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ON SEQUENCES OF $C_b^{k,\delta}$ MAPS WHICH CONVERGE IN THE UNIFORM C^0-NORM

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ABSTRACT. We study maps $f \in C_b^{k,\delta}(U,Y)$ and give detailed estimates on $\|D^k f(x)\|, x \in U$, in terms of $\|f\|$ and $\|f\|_{k,\delta}$. These estimates are used to prove a lemma by D. Henry for the case $k \geq 2$. Here $U \subset X$ is an open subset and X and Y are Banach spaces.

1. INTRODUCTION

Let X and Y be Banach spaces. Let $U \subset X$ be such that U is open. Let $0 < \delta \leq 1, k = 0, 1, 2, \cdots$ and $b > 0$. Define the following:

$$C_b^{k,\delta}(U,Y) = \{u \in C^{k,\delta}(U,Y) \ \mid \ \|u\|_{k,\delta} \leq b\},$$

$$\|u\|_{k,\delta} = \max\{\|u\|, \|Du\|, \cdots, \|D^k u\|, H_\delta(D^k u)\},$$

$$H_\delta(D^k u) = \sup\left\{\frac{\|D^k u(x) - D^k u(x')\|}{|x - x'|^\delta} \ \mid \ x \neq x', x \in U, x' \in U\right\}.$$

Suppose we have a sequence of maps $u_n \in C_b^{k,\delta}(U,Y), n \geq 1$, which converges to a map $u : U \to Y$ in the uniform C^0-norm. That is, $\|u - u_n\| \to 0$ as $n \to \infty$. In this case one would say that $u \in C_b^{k,\delta}(U,Y)$ and that for $x \in U, \|D^k u(x) - D^k u_n(x)\| \to 0$ as $n \to \infty$. D. Henry showed that this is true for $k = 1$ and that $\|D^1 u(x) - D^1 u_n(x)\| \to 0$ uniformly on any subset $U_o \subset U$ which is uniformly bounded away from the boundary of U. See [3, page 151] and [1, pages 305, 319].

This situation arises frequently when one is trying to obtain a local $C^{k,\delta}$ invariant manifold as a fixed point of a transformation $\Gamma : C_b^{k,\delta}(U,Y) \to C_b^{k,\delta}(U,Y)$ by showing that Γ is a contraction. It is obvious that it is much easier to show that Γ is a contraction on a bounded closed subset of $C_b^{0,1}(U,Y)$ in the norm $\|\cdot\|$ rather than working with the norm $\|\cdot\|_{k,\delta}$. We use these results in studying the dependence of invariant manifolds on parameters [2].

In this work (Lemma 2.1) we give detailed estimates on $D^k f(x), x \in U$, in terms of $\|f\|$ and $\|f\|_{k,\delta}$ for any $f \in C_b^{k,\delta}(U,Y)$. These estimates are important in their own right. We point out that these estimates (H1) and (Hk) are homogeneous in $\|f\|$ and $\|f\|_{k,\delta}$ in the sense that the sum of the exponents of $\|f\|$ and $\|f\|_{k,\delta}$ is 1. In addition, they lead to a proof of Henry's lemma for $k \geq 2$. The case $k = 2$ does

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not follow by induction from the case $k = 1$. See Remark 2.3(3). In Proposition 2.5 we will show that in fact $\|D^k u - D^k u_n\|_{k,\beta} \to 0$ on U_o for any $0 < \beta < \delta$.

2. Main results

2.1. Lemma (Estimate H)**.** *Let X and Y be Banach spaces. Let $U \subset X$ be such that U is open. Let $k = 1, 2, 3, \cdots$ and $0 < \delta \leq 1$. Let ∂U be the boundary of U. For $x \in U$, let $d(x) := \min\{1, dist(x, \partial U)\}$. For all $f \in C^{k,\delta}(U, Y)$ and all $x \in U$ the following estimates are true:*

$$d(x)\|Df(x)\| \leq N_1 \|f\|^{\frac{\delta}{1+\delta}} \|f\|_{1,\delta}^{\frac{1}{1+\delta}}, \qquad k = 1, \tag{H1}$$

$$d(x)^{a_k} \|D^k f(x)\| \leq N_k \|f\|^{\beta_k} \|f\|_{k,\delta}^{1-\beta_k}, \qquad k \geq 2, \tag{Hk}$$

where $N_1 = 4, a_1 = 1$. If $k = 1, \beta_1 = \delta/(1+\delta)$. If $k \geq 2, \beta_1 = 1/2$ and

$$\begin{aligned} N_j &= 2(j!) \times \left[2 + \frac{N_1}{1!} + \frac{N_2}{2!} \cdots + \frac{N_{j-1}}{(j-1)!}\right] \\ &= 4(j!)(3^{j-1}), \qquad j = 2, \cdots, k, \\ a_j &= a_{j-1} + j = \frac{j(j+1)}{2}, \qquad j = 2, \cdots, k, \\ \beta_j &= \frac{1}{j+1}\beta_{j-1} = \frac{1}{(j+1)!}, \qquad j = 2, \cdots, k-1, \\ \beta_k &= \frac{\delta}{k+\delta}\beta_{k-1} = \frac{\delta}{(k+\delta)(k!)}. \end{aligned}$$

We prove this lemma in Section 3.

