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ON THE DENSITY OF THE SET OF GENERATORS OF A POLYNOMIAL ALGEBRA

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ABSTRACT. Let $K[X] = K[x_1, ..., x_n]$, $n \geq 2$, be the polynomial algebra over a field K of characteristic 0. We call a polynomial $p \in K[X]$ *coordinate* (or a generator) if $K[X] = K[p, p_2, ..., p_n]$ for some polynomials $p_2, ..., p_n$. In this note, we give a simple proof of the following interesting fact: for any polynomial h of the form $(x_i + q)$, where q is a polynomial without constant and linear terms, and for any integer $m \geq 2$, there is a coordinate polynomial p such that the polynomial $(p - h)$ has no monomials of degree $\leq m$. A similar result is valid for *coordinate k-tuples* of polynomials, for any $k < n$. This contrasts sharply with the situation in other algebraic systems.

On the other hand, we establish (in the two-variable case) a result related to a different kind of density. Namely, we show that given a *non-coordinate* two-variable polynomial, any sufficiently small perturbation of its non-zero coefficients gives another non-coordinate polynomial.

1. INTRODUCTION

Let $K[X] = K[x_1, ..., x_n]$, $n \geq 2$, be the polynomial algebra over a field K of characteristic 0. We denote by $\text{mindeg}(p)$ the minimal degree of non-zero monomials of $p \in K[X]$.

We call automorphic images of x_1 *coordinate polynomials* to simplify the language. Similarly, a k-tuple of polynomials $(p_1, \ldots, p_k)$, $p_i \in K[X]$, $k \leq n$, is *coordinate* if there exists an automorphism of $K[X]$ which sends $x_1, \ldots, x_k$ respectively to $p_1, \ldots, p_k$. Equivalently, a k-tuple $(p_1, \ldots, p_k)$ is coordinate if there are polynomials $p_{k+1}, \ldots, p_n \in K[X]$ such that $K[p_1, \ldots, p_k, p_{k+1}, \ldots, p_n] = K[X]$.

In this note, we give a simple proof of the following interesting fact: the set of coordinate polynomials is dense (in the formal power series topology) in the set of polynomials of the form $(x_i + q)$, $\text{mindeg}(q) \geq 2$. That is, any polynomial of this form can be completed to a coordinate polynomial by monomials of arbitrarily high degree. Actually, our proof yields a somewhat stronger result:

Theorem 1.1. *For any $(n-1)$-tuple of polynomials $(h_1, \ldots, h_{n-1})$ of the form $h_i = x_i + r_i, \text{mindeg}(r_i) \geq 2$, $i = 1, \ldots, n-1$, and any integer $m \geq 2$, there*

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exists a coordinate $(n-1)$-tuple $(p_1, \ldots, p_{n-1})$ such that $\mathrm{mindeg}(p_i - h_i) > m$, $i = 1, \ldots, n-1$.

There is a non-commutative version of Theorem 1.1 (see [4]) which involves machinery from representation theory of the general linear group $GL_n(K)$. Our proof here is based on simpler ideas and is a consequence of a result of Anick [1].

Theorem 1.1 contrasts sharply with the situation in other, non-commutative, algebras. For example, the set of primitive elements (that is what generators are usually called in a non-commutative setting) of a free Lie algebra of rank 2 is not dense because by a theorem of Cohn [2], all automorphisms of this algebra are linear. Moreover, although the automorphism groups of $K[x_1, x_2]$ and $K\langle x_1, x_2\rangle$ (the free associative algebra of rank 2) are isomorphic (see e.g. [3]), we have:

Proposition 1.2. *The element $u = x_1 + x_1x_2$ cannot be completed to a primitive element of $K\langle x_1, x_2\rangle$ by monomials of degree higher than 2.*

The proof of Proposition 1.2 is based on a characterization of generators of $K\langle x_1, x_2\rangle$ as it appears in [6].

Finally, we establish (in the two-variable case) a result related to a different kind of density:

Theorem 1.3. *Let $p(x,y) = \sum_{i,j=1}^{m} c_{ij} \cdot x^i y^j$, $c_{ij} \in K$, be a non-coordinate polynomial from $K[x,y]$. Let $K = \mathbf{R}$ or $\mathbf{C}$. Then there is an $\varepsilon > 0$ such that every polynomial $q(x,y) = \sum_{i,j=1}^{m} c'_{ij} \cdot x^i y^j$ with $\mid c_{ij} - c'_{ij} \mid < \varepsilon$ if $c_{ij} \neq 0$ and $c'_{ij} = 0$ if $c_{ij} = 0$, is non-coordinate as well.*

