ON THE POINTWISE MAXIMUM OF CONVEX FUNCTIONS

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This paper is dedicated to Professor Robert Phelps

Abstract. We study the conjugate of the maximum, \( f \vee g \), of \( f \) and \( g \) when \( f \) and \( g \) are proper convex lower semicontinuous functions on a Banach space \( E \). We show that \( (f \vee g)^* = f^* \vee g^* \) on the bidual, \( E^{**} \), of \( E \) provided that \( f \) and \( g \) satisfy the Attouch-Brezis constraint qualification, and we also derive formulae for \( (f \vee g)^* \) and for the “preconjugate” of \( f^* \vee g^* \).

INTRODUCTION

Let \( E \) be a real nontrivial Banach space. If \( f : E \to \mathbb{R} \cup \{ \infty \} \), we write

\[
\text{dom } f := \{ x \in E : f(x) \in \mathbb{R} \},
\]

the “effective domain” of \( f \). We write \( PCLSC(E) \) for the set of all convex lower semicontinuous functions \( f : E \to \mathbb{R} \cup \{ \infty \} \) such that \( \text{dom } f \neq \emptyset \). (The “P” stands for “proper”, which is the adjective frequently used to denote the fact that the effective domain of a function is nonempty.)

We write \( E^* \) for the dual space of \( E \). If \( f \in PCLSC(E) \), we define \( f^* : E^* \to \mathbb{R} \cup \{ \infty \} \) by

\[
f^*(x^*) := \sup_{E} (x^* - f),
\]

the conjugate of \( f \). Then (see [5], p. 210) \( f^* \in PCLSC(E^*) \).

We define the biconjugate, \( f^{**} \), of \( f \) by

\[
f^{**}(x^{**}) := (f^*)^*(x^{**}) \quad (x^{**} \in E^{**}).
\]

From what we have observed above, \( f^{**} \in PCLSC(E^{**}) \). In fact, \( f^{**} \) is lower semicontinuous with respect to the weak* topology of \( E^{**} \) and (see [5], p. 210 again)

\[
(0.1) \quad \text{for all } x \in E, \quad f^{**}(\hat{x}) = f(x),
\]

where \( \hat{x} \) is the canonical image of \( x \) in \( E^{**} \).
Let \( f, g \in \mathcal{PCLSC}(E) \). We say that \( f \) and \( g \) satisfy the *Attouch–Brézis constraint qualification* if
\[
(\text{AB}) \quad \bigcup_{\lambda > 0} \lambda (\text{dom } f - \text{dom } g) \quad \text{is a closed subspace of } E.
\]

It is well known that if \( f, g \in \mathcal{PCLSC}(E) \) and \( f \) and \( g \) satisfy \((\text{AB})\), then
\[
(f + g)^{*} = f^{**} + g^{**} \quad \text{on } E^{**}.
\]

In fact, Rockafellar used the equality \((0.2)\) (under a stronger constraint qualification) in his proof in [5], Proposition 1, pp. 211–212 that the subdifferential of an element of \( \mathcal{PCLSC}(E) \) is maximal monotone. The equality \((0.2)\) follows easily from the "\( \inf \)-convolution" formula for \( (f + g)^{*} \); namely that, for all \( w^{*} \in E^{*} \),
\[
(f + g)^{*}(w^{*}) = \min_{y^{*}, z^{*} \in E^{*}, y^{*} + z^{*} = w^{*}} \left[ f^{*}(y^{*}) + g^{*}(z^{*}) \right],
\]
which was established by Attouch–Brézis in [1], Corollary 2.3, pp. 131–132.

In this paper, we consider the corresponding problem with \( f \) and \( g \) replaced by \( f \vee g \), where, for all \( f, g \in \mathcal{PCLSC}(E) \), \( f \vee g \) is defined by
\[
(f \vee g)(x) := \max\{f(x), g(x)\} \quad (x \in E).
\]
Indeed, we will prove in Theorem 6 that if \( f, g \in \mathcal{PCLSC}(E) \) and \( f \) and \( g \) satisfy \((\text{AB})\), then
\[
(f \vee g)^{*} = f^{**} \vee g^{**} \quad \text{on } E^{**}.
\]

We will complement this in Remark 8 by giving an example showing that the equality \((0.4)\) can fail when \((\text{AB})\) is not satisfied, even if \( f \vee g \in \mathcal{PCLSC}(E) \). Now \((0.4)\) would follow easily from the equality that, for all \( w^{*} \in E^{*} \),
\[
(f \vee g)^{*}(w^{*}) = \inf_{\rho \in [0,1]} \left[ \rho f^{*}(u^{*}) + (1 - \rho)g^{*}(v^{*}) \right].
\]

