ON THE DENSITY OF THE SET OF GENERATORS OF A POLYNOMIAL ALGEBRA

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Abstract. Let $K[X] = K[x_1, \ldots, x_n]$, $n \geq 2$, be the polynomial algebra over a field $K$ of characteristic 0. We call a polynomial $p \in K[X]$ coordinate (or a generator) if $K[X] = K[p, p_2, \ldots, p_n]$ for some polynomials $p_2, \ldots, p_n$. In this note, we give a simple proof of the following interesting fact: for any polynomial $h$ of the form $(x_i + q)$, where $q$ is a polynomial without constant and linear terms, and for any integer $m \geq 2$, there is a coordinate polynomial $p$ such that the polynomial $(p - h)$ has no monomials of degree $\leq m$. A similar result is valid for coordinate $k$-tuples of polynomials, for any $k < n$. This contrasts sharply with the situation in other algebraic systems.

On the other hand, we establish (in the two-variable case) a result related to a different kind of density. Namely, we show that given a non-coordinate two-variable polynomial, any sufficiently small perturbation of its non-zero coefficients gives another non-coordinate polynomial.

1. Introduction

Let $K[X] = K[x_1, \ldots, x_n]$, $n \geq 2$, be the polynomial algebra over a field $K$ of characteristic 0. We denote by $\text{mindeg}(p)$ the minimal degree of non-zero monomials of $p \in K[X]$.

We call automorphic images of $x_1$ coordinate polynomials to simplify the language. Similarly, a $k$-tuple of polynomials $(p_1, \ldots, p_k)$, $p_i \in K[X]$, $k \leq n$, is coordinate if there exists an automorphism of $K[X]$ which sends $x_1, \ldots, x_k$ respectively to $p_1, \ldots, p_k$. Equivalently, a $k$-tuple $(p_1, \ldots, p_k)$ is coordinate if there are polynomials $p_{k+1}, \ldots, p_n \in K[X]$ such that $K[p_1, \ldots, p_k, p_{k+1}, \ldots, p_n] = K[X]$.

In this note, we give a simple proof of the following interesting fact: the set of coordinate polynomials is dense (in the formal power series topology) in the set of polynomials of the form $(x_i + q)$, $\text{mindeg}(q) \geq 2$. That is, any polynomial of this form can be completed to a coordinate polynomial by monomials of arbitrarily high degree. Actually, our proof yields a somewhat stronger result:

Theorem 1.1. For any $(n - 1)$-tuple of polynomials $(h_1, \ldots, h_{n-1})$ of the form $h_i = x_i + r_i$, $\text{mindeg}(r_i) \geq 2$, $i = 1, \ldots, n - 1$, and any integer $m \geq 2$, there
exists a coordinate \((n - 1)\)-tuple \((p_1, \ldots, p_{n-1})\) such that \(\text{mindeg}(p_i - h_i) > m, \ i = 1, \ldots, n - 1\).

There is a non-commutative version of Theorem 1.1 (see [4]) which involves machinery from representation theory of the general linear group \(GL_n(K)\). Our proof here is based on simpler ideas and is a consequence of a result of Anick [1].

Theorem 1.1 contrasts sharply with the situation in other, non-commutative, algebras. For example, the set of primitive elements (that is what generators are usually called in a non-commutative setting) of a free Lie algebra of rank 2 is not dense because by a theorem of Cohn [2], all automorphisms of this algebra are linear. Moreover, although the automorphism groups of \(K[x_1, x_2]\) and \(K\langle x_1, x_2 \rangle\) (the free associative algebra of rank 2) are isomorphic (see e.g. [3]), we have:

**Proposition 1.2.** The element \(u = x_1 + x_1x_2\) cannot be completed to a primitive element of \(K\langle x_1, x_2 \rangle\) by monomials of degree higher than 2.

The proof of Proposition 1.2 is based on a characterization of generators of \(K\langle x_1, x_2 \rangle\) as it appears in [1].

Finally, we establish (in the two-variable case) a result related to a different kind of density:

**Theorem 1.3.** Let \(p(x, y) = \sum_{i,j=1}^{m} c_{ij} \cdot x^iy^j, \ c_{ij} \in K, \) be a non-coordinate polynomial from \(K[x, y]\). Let \(K = \mathbb{R}\) or \(\mathbb{C}\). Then there is an \(\varepsilon > 0\) such that every polynomial \(q(x, y) = \sum_{i,j=1}^{m} c'_{ij} \cdot x^iy^j\) with \(|c_{ij} - c'_{ij}| < \varepsilon\) if \(c_{ij} \neq 0\) and \(c'_{ij} = 0\) if \(c_{ij} = 0\), is non-coordinate as well.