2.2. Lemma (Henry's lemma)**.** *The following are true:*

(1) *Any closed bounded subset $C_b^{k,\delta}(U, Y)$ is a closed and bounded subset of $(C^0(U, Y), \|\cdot\|)$. That is, if we have a sequence $\{u_n \in C_b^{k,\delta}(U, Y) \mid 0 \leq n < \infty\}$ and a map $u : U \to Y$ such that $\|u_n - u\| \to 0$ as $n \to \infty$, then $u \in C_b^{k,\delta}(U, Y)$. Moreover, $D^k u_n(x) \to D^k u(x)$ point-wise as $n \to \infty$.*

(2) *Let $U_o \subset U$ be a subset which is uniformly bounded away from the boundary of U. Then $D^k u_n(x) \to D^k u(x)$ uniformly on U_o.*

Proof. (1) If $k = 0$, then for any $x \neq x'$,

$$\begin{aligned} |u(x) - u(x')| &\leq |u_n(x) - u_n(x')| + 2\|u_n - u\| \\ &\leq b|x - x'|^\delta + 2\|u_n - u\| \to b|x - x'|^\delta \end{aligned}$$

as $n \to \infty$.

When $k \geq 1$, let $f := u_n - u_m$. It follows from estimate (Hk) of Lemma 2.1 that

$$d(x)^{a_k} \|D^k u_n(x) - D^k u_m(x)\| \leq N_k \|u_n - u_m\|^{\beta_k} \|u_n - u_m\|_{k,\delta}^{1-\beta_k}. \tag{*}$$

Thus u is $C^{k,\delta}$and $D^k u_n(x) \to D^k u(x)$ for each $x \in U$. It also follows that $\|u\|_{k,\delta} \leq b$.

Part (2) follows immediately from estimate (*) and the fact that $U_o \subset U$ is uniformly bounded away from the boundary of U. □

2.3. *Remarks.* 1. The proof of Henry's lemma 2.2 is based on estimates (H1) and (Hk) of Lemma 2.1.

2. Estimate (H1) and the proof of Henry's lemma for $k = 1$ are given in [3, page 151] and [1, page 319] without the explicit dependence on $\|f\|_{1,\delta}$.

3. The $k \geq 2$ case of Henry's lemma does not follow by induction from the $C^{1,\delta}$ case without estimate (Hk). This is because the convergence in the latter is point-wise and the uniform convergence holds only on any proper subset $U_o \subset U$ which is uniformly bounded away from the boundary of U.

4. Moreover, estimates (H1) and (Hk) of Lemma 2.1 are of interest by themselves. In particular, notice that in estimates (H1) and (Hk) the sum of the exponents of the different norms of the function that appear on the right-hand side is always one. In other words, these inequalities are homogeneous so to speak. This is not the case with the estimates in [3, page 151] and [1, page 319].

5. In [4, page 182] Lanford gives a lemma similar to Henry's lemma. In Lanford's lemma, all limits, in the hypothesis and the conclusion, are pointwise in the weak topology.

The following is a classical result which we will need later. We present a proof here for completeness.

2.4. Lemma. (1) *Let $g \in C^{0,\delta}(U_o, Y)$ and $0 < \beta < \delta \leq 1$. Then*

$$\|g\|_{0,\beta} \leq 2^{1-(\beta/\delta)} \|g\|^{1-(\beta/\delta)} \|g\|_{0,\delta}^{\beta/\delta}. \tag{2.1}$$

(2) *Let f_n and f be $C_b^{0,\delta}(U_o, Y)$ and assume that $\|f_n - f\| \to 0$. Then $\|f_n - f\|_{0,\beta} \to 0$ for all $0 \leq \beta < \delta$.*

Proof. Notice that part (2) follows from part (1) by setting $g = f_n - f$. So, we prove part (1).