Unfortunately, \((0.5)\) fails even if \( E = \mathbb{R}^{2} \), \( g \in \mathcal{CC}(E) \) and \( f \) and \( g \) satisfy \((\text{AB})\), where \( \mathcal{CC}(E) \) stands for the set of all real convex continuous functions on \( E \). We give an example of this in Remark 3. The actual formula for \((f \vee g)^{*}\) is much more complicated. In fact, we give two such formulae. The first, in (2.3), appears in Traoré and Volle, [7], Section 7, p. 149 and does not seem to lead easily to \((0.4)\).

We now give the background for the second, much more complicated formula, which appears in (2.1), and *does* lead easily to \((0.4)\). Let \( F \) be a nontrivial Banach space. (The reason why we also introduce the symbol \( F \) to represent a Banach space is that we will be applying these concepts with \( F := E^{*} \).) If \( w \in F \) and \( \delta > 0 \), let \( B(w, \delta) := \{x \in F : \|x - w\| < \delta\} \) and
\[
L(w, \delta) := \{\rho \in [0,1], \rho u + \sigma v \in B(w, \delta) : \rho > 0, \sigma > 0, u, v \in F, \rho + \sigma = 1, \rho u + \sigma v \in B(w, \delta)\}.
\]

Suppose that \( f, g \in \mathcal{PCLSC}(F) \). If \( w \in F \), write
\[
(f \wedge g)(w) := \inf_{(\rho, \sigma, u, v) \in L(w, \delta)} \left[ \rho f(u) + \sigma g(v) \right] \quad (\delta > 0) \quad \text{and} \quad (f \wedge g)(w) := \sup_{\delta > 0} (f \wedge g)(w) = \lim_{\delta \to 0} (f \wedge g)(w).
\]

Then the formula that we shall give in (2.1) is that if \( w^{*} \in E^{*} \), then
\[
(f \vee g)^{*}(w^{*}) = (f \wedge g)^{*}(w^{*}).
\]
Incidentally, the equality (0.4) is closely related to the result proved by Gossez in [3], Lemme 2.1, p. 375 that the subdifferential of an element of $\mathcal{PCLSC}(E)$ is maximal monotone of "dense type". Unfortunately, it would take us much too far afield to dwell on this issue any further.

Up to this point, we have presented the quantity $(f \wedge g)(w)$ simply as a number which appears as the result of certain computations. In fact, $f \wedge g$ has much more significance when we consider it as a function. We shall show in Theorem 11 that if $f, g \in \mathcal{PCLSC}(F)$ and $\text{dom} f^* \cap \text{dom} g^* \neq \emptyset$, then

$$f \wedge g \in \mathcal{PCLSC}(F) \quad \text{and} \quad (f \wedge g)^* = f^* \vee g^* \text{ on } F^*.$$  

In other words, $f \wedge g$ is the "preconjugate" of $f^* \vee g^*$. We shall use this result in Theorem 12 to give a precise description of when (0.4) occurs. Namely, if $f, g \in \mathcal{PCLSC}(E)$ and $\text{dom} f \cap \text{dom} g \neq \emptyset$, then (0.4) occurs if, and only if,

$$(f \vee g)^* = f^* \wedge g^* \text{ on } E^*.$$  

In the proof of Theorem 2, we shall use the minimax theorem below, which follows from a result of Fan (see [2]). (See also [4] and [6] for simple generalizations of Fan’s result.)

**Theorem 1.** Let $A$ be a nonempty convex subset of a vector space, and let $B$ be a nonempty compact convex subset of a topological vector space. Let $h : A \times B \to \mathbb{R}$ be convex on $A$, and concave and upper semicontinuous on $B$. Then

$$\inf_A \max_B h = \max_B \inf_A h.$$  

**The Conjugate of a Maximum**

**Theorem 2.** Suppose that $f, g \in \mathcal{PCLSC}(E)$, $f$ and $g$ satisfy $(AB)$ and $w^* \in E^*$. Then:

$$(f \vee g)^* (w^*) = (f^* \wedge g^*)(w^*).$$

**Proof.** We first prove that if $\rho, \sigma > 0$, then there exist $u^*, v^* \in E^*$ such that

$$\rho u^* + \sigma v^* = w^* \quad \text{and} \quad \rho f^*(u^*) + \sigma g^*(v^*) = \sup_A \left[ w^* - \rho f - \sigma g \right],$$

