{n_k\}_{k \in \mathbb{N}}$ ,  $\{j_k\}_{k \in \mathbb{N}}$ ,  $\{q_k\}_{k \in \mathbb{N}}$ , and  $\{s_k\}_{k \in \mathbb{N}}$  by induction. Fix  $k \in \mathbb{N}$  and suppose that  $n_i, j_i, q_i, s_i$  are defined for all  $i < k$ . Find  $r \geq 2$  such that

- (a)  $2^{-r2^{p_k}} \leq 2^{-k-1}\theta$ ;
- (b)  $r2^{p_k} > 2^{n_{k-1}+1}$  if  $k \geq 1$ ;
- (c)  $2^{-r2^{p_k}} \leq 2^{i-k} 2^{-2^{n_i+1}}$  for all  $i < k$ ;
- (d)  $2^{-r2^{p_k}} \leq 2^{i-k} \varepsilon_i / m_{j_i}$  for all  $i < k$ .

Let  $l \in \mathbb{N}$  be such that  $l > j_{k-1}$  if  $k \geq 1$ , and  $\alpha(m_j) \geq q_{k-1} 2^{p_{k-1}}$  for all  $j \geq l$ . By Lemma 8 there exist  $j_k \geq l$ ,  $q_k \geq r$ , and  $s_k \in \{-1, 1\}$  such that

$$(e) \quad \left\| m_{j_k} s_k 2^{-q_k 2^{-p_k}} + a_{j_k} \right\| \geq 2^{-2^{p_k}}.$$

Put  $n_k = \max\{i : 2^i < q_k 2^{p_k}\}$ .

Let  $x = \sum_{k \in \mathbb{N}} s_k 2^{-q_k 2^{p_k}}$ . By (a),  $|x| \leq \sum_{k \in \mathbb{N}} 2^{-q_k 2^{p_k}} \leq \sum_{k \in \mathbb{N}} 2^{-k-1} \theta = \theta$ , and thus (1) holds.

To show (2), fix  $k \in \mathbb{N}$ . If  $k \geq 1$ , then by (b),  $q_k 2^{p_k} > 2^{n_{k-1}+1}$ . Thus  $n_k > n_{k-1}$  and hence  $q_i 2^{p_i} < 2^{n_k}$  for all  $i < k$ . Since  $q_k \geq 2$ , we have  $2^{n_k+1} \geq q_k 2^{p_k} \geq 2^{p_k+1}$  and thus  $n_k \geq p_k$ . Hence  $2^{n_k}$  is a multiple of  $2^{p_k}$  and we have  $q_k 2^{p_k} \geq 2^{n_k} + 2^{p_k}$ . Using (c) we obtain that

$$(f) \quad \begin{aligned} \sum_{i \geq k} 2^{-q_i 2^{p_i}} &\leq 2^{-q_k 2^{p_k}} + \sum_{i > k} 2^{k-i} 2^{-2^{n_k+1}} \\ &\leq 2^{-2^{n_k} - 2^{p_k}} + 2^{-2^{n_k+1}} = 2^{-2^{n_k}} \left( 2^{-2^{p_k}} + 2^{-2^{n_k}} \right). \end{aligned}$$

Hence  $\|2^{2^{n_k}} x\| \leq 2^{2^{n_k}} \sum_{i \geq k} 2^{-q_i 2^{p_i}} \leq 2^{-2^{p_k}} + 2^{-2^{n_k}}$ , and thus the first part of (2) holds. To show the latter one, let  $n \notin \{n_k : k \in \mathbb{N}\}$  and put  $k = \min\{i : n_i > n\}$ . Then for all  $i < k$  we have  $n > n_i$  and thus  $q_i 2^{p_i} \leq 2^n$ . Hence by (f),

$$\begin{aligned} \|2^{2^n} x\| &\leq 2^{2^n} \sum_{i \geq k} 2^{-q_i 2^{p_i}} \\ &\leq 2^{2^n - 2^{n_k}} \left( 2^{-2^{p_k}} + 2^{-2^{n_k}} \right) \leq 2^{2^n - 2^{n+1}} = 2^{-2^n}. \end{aligned}$$