2. Preliminaries

For background on polynomial automorphisms we refer to the book [3]. Anick [1] proved that, with respect to the formal power series topology, the set $\mathbf{J}$ of endomorphisms of $K[X]$ with an invertible Jacobian matrix is a closed set, and the group of tame automorphisms of $K[X]$ is dense in $\mathbf{J}$. This means that for any polynomial mapping $F = (f_1, \ldots, f_n)$ with invertible Jacobian matrix $J(F)$ (i.e. $0 \neq J(F) \in K$) and for any positive integer m, there is a tame automorphism $G = (g_1, \ldots, g_n)$ such that the polynomials $f_i - g_i$ contain no monomials of degree less than m. An interpretation of the result of Anick in the language we need is given in [5, Theorem 4.2.7]. We recall some details here briefly.

Let P_k be the K-vector space of all homogeneous polynomials of degree $k \geq 0$. Let I_k, $k \geq 2$, be the semigroup of all polynomial endomorphisms $F = (f_1, \ldots, f_n)$ such that x_i is the only monomial of f_i of degree less than k; $i = 1, \ldots, n$. We write

$$f_i = x_i + g_i + h_i,$$

where $g_i \in P_k$ is the homogeneous component of f_i of degree k, and $\mathrm{mindeg}(h_i) > k$.

It turns out that there is a homomorphism ϕ of I_k onto the direct sum of additive groups $P_k^{\oplus n} \cong P_k \oplus \ldots \oplus P_k$ such that $\phi(F) = (g_1, \ldots, g_n)$.

Let T be the group of tame automorphisms of the algebra $K[X]$, and let $\mathbf{S}_k$ be the set of all polynomial mappings $S = (s_1, \ldots, s_n) \in I_k$ such that $s_i = x_i + g_i + h_i$, where $g_i \in P_k$ and $h_i \in \sum_{l>k} P_l$, with the property

$$\sum_{i=1}^{n} \frac{\partial g_i}{\partial x_i} = 0.$$

The main step of Anick's proof (see [5], Step 2 of the proof of Theorem 4.2.7) is to show that

$$\phi(T \cap I_k) = \phi(\mathbf{S}_k).$$

This yields

Lemma 2.1. *For any $S \in \mathbf{S}_k \subseteq I_k$, there is a tame automorphism G_k of $K[X]$ such that $S \circ G_k^{-1}$ is in I_{k+1}.*

3. Proofs

Proof of Theorem 1.1. Let $u_1, \dots, u_{n-1}$ be $n-1$ polynomials without constant and linear terms, and let u_{ij} be the homogeneous component of degree j of the polynomial u_i; $j = 2, \dots, m$. Let k be the smallest integer such that $u_{ik} \neq 0$ for some i. Let, for example, $i = 1$.

Write the partial derivative of u_{1k} with respect to x_1 in the form

$$\frac{\partial u_{1k}}{\partial x_1} = \sum_{j=0}^{k-1} a_j x_n^j,$$

where the polynomials a_j do not depend on x_n. There is a homogeneous polynomial $w_{1k} \in P_k$ such that

$$\frac{\partial w_{1k}}{\partial x_2} = -\sum_{j=0}^{k-1} a_j x_n^j.$$

Consider an endomorphism F_{1k} of the algebra $K[X]$ defined by

$$F_{1k} = (x_1 + u_{1k},\ x_2, \dots,\ x_{n-1},\ x_n + w_{1k}).$$

Clearly, $F_{1k} \in I_k$ and, because of the choice of w_{1k}, also $F_{1k} \in \mathbf{S}_k$. Similarly, we construct endomorphisms

$$F_{ik} = (x_1, \dots,\ x_{i-1},\ x_i + u_{ik},\ x_{i+1}, \dots,\ x_{n-1},\ x_n + w_{ik})$$

for some $w_{ik} \in P_k$, $i = 2, \dots, n-1$, such that $F_{ik} \in \mathbf{S}_k$. Hence the composition $F_k = F_{1,k} \circ \dots \circ F_{n-1,k}$ also belongs to $\mathbf{S}_k$ and by Lemma 2.1 there exists a tame automorphism $G_k = (g_{1,k}, \dots, g_{n,k}) \in I_k$ such that $F_{k+1} = F_k \circ G_k^{-1} \in I_{k+1}$. Therefore, $g_{i,k} = x_i + u_{i,k} + v_{i,k+1} + v_i$, $i = 1, \dots, n-1$, where $v_{i,k+1}$ is homogeneous of degree $k+1$, and $v_i \in \sum_{l \geq k+2} P_l$.