where $A$ is the nonempty convex set $\text{dom} f \cap \text{dom} g$. To this end, let $\rho, \sigma > 0$. Clearly $\rho f$ and $\sigma g$ also satisfy $(AB)$; consequently, from the Attouch–Brézis formula for the conjugate of a sum (see (0.3) above), there exist $y^* \in E^* \quad \text{and} \quad z^* \in E^*$ such that

$$y^* + z^* = w^* \quad \text{and} \quad (\rho f)^*(y^*) + (\sigma g)^*(z^*) = (\rho f + \sigma g)^*(w^*).$$

We now put $u^* := y^*/\rho$ and $v^* := z^*/\sigma$, and obtain (2.2) since we then have $(\rho f)^*(y^*) = \rho f^*(u^*)$, $(\sigma g)^*(z^*) = \sigma g^*(v^*)$ and

$$(\rho f + \sigma g)^*(w^*) = \sup_A \left[ w^* - \rho f - \sigma g \right].$$

We next prove that

$$(f \vee g)^* (w^*) = \min_{\lambda \in [0,1]} \sup_A \left[ w^* - \lambda f - (1-\lambda)g \right].$$
This follows from the minimax theorem, Theorem 1, with \( B := [0, 1] \), since
\[
(f \vee g)^*(w^*) = \sup_{x \in A} [(x, w^*) - (f \vee g)(x)]
= \sup_{x \in A} \min_{\lambda \in [0,1]} \left[(x, w^*) - \lambda f(x) - (1 - \lambda)g(x)\right].
\]

We now prove the inequality "\( \geq \)" in (2.1). Since this is trivially true if \((f \vee g)^*(w^*) = \infty\), we can and will suppose that \((f \vee g)^*(w^*) \in \mathbb{R}\). Let \( \delta, \varepsilon > 0 \). We shall prove that there exists \((\rho, \sigma, u^*, v^*) \in L(w^*, \delta)\) such that
\[
\rho f^*(u^*) + \sigma g^*(v^*) \leq (f \vee g)^*(w^*) + \varepsilon.
\]
The desired inequality will then follow by taking the infimum over \((\rho, \sigma, u^*, v^*) \in L(w^*, \delta)\) and then letting \( \delta \to 0 \) and \( \varepsilon \to 0 \). From (2.3), there exists \( \lambda \in [0, 1] \) such that
\[
\sup_{A} [w^* - \lambda f - (1 - \lambda)g] = (f \vee g)^*(w^*).
\]

**Case 1** (\( \lambda \in (0, 1) \)). From (2.2), there exist \( u^*, v^* \in E^* \) such that
\[
\lambda u^* + (1 - \lambda)v^* = w^* \quad \text{and} \quad \lambda f^*(u^*) + (1 - \lambda)g^*(v^*) = (f \vee g)^*(w^*)
\]
and (2.4) is immediate with \( \rho := \lambda \) and \( \sigma := 1 - \lambda \).

**Case 2** (\( \lambda = 0 \)). Here we have
\[
\sup_{A} [w^* - g] = (f \vee g)^*(w^*).
\]
As we have already observed, \( f^* \in PC\mathscr{L}\mathcal{C}(E^*) \). Hence there exists \( x^* \in E^* \) such that \( f^*(x^*) \in \mathbb{R} \). If \( \rho > 0, \sigma > 0, \rho + \sigma = 1 \) and \((\rho, \sigma)\) is sufficiently close to \((0, 1)\), then
\[
(\rho, \sigma, x^*, w^*) \in L(w^*, \delta) \quad \text{and} \quad \rho f^*(x^*) \leq \rho(f \vee g)^*(w^*) + \varepsilon.
\]
Using (2.2) again, there exist \( u^*, v^* \in E^* \) such that
\[
\rho u^* + \sigma v^* = \rho x^* + \sigma w^*
\]
and
\[
\rho f^*(u^*) + \sigma g^*(v^*) = \sup_{A} [\rho x^* + \sigma w^* - \rho f - \sigma g]
= \sup_{A} [\rho(x^* - f) + \sigma(w^* - g)]
\leq \rho \sup_{A} [x^* - f] + \sigma \sup_{A} [w^* - g]
\leq \rho f^*(x^*) + \sigma \sup_{A} [w^* - g].
\]

Thus, from (2.6) and (2.5),
\[
\rho f^*(u^*) + \sigma g^*(v^*) \leq [\rho(f \vee g)^*(w^*) + \varepsilon] + \sigma(f \vee g)^*(w^*)
= (f \vee g)^*(w^*) + \varepsilon.
\]
We now obtain (2.4) since, from (2.6) and (2.7), \((\rho, \sigma, u^*, v^*) \in L(w^*, \delta)\).