2. Preliminaries

For background on polynomial automorphisms we refer to the book [3]. Anick [1] proved that, with respect to the formal power series topology, the set \(J\) of endomorphisms of \(K[X]\) with an invertible Jacobian matrix is a closed set, and the group of tame automorphisms of \(K[X]\) is dense in \(J\). This means that for any polynomial mapping \(F = (f_1, \ldots, f_n)\) with invertible Jacobian matrix \(J(F)\) (i.e. \(0 \neq J(F) \in K\)) and for any positive integer \(m\), there is a tame automorphism \(G = (g_1, \ldots, g_n)\) such that the polynomials \(f_i - g_i\) contain no monomials of degree less than \(m\). An interpretation of the result of Anick in the language we need is given in [5] Theorem 4.2.7. We recall some details here briefly.

Let \(P_k\) be the \(K\)-vector space of all homogeneous polynomials of degree \(k \geq 0\). Let \(I_k, k \geq 2,\) be the semigroup of all polynomial endomorphisms \(F = (f_1, \ldots, f_n)\) such that \(x_i\) is the only monomial of \(f_i\) of degree less than \(k; \ i = 1, \ldots, n\). We write

\[ f_i = x_i + g_i + h_i, \]

where \(g_i \in P_k\) is the homogeneous component of \(f_i\) of degree \(k\), and \(\text{mindeg}(h_i) > k\).

It turns out that there is a homomorphism \(\phi\) of \(I_k\) onto the direct sum of additive groups \(P_k \oplus I_k \oplus \ldots \oplus I_k\) such that \(\phi(F) = (g_1, \ldots, g_n)\).

Let \(T\) be the group of tame automorphisms of the algebra \(K[X]\), and let \(S_k\) be the set of all polynomial mappings \(S = (s_1, \ldots, s_n) \in I_k\) such that \(s_i = x_i + g_i + h_i\), where \(g_i \in P_k\) and \(h_i \in \sum_{l > k} P_l\), with the property

\[ \sum_{i=1}^{n} \frac{\partial g_i}{\partial x_i} = 0. \]
The main step of Anick’s proof (see [5], Step 2 of the proof of Theorem 4.2.7) is to show that

$$\phi(T \cap I_k) = \phi(S_k).$$

This yields

**Lemma 2.1.** For any $S \in S_k \subseteq I_k$, there is a tame automorphism $G_k$ of $K[X]$ such that $S \circ G_k^{-1}$ is in $I_{k+1}$.

3. Proofs

**Proof of Theorem 1.1.** Let $u_1, \ldots, u_{n-1}$ be $n - 1$ polynomials without constant and linear terms, and let $u_{ij}$ be the homogeneous component of degree $j$ of the polynomial $u_i; j = 2, \ldots, m$. Let $k$ be the smallest integer such that $u_{ik} \neq 0$ for some $i$. Let, for example, $i = 1$.

Write the partial derivative of $u_{1k}$ with respect to $x_1$ in the form

$$\frac{\partial u_{1k}}{\partial x_1} = \sum_{j=0}^{k-1} a_j x_n^j,$$

where the polynomials $a_j$ do not depend on $x_n$. There is a homogeneous polynomial $w_{1k} \in P_k$ such that

$$\frac{\partial w_{1k}}{\partial x_2} = -\sum_{j=0}^{k-1} a_j x_n^j.$$

Consider an endomorphism $F_{1k}$ of the algebra $K[X]$ defined by

$$F_{1k} = (x_1 + u_{1k}, x_2, \ldots, x_{n-1}, x_n + w_{1k}).$$

Clearly, $F_{1k} \in I_k$ and, because of the choice of $w_{1k}$, also $F_{1k} \in S_k$. Similarly, we construct endomorphisms

$$F_{ik} = (x_1, \ldots, x_{i-1}, x_i + u_{ik}, x_{i+1}, \ldots, x_{n-1}, x_n + w_{ik})$$

for some $w_{ik} \in P_k$, $i = 2, \ldots, n - 1$, such that $F_{ik} \in S_k$. Hence the composition $F_k = F_{1k} \circ \cdots \circ F_{n-1k}$ also belongs to $S_k$ and by Lemma 2.3 there exists a tame automorphism $G_k = (g_{1,k}, \ldots, g_{n,k}) \in I_k$ such that $F_{k+1} = F_k \circ G_k^{-1} \in I_{k+1}$. Therefore, $g_{i,k} = x_i + u_{ik} + v_{i,k+1} + v_i; i = 1, \ldots, n - 1$, where $v_{i,k+1}$ is homogeneous of degree $k + 1$, and $v_i \in \sum_{l \geq k+2} P_l$.

Continuing this way, we obtain a tame automorphism $G_{k+1} \in I_{k+1}$ such that $g_{i,k+1} = x_i + (u_{i,k+1} - v_{i,k+1}) + w_i; i = 1, \ldots, n - 1$, where $w_i \in \sum_{l \geq k+2} P_l$.