To show (3), fix again  $k \in \mathbb{N}$ . From the choice of  $l$  it follows that  $j_k > j_{k-1}$  if  $k \geq 1$ , and  $\alpha(m_{j_k}) \geq q_i 2^{p_i}$  for all  $i < k$ . Hence by (e) and (d),

$$\begin{aligned} \|m_{j_k} x + a_{j_k}\| &\geq \left\| m_{j_k} s_k 2^{-q_k 2^{p_k}} + a_{j_k} \right\| - m_{j_k} \sum_{i > k} 2^{-q_i 2^{p_i}} \\ &\geq 2^{-2^{p_k}} - m_{j_k} \sum_{i > k} 2^{k-i} \varepsilon_k / m_{j_k} = 2^{-2^{p_k}} - \varepsilon_k. \end{aligned}$$

□

For  $n \in \mathbb{N}$ , denote  $\phi(n) = \sum_{k \geq n} 2^{-2^k}$ ,  $\psi(0) = 1$ , and  $\psi(n) = 2^{-2^n} \sum_{k < n} 2^{2^k}$  if  $n \geq 1$ . It can be easily shown that both  $\{\phi(n)\}_{n \in \mathbb{N}}$  and  $\{\psi(n)\}_{n \in \mathbb{N}}$  are decreasing and converging to 0.

**Proposition 10.** *For every  $c > 0$  there exists  $X \subseteq \mathbb{T}$  such that*

- (1) *there exists an increasing sequence  $\{n_k\}_{k \in \mathbb{N}}$  of natural numbers such that  $\sum_{k \in \mathbb{N}} \|n_k x\| \leq c$  for all  $x \in X$ ;*
- (2) *for every increasing sequence  $\{m_j\}_{j \in \mathbb{N}}$  of natural numbers, every sequence  $\{a_j\}_{j \in \mathbb{N}}$  of elements of  $\mathbb{T}$ , and for every  $\eta > 0$  there exists  $x \in X$  such that  $\sum_{j \in \mathbb{N}} \|m_j x + a_j\| > c - \eta$ .*

*Proof.* Put  $X = \{x \in \mathbb{T} : \sum_{n \in \mathbb{N}} \|2^{2^n} x\| \leq c\}$ . Clearly  $X$  satisfies the condition (1).

To show (2), fix  $\eta > 0$  and sequences  $\{m_j\}_{j \in \mathbb{N}}$ ,  $\{a_j\}_{j \in \mathbb{N}}$ . We will find  $x \in X$  such that  $\sum_{j \in \mathbb{N}} \|m_j x + a_j\| > c - \eta$ . We will consider two cases.

(a) If  $\lim_{j \rightarrow \infty} \alpha(m_j) = \infty$ , then we can find  $N \in \mathbb{N}$ , a sequence  $\{p_k\}_{k \in \mathbb{N}}$  of positive integers and a sequence  $\{\varepsilon_k\}_{k \in \mathbb{N}}$  of positive reals such that  $\sum_{k \in \mathbb{N}} 2^{-2^{p_k}} - \sum_{k \in \mathbb{N}} \varepsilon_k > c - \eta$  and  $\sum_{k \in \mathbb{N}} 2^{-2^{p_k}} + \phi(N) + \psi(N) \leq c$ . Put  $\theta = 2^{-2^N}$ . There exists

$x \in \mathbb{T}$  satisfying the conditions (1)–(3) of Lemma 9. We have

$$\sum_{n \in \mathbb{N}} \|2^{2^n} x\| = \sum_{n < N} \|2^{2^n} x\| + \sum_{n \geq N} \|2^{2^n} x\| \leq \sum_{n < N} 2^{2^n} \theta + \sum_{k \in \mathbb{N}} 2^{-2^{p_k}} + \sum_{n \geq N} 2^{-2^n} \leq c$$

and

$$\sum_{j \in \mathbb{N}} \|m_j x + a_j\| \geq \sum_{k \in \mathbb{N}} \|m_{j_k} x + a_{j_k}\| \geq \sum_{k \in \mathbb{N}} (2^{-2^{p_k}} - \varepsilon_k) > c - \eta.$$