Notice that

$$\Delta := \frac{|g(x+h) - g(x)|}{|h|^\beta} \leq \min\left\{ 2\frac{\|g\|}{|h|^\beta} \ , \ |h|^{\delta-\beta} \|g\|_{0,\delta} \right\}. \tag{\#}$$

The two terms on the right-hand side of (#) are equal iff

$$|h| = \left(\frac{2\|g\|}{\|g\|_{0,\delta}} \right)^{1/\delta} := A.$$

When $|h| > A$, we use the first term on the right-hand side of (#) to estimate Δ. When $|h| \leq A$ we use the second term. In either case we obtain

$$\Delta \leq 2^{1-(\beta/\delta)} \|g\|^{1-(\beta/\delta)} \|g\|_{0,\delta}^{\beta/\delta}$$

which proves (1). □

2.5. Proposition. *Let u_n be a sequence in $C_b^{k,\delta}(U, Y)$ and $u : U \to Y$. Assume that $\|u_n - u\| \to 0$. Let $U_o \subset U$ be an open subset which is uniformly bounded away from the boundary of U. Let $0 < \beta < \delta$. Then $u \in C_b^{k,\delta}(U, Y)$. Moreover, on U_o, $\|u_n - u\|_{k,\beta} \to 0$ as $n \to \infty$.*

Proof. Applying Lemma 2.2, we can see that $\|D^j u_n - D^j u\| \to 0$ on U_o as $n \to \infty$ for $j = 0, 1, \cdots, k$. Notice that for $f \in C^{k,\delta}(U, Y)$ we have $D^k f \in C^{0,\delta}(U, \mathcal{S}(X^k, Y))$ where $\mathcal{S}(X^k, Y)$ is the Banach space of symmetric k-multilinear maps from X^k to Y. Thus we can apply assertion (2) of Lemma 2.4 to $D^k u_n$ and $D^k u$ and obtain

$$\|D^k u_n - D^k u\|_{0,\beta} \to 0$$

as $n \to \infty$ on U_o. □

3. Proof of Estimate H

Let $x \in U$ be fixed but arbitrary. Let $v \in X$ and $|v| = 1$ and let $0 < t < d(x)$. Then

$$\begin{aligned} |Df(x)tv| &\le |f(x) - f(x+tv)| + |f(x+tv) - f(x) - Df(x)tv| \\ &\le 2\|f\| + \frac{\|f\|_{1,\delta}}{1+\delta} t^{1+\delta} \le 2\|f\| + \|f\|_{1,\delta} t^{1+\delta}. \end{aligned}$$

Thus

$$(*) \qquad \|Df(x)\| \le \frac{2\|f\|}{t} + \|f\|_{1,\delta} t^{\delta}, \qquad t < d(x) \le 1.$$

The two terms on the right-hand side of (*) are equal iff

$$(**) \qquad \begin{aligned} t &= \left(\frac{R}{S}\right)^{\frac{1}{1+\delta}} =: A, \\ R &= 2\|f\|, \\ S &= \|f\|_{1,\delta}. \end{aligned}$$

We have two cases. Case 1: $d(x) \le A$. Then for all $0 < t < d(x)$

$$t\|Df(x)\| \le 2R \le 4\|f\| \le 4\|f\|^{\frac{\delta}{1+\delta}} \|f\|_{1,\delta}^{\frac{1}{1+\delta}}.$$

Taking the supremum over $0 < t < d(x)$, we obtain

$$\text{(a1)} \qquad d(x)\|Df(x)\| \le 4\|f\|^{\frac{\delta}{1+\delta}} \|f\|_{1,\delta}^{\frac{1}{1+\delta}}.$$

Case 2: $d(x) > A$. In this case we evaluate the right-hand side of (*) at $t = A$. Thus

$$\text{(b1)} \qquad d(x)\|Df(x)\| \le \|Df(x)\| \le 2R^{\frac{\delta}{1+\delta}} S^{\frac{1}{1+\delta}} \le 4\|f\|^{\frac{\delta}{1+\delta}} \|f\|_{1,\delta}^{\frac{1}{1+\delta}}.$$

It follows from (a1) and (b1) that estimate (H1) holds.

Now we consider the case $k \ge 2$. In this case (H1) holds with $\delta = 1$. Assume that our estimate is true for $i = 1, 2, \cdots, k-1$. Since $\|f\|_{i,\delta} \le \|f\|_{l,\delta}$ and $\beta_l < \beta_i$ for $i < l$, it follows that

$$\text{(3.1)} \qquad \|f\|^{\beta_i} \|f\|_{i,\delta}^{1-\beta_i} \le \|f\|^{\beta_l} \|f\|_{l,\delta}^{1-\beta_l}.$$

Let $v^{(i)} = (v, \cdots, v)$ with i components. For $i = 1, 2, \cdots, k$, let

$$\begin{aligned} \frac{t^i}{i!} D^i f(x) v^{(i)} &= \mathcal{F}_{i-1} - \mathcal{F}_i(t), \\ \mathcal{F}_i(t) &:= f(x+tv) - f(x) - \frac{t}{1!} Df(x)v \\ &\qquad - \frac{t^2}{2!} D^2 f(x) v^{(2)} - \cdots - \frac{t^i}{i!} D^i f(x) v^{(i)}, \\ \mathcal{A}_i(t) &:= \|\mathcal{F}_i\|. \end{aligned}$$