If we act by the automorphism $G_{k+1}$ on the polynomial $u_{ik}$, we get a polynomial of the form $u_{ik} + s_i$, where $s_i$ has no homogeneous components of degree less than $(k - 1) + (k + 1) = 2k > k + 1$. Therefore, the automorphism $G_{k+1} \circ G_k$ takes $x_i$ to $x_i + u_{ik} + u_{i,k+1} + (\text{terms of higher degree})$, $i = 1, \ldots, n - 1$.

In a finite number of steps, we obtain a tame automorphism $G = (g_1, \ldots, g_n)$ such that

$$g_i = x_i + u_i + (\text{terms of higher degree}), i = 1, \ldots, n - 1.$$

This completes the proof of Theorem 1.1. \hfill \Box
Proof of Proposition 1.2. By way of contradiction, suppose there is an element $w$ without monomials of degree lower than 3, such that $u = x_1 + x_1 x_2 + w$ is a primitive element of $K\langle x_1, x_2 \rangle$.

By Corollary 1.5 of [6], every primitive element of $K\langle x_1, x_2 \rangle$ is palindromic, i.e., is invariant under the operator $\overline{\cdot}$ that re-writes every monomial backwards. For example, $(x_1 x_2)^\overline{\cdot} = x_2 x_1$; $(x_1 x_2 x_1 x_2^3)^\overline{\cdot} = x_2^3 x_1 x_2 x_1$, etc.

It is clear that if an element of $K\langle x_1, x_2 \rangle$ is palindromic, then its every homogeneous component is palindromic, too. Since the homogeneous component of degree 2 of our element $u$ is not palindromic, this yields a contradiction. \hfill \Box

Remark. Clearly, the statement of Proposition 1.2 holds for any element $u = x_1 + a \cdot x_1 x_2 + b \cdot x_2 x_1 \in K\langle x_1, x_2 \rangle$, where $a, b \in K$ and $a \neq b$.

A combination of Theorem 1.1 and Proposition 1.2 calls for an example of a coordinate polynomial $p \in K[x_1, x_2]$ of the form $x_1 + x_1 x_2 + (\text{terms of higher degree})$; an example like that is given below.

Example. The polynomial

$$p = x_1 + x_1 x_2 + \frac{1}{4}(x_1 x_2^2 - x_1^2 x_2 - x_1^3 + x_2^3) - \frac{1}{16}(x_1 + x_2)^4$$

is coordinate since it is the image of $x_1$ under the automorphism $\alpha \beta \alpha^{-1} \beta \alpha$, where $\alpha$ takes $x_1$ to $(x_1 + x_2)$ and fixes $x_2$, and $\beta$ fixes $x_1$ and takes $x_2$ to $(x_2 - \frac{x_1^3}{4})$.

Proof of Theorem 1.3. Here we use a characterization of two-variable polynomial automorphisms given in [3, Theorem 6.8.5], which implies, in particular, that if $p(x, y) \in K[x, y]$ is a (non-linear) coordinate polynomial, then there is an elementary automorphism of the form $\{x \rightarrow x + \lambda \cdot y^m; y \rightarrow y\}$ or $\{x \rightarrow x; y \rightarrow y + \lambda \cdot x^m\}$, $\lambda \in K^\ast$, that decreases the degree of $p(x, y)$.

Let $p(x, y) = \sum c_{ij} x^i y^j$, $c_{ij} \in K^\ast$, be a non-linear polynomial. Then the condition in the previous paragraph translates into a system of homogeneous polynomial equations, where $c_{ij}$ are considered indeterminates, and (polynomial) functions of $\lambda$ are considered coefficients (every equation in this system expresses the condition on the coefficient at a particular monomial to be equal to zero). Thus, the set of solutions of this system is a subset of $K^\ast$, where $s$ is the number of non-zero coefficients of our polynomial $p(x, y)$.

If $p(x, y)$ is a coordinate polynomial, then this system has non-zero solutions. If it is not, then the system might or might not have non-zero solutions. If it does not, then, since the set of non-solutions of a polynomial system is an open set (because it is the complement of a closed set), the result follows. Note that the presence of “unfit” solutions (where some of $c_{ij}$ are equal to zero) does not change the openness of the set of non-solutions since it is equivalent to adding to this set several sets of a smaller dimension, and those are always closed sets.

If our system has non-zero solutions, then we apply an elementary automorphism to the polynomial $p(x, y)$ and reduce its degree. The new polynomial is still non-coordinate, and its coefficients are polynomial functions of $c_{ij}$ and $\lambda$. Now applying the same argument to this new polynomial yields the result because continuing the reduction of the degree, we eventually obtain a system without non-zero solutions. \hfill \Box
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