(b) If the sequence  $\{\alpha(m_j)\}_{j \in \mathbb{N}}$  does not tend to infinity, then we will find  $x \in X$  such that the sequence  $\{\|m_j x + a_j\|\}_{j \in \mathbb{N}}$  does not tend to 0. We can suppose that  $\lim_{j \rightarrow \infty} \|a_j\| = 0$  (otherwise take  $x = 0$ ). There exist  $M \in \mathbb{N}$  and an increasing sequence  $\{j_k\}_{k \in \mathbb{N}}$  of natural numbers such that  $\alpha(m_{j_k}) = M$  for all  $k$ . Find  $N \in \mathbb{N}$  such that  $2^N > M$  and  $\psi(N) \leq c$ . For  $x = 2^{-2^N}$  we get

$$\sum_{n \in \mathbb{N}} \|2^{2^n} x\| = \sum_{n < N} 2^{2^n - 2^N} = \psi(N) \leq c.$$

Since  $\|m_{j_k} x + a_{j_k}\| \geq \|m_{j_k} x\| - \|a_{j_k}\| \geq 2^{M-2^N} - \|a_{j_k}\|$  for all  $k \in \mathbb{N}$ , the sequence  $\{\|m_j x + a_j\|\}_{j \in \mathbb{N}}$  does not converge to 0. □

**Proposition 11.** *For every  $c > 0$  there exists  $X \subseteq \mathbb{T}$  such that*

- (1) *for every  $\eta > 0$  there exists an increasing sequence of natural numbers  $\{n_k\}_{k \in \mathbb{N}}$  such that  $\sum_{k \in \mathbb{N}} \|n_k x\| < c + \eta$  for all  $x \in X$ ;*
- (2) *for every increasing sequence  $\{m_j\}_{j \in \mathbb{N}}$  of natural numbers and for every sequence  $\{a_j\}_{j \in \mathbb{N}}$  of elements of  $\mathbb{T}$  there exists  $x \in X$  such that  $\sum_{j \in \mathbb{N}} \|m_j x + a_j\| \geq c$ .*

*Proof.* Put

$$X = \left\{ x \in \mathbb{T} : \sum_{n \geq N} \|2^{2^n} x\| \leq c + \phi(N) + \psi(N) \text{ for all } N \in \mathbb{N} \right\}.$$

Clearly  $X$  satisfies condition (1).

To show (2), fix sequences  $\{m_j\}_{j \in \mathbb{N}}$ ,  $\{a_j\}_{j \in \mathbb{N}}$ . We will find  $x \in X$  such that  $\sum_{j \in \mathbb{N}} \|m_j x + a_j\| \geq c$ . Again let us discuss two cases.

(a) Assume that  $\lim_{j \rightarrow \infty} \alpha(m_j) = \infty$ . Let us choose a sequence  $\{p_k\}_{k \in \mathbb{N}}$  of positive integers such that  $\sum_{k \in \mathbb{N}} 2^{-2^{p_k}} = c$ . Find  $t \in \mathbb{N}$  such that  $\|m_0 2^{-2^t} + a_0\| > 0$  and put  $\theta = \|m_0 2^{-2^t} + a_0\|/2$ . Fix a sequence  $\{\varepsilon_k\}_{k \in \mathbb{N}}$  of positive reals such that  $\sum_{k \in \mathbb{N}} \varepsilon_k = \theta$  and find  $l \in \mathbb{N}$  such that  $\alpha(m_j) \geq 2^t$  for all  $j \geq l$ . By Lemma 9 there exists  $x' \in \mathbb{T}$  such that

- (1')  $\|x'\| \leq \theta/m_0$ ;
- (2') there exists an increasing sequence  $\{n_k\}_{k \in \mathbb{N}}$  of natural numbers such that  $\|2^{2^{n_k}} x'\| \leq 2^{-2^{p_k}} + 2^{-2^{n_k}}$  for all  $k \in \mathbb{N}$ , and  $\|2^{2^n} x'\| \leq 2^{-2^n}$  for all  $n \notin \{n_k : k \in \mathbb{N}\}$ ;
- (3') there exists an increasing sequence  $\{j_k\}_{k \in \mathbb{N}}$  of natural numbers such that  $j_k \geq l$  and  $\|m_{j_k} x' + a_{j_k}\| \geq 2^{-2^{p_k}} - \varepsilon_k$  for all  $k \in \mathbb{N}$ .