It follows that

$$\mathcal{F}_k(t) = \int_0^t \int_0^{s_0} \cdots \int_0^{s_{k-1}} [D^k f(x + s_k v) - D^k f(x)] v^{(k)} \, ds_k \cdots ds_1 ds_0.$$

Thus

$$\begin{aligned}\mathcal{A}_k(t) &\le H_\delta(D^k f)\int_0^t\int_0^{s_0}\cdots\int_0^{s_{k-1}} s_k^\delta ds_k\cdots ds_1 ds_0 \\ &= \frac{t^{k+\delta}}{(k+\delta)!}H_\delta(D^k f) \le \frac{t^{k+\delta}}{(k+\delta)!}\|f\|_{k,\delta} \le \|f\|_{k,\delta}t^{k+\delta}.\end{aligned} \tag{3.2}$$

Using (3.1) we obtain

$$\begin{aligned}\mathcal{A}_{k-1}(t) &\le 2\|f\| + \frac{tN_1}{1!}\|f\|^{\beta_1}\|f\|_{k,\delta}^{1-\beta_1}d(x)^{-1} \\ &\quad + \frac{t^2N_2}{2!}\|f\|^{\beta_2}\|f\|_{k,\delta}^{1-\beta_2}d(x)^{-a_2} \\ &\quad + \cdots + \frac{t^{k-1}N_{k-1}}{(k-1)!}\|f\|^{\beta_{k-1}}\|f\|_{k,\delta}^{1-\beta_{k-1}}d(x)^{-a_{k-1}} \\ &\le \frac{C_k}{d(x)^{a_{k-1}}}\|f\|^{\beta_{k-1}}\|f\|_{k,\delta}^{1-\beta_{k-1}}, \\ C_k &= 2 + \frac{N_1}{1!} + \frac{N_2}{2!} + \cdots + \frac{N_{k-1}}{(k-1)!}.\end{aligned} \tag{3.3}$$

Using (3.2) and (3.3) we obtain, for $0 < t < d(x)$,

$$\begin{aligned}\frac{t^k\|D^k f(x)\|}{k!} &\le \mathcal{A}_{k-1}(t) + \mathcal{A}_k(t), \\ \frac{\|D^k f(x)\|}{k!} &\le \frac{R}{t^k} + St^\delta, \\ R &:= \frac{C_k}{d(x)^{a_{k-1}}}\|f\|^{\beta_{k-1}}\|f\|_{k,\delta}^{1-\beta_{k-1}}, \\ S &:= \|f\|_{k,\delta}.\end{aligned} \tag{3.4}$$

The two quantities on the right-hand side of (3.4) are equal iff

$$t = \left(\frac{R}{S}\right)^{1/(k+\delta)} =: B.$$

As before, there are two cases. Case 1: $d(x) \le B$. Then $0 \le t < d(x) \le B$ and

$$\begin{aligned}d(x)^k\frac{\|D^k f(x)\|}{k!} &\le 2R \le \frac{2C_k}{d(x)^{a_{k-1}}}\|f\|^{\beta_{k-1}}\|f\|_{k,\delta}^{1-\beta_{k-1}} \\ &\le \frac{2C_k}{d(x)^{a_{k-1}}}\|f\|^{\beta_k}\|f\|_{k,\delta}^{1-\beta_k}.\end{aligned} \tag{ak}$$

Case 2: $d(x) > B$. In this case we evaluate (3.4) at $t = B$ and obtain

$$\text{(bk)} \qquad \begin{aligned} \frac{\|D^k f(x)\|}{k!} &\le 2R^{\delta/(k+\delta)} S^{k/(k+\delta)} \\ &\le \frac{2C_k^{\delta/(k+\delta)}}{d(x)^h [(k+\delta)!]^{k/(k+\delta)}} \|f\|^r \|f\|_{k,\delta}^s \\ &\le \frac{2C_k}{d(x)^h} \|f\|^{\beta_k} \|f\|_{k,\delta}^{1-\beta_k}, \\ h &= \frac{\delta a_{k-1}}{k+\delta} < a_{k-1}, \\ r &= \frac{\delta \beta_{k-1}}{k+\delta} = \beta_k, \\ s &= \frac{\delta(1-\beta_{k-1})}{k+\delta} + \frac{k}{k+\delta} = 1 - \beta_k. \end{aligned}$$

It follows from (ak) and (bk) that

$$d(x)^{k+a_{k-1}} \|D^k f(x)\| \le 2(k!) C_k \|f\|^{\beta_k} \|f\|_{k,\delta}^{1-\beta_k}$$

which finishes the proof of the lemma.

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