Continuing this way, we obtain a tame automorphism $G_{k+1} \in I_{k+1}$ such that $g_{i,k+1} = x_i + (u_{i,k+1} - v_{i,k+1}) + w_i$, $i = 1, \dots, n-1$, where $w_i \in \sum_{l \geq k+2} P_l$.

If we act by the automorphism G_{k+1} on the polynomial u_{ik}, we get a polynomial of the form $u_{ik} + s_i$, where s_i has no homogeneous components of degree less than $(k-1) + (k+1) = 2k > k+1$. Therefore, the automorphism $G_{k+1} \circ G_k$ takes x_i to $x_i + u_{ik} + u_{i,k+1} +$ (terms of higher degree), $i = 1, \dots, n-1$.

In a finite number of steps, we obtain a tame automorphism $G = (g_1, \dots, g_n)$ such that

$$g_i = x_i + u_i + \text{(terms of higher degree)},\ i = 1, \dots, n-1.$$

This completes the proof of Theorem 1.1. □

Proof of Proposition 1.2. By way of contradiction, suppose there is an element w without monomials of degree lower than 3, such that $u = x_1 + x_1x_2 + w$ is a primitive element of $K\langle x_1, x_2\rangle$.

By Corollary 1.5 of [6], every primitive element of $K\langle x_1, x_2\rangle$ is palindromic, i.e., is invariant under the operator $^{\leftarrow}$ that re-writes every monomial backwards. For example, $(x_1x_2)^{\leftarrow} = x_2x_1$; $(x_1x_2x_1x_2^2)^{\leftarrow} = x_2^2x_1x_2x_1$, etc.

It is clear that if an element of $K\langle x_1, x_2\rangle$ is palindromic, then its every homogeneous component is palindromic, too. Since the homogeneous component of degree 2 of our element u is not palindromic, this yields a contradiction. $\square$

Remark. Clearly, the statement of Proposition 1.2 holds for any element $u = x_1 + a \cdot x_1x_2 + b \cdot x_2x_1 \in K\langle x_1, x_2\rangle$, where $a, b \in K$ and $a \neq b$.

A combination of Theorem 1.1 and Proposition 1.2 calls for an example of a coordinate polynomial $p \in K[x_1, x_2]$ of the form $x_1 + x_1x_2 +$ (terms of higher degree); an example like that is given below.

Example. The polynomial

$$p = x_1 + x_1x_2 + \frac{1}{4}(x_1x_2^2 - x_1^2x_2 - x_1^3 + x_2^3) - \frac{1}{16}(x_1 + x_2)^4$$

is coordinate since it is the image of x_1 under the automorphism $\alpha\beta\alpha^{-2}\beta\alpha$, where α takes x_1 to $(x_1 + x_2)$ and fixes x_2, and β fixes x_1 and takes x_2 to $(x_2 - \frac{x_1^2}{4})$.

Proof of Theorem 1.3. Here we use a characterization of two-variable polynomial automorphisms given in [3, Theorem 6.8.5], which implies, in particular, that if $p(x, y) \in K[x, y]$ is a (non-linear) coordinate polynomial, then there is an elementary automorphism of the form $\{x \to x + \lambda \cdot y^m;\ y \to y\}$ or $\{x \to x;\ y \to y + \lambda \cdot x^m\}$, $\lambda \in K^*$, that decreases the degree of $p(x, y)$.

Let $p(x, y) = \sum c_{ij} \cdot x^i y^j$, $c_{ij} \in K^*$, be a non-linear polynomial. Then the condition in the previous paragraph translates into a system of homogeneous polynomial equations, where c_{ij} are considered indeterminates, and (polynomial) functions of λ are considered coefficients (every equation in this system expresses the condition on the coefficient at a particular monomial to be equal to zero). Thus, the set of solutions of this system is a subset of K^s, where s is the number of non-zero coefficients of our polynomial $p(x, y)$.

If $p(x, y)$ is a coordinate polynomial, then this system has non-zero solutions. If it is not, then the system might or might not have non-zero solutions. If it does not, then, since the set of non-solutions of a polynomial system is an open set (because it is the complement of a closed set), the result follows. Note that the presence of "unfit" solutions (where some of c_{ij} are equal to zero) does not change the openness of the set of non-solutions since it is equivalent to adding to this set several sets of a smaller dimension, and those are always closed sets.

If our system has non-zero solutions, then we apply an elementary automorphism to the polynomial $p(x, y)$ and reduce its degree. The new polynomial is still non-coordinate, and its coefficients are polynomial functions of c_{ij} and λ. Now applying the same argument to this new polynomial yields the result because continuing the reduction of the degree, we eventually obtain a system without non-zero solutions. $\square$

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