Put  $x = x' + 2^{-2^t}$ . Then for every  $N \in \mathbb{N}$  we have

$$\begin{aligned} \sum_{n \geq N} \|2^{2^n} x\| &\leq \sum_{n \geq N} \|2^{2^n} x'\| + \sum_{n \geq N} \|2^{2^n - 2^t}\| \\ &\leq \sum_{k \in \mathbb{N}} 2^{-2^{p_k}} + \sum_{n \geq N} 2^{-2^n} + \sum_{N \leq n < t} 2^{2^n - 2^t} \leq c + \phi(N) + \psi(N), \end{aligned}$$

since the last sum equals 0 if  $t \leq N$  and is not greater than  $\psi(t) < \psi(N)$  if  $t > N$ . Moreover,

$$\begin{aligned} \sum_{j \in \mathbb{N}} \|m_j x + a_j\| &\geq \|m_0 x + a_0\| + \sum_{k \in \mathbb{N}} \|m_{j_k} x + a_{j_k}\| \\ &\geq \|m_0 2^{-2^t} + a_0\| - \|m_0 x'\| + \sum_{k \in \mathbb{N}} \|m_{j_k} x' + a_{j_k}\| \\ &\geq \theta + \sum_{k \in \mathbb{N}} 2^{-2^{p_k}} - \sum_{k \in \mathbb{N}} \varepsilon_k = c. \end{aligned}$$

(b) If the sequence  $\{\alpha(m_j)\}_{j \in \mathbb{N}}$  does not tend to infinity, then similarly as in the proof of Proposition 10 there exists  $x \in \mathbb{T}$  such that  $\sum_{n \in \mathbb{N}} \|2^{2^n} x\| \leq c$  and the sequence  $\{\|m_j x + a_j\|\}_{j \in \mathbb{N}}$  does not tend to 0.  $\square$

The following theorem summarizes our results concerning the hierarchy of the families  $\mathcal{B}_0^c$  and  $\mathcal{B}_0^{*c}$ .

**Theorem 12.** *Let  $c$  be a positive real. Then*

$$\bigcup_{d < c} \mathcal{B}_0^d = \bigcup_{d < c} \mathcal{B}_0^{*d} \subsetneq \mathcal{B}_0^c \subseteq \mathcal{B}_0^{*c} \subsetneq \bigcap_{d > c} \mathcal{B}_0^d = \bigcap_{d > c} \mathcal{B}_0^{*d}.$$

**Corollary 13.** *If  $c < d$ , then  $\mathcal{B}_0^c \subsetneq \mathcal{B}_0^d$  and  $\mathcal{B}_0^{*c} \subsetneq \mathcal{B}_0^{*d}$ .*

Let us note that the proofs of Propositions 10 and 11 can be easily adopted for showing the following statements.

**Proposition 14.** *For every  $c > 0$  there exists  $X \subseteq \mathbb{T}$  such that*

- (1) *there exists an increasing sequence  $\{n_k\}_{k \in \mathbb{N}}$  of natural numbers such that  $\sum_{k \in \mathbb{N}} |\sin \pi n_k x| \leq c$  for all  $x \in X$ ;*
- (2) *for every increasing sequence  $\{m_j\}_{j \in \mathbb{N}}$  of natural numbers, every sequence  $\{a_j\}_{j \in \mathbb{N}}$  of elements of  $\mathbb{T}$ , and for every  $\eta > 0$  there exists  $x \in X$  such that  $\sum_{j \in \mathbb{N}} |\sin \pi(m_j x + a_j)| > c - \eta$ .*

**Proposition 15.** *For every  $c > 0$  there exists  $X \subseteq \mathbb{T}$  such that*

- (1) *for every  $\eta > 0$  there exists an increasing sequence of natural numbers  $\{n_k\}_{k \in \mathbb{N}}$  such that  $\sum_{k \in \mathbb{N}} |\sin \pi n_k x| < c + \eta$  for all  $x \in X$ ;*
- (2) *for every increasing sequence  $\{m_j\}_{j \in \mathbb{N}}$  of natural numbers and for every sequence  $\{a_j\}_{j \in \mathbb{N}}$  of elements of  $\mathbb{T}$  there exists  $x \in X$  such that*

$$\sum_{j \in \mathbb{N}} |\sin \pi(m_j x + a_j)| \geq c.$$

Hence it is possible to prove results analogous to Theorem 12 and Corollary 13, concerning the families of sets obtained by replacing the expression  $\sum_{k \in \mathbb{N}} \|n_k x\|$  in Definition 5 by  $\sum_{k \in \mathbb{N}} |\sin \pi n_k x|$ .

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