**Case 3** (\( \lambda = 1 \)). The proof of this is similar to that of Case 2, except that the roles of \( f \) and \( g \) are reversed. This completes the proof of the inequality "\( \geq \)" in (2.1).
We now prove the reverse inequality. Let \( x \in A \) and \((\rho, \sigma, u^*, \nu^*) \in L(w^*, \delta)\). Then
\[
\rho f^*(u^*) + \sigma g^*(\nu^*) \geq \rho[x, u^*] - f(x) + \sigma[x, \nu^*] - g(x)
\]
\[
= \langle x, \rho u^* + \sigma \nu^* \rangle - \rho f(x) - \sigma g(x)
\]
\[
\geq \langle x, w^* \rangle - \delta\|x\| - (f \lor g)(x).
\]
Taking the infimum over \((\rho, \sigma, u^*, \nu^*) \in L(w^*, \delta)\), we obtain
\[
(f^* \land g^*)(w^*) \geq \langle x, w^* \rangle - \delta\|x\| - (f \lor g)(x).
\]
Letting \( \delta \to 0 \),
\[
(f^* \land g^*)(w^*) \geq \langle x, w^* \rangle - (f \lor g)(x).
\]
The inequality “\( \leq \)” in (2.1) now follows by taking the supremum of the right hand side over \( x \in A \). (Note: this can also be deduced from Lemma 10(a), which is independent of the analysis in this Theorem.)

This completes the proof of Theorem 2. \( \square \)

If \( C \subset E \), the indicator function of \( C \) is the function \( I_C : E \to \mathbb{R} \cup \{ \infty \} \) defined by
\[
I_C(x) := \begin{cases} 0 & \text{if } x \in C; \\ \infty & \text{otherwise.} \end{cases}
\]

**Remark 3.** We now give the promised example where \( f, g \in \mathcal{PCLSC}(E) \) and \( f \) and \( g \) satisfy (\( \mathcal{AB} \)), but (0.5) fails. (We leave it to the reader to check that (0.5) does not hold if both \( f \in \mathcal{CC}(E) \) and \( g \in \mathcal{CC}(E) \).) Here is the example. Define \( f \in \mathcal{PCLSC}(\mathbb{R}^2) \) and \( g \in \mathcal{CC}(\mathbb{R}^2) \) by
\[
f(x_1, x_2) := \begin{cases} x_2 & \text{if } x_1 \geq 0; \\ \infty & \text{otherwise;}; \end{cases}
\]
and
\[
g(x_1, x_2) := x_1.
\]
Then \((f \lor g)^*(0) = -\inf(f \lor g) = 0\). On the other hand, \( f^* \) is the indicator function of \((\mathbb{R} \times \{1\}) \cap \{0\}\times \{1\}\) and \( g^* \) is the indicator function of \(\{1, 0\}\). Consequently, if \( \rho \in [0, 1], u^* \in \mathbb{R}^2, \nu^* \in \mathbb{R}^2 \) and \( \rho u^* + (1 - \rho)\nu^* = 0 \), then \( \rho f^*(u^*) + (1 - \rho)g^*(\nu^*) = \infty \), and so (0.5) fails. We note that \((\mathcal{AB})\) is satisfied in this example because \( g \in \mathcal{CC}(\mathbb{R}^2) \).

**Remark 4.** Let \( f, g \in \mathcal{PCLSC}(E) \), \( f, g \) satisfy \((\mathcal{AB})\), \( x \in E \) and \( f(x) = g(x) \in \mathbb{R} \). We briefly discuss the problem of finding a formula for \( \partial(f \lor g)(x) \). Suppose first that, for all \( w^* \in \partial(f \lor g)(x) \), the following “exact” version of (0.5) holds:
\[
(4.1) \quad (f \lor g)^*(w^*) = \min_{\rho \in [0,1], u^*, \nu^* \in E^*} \left[ \rho f^*(u^*) + (1 - \rho)g^*(\nu^*) \right].
\]
Then it is easily seen that
\[
\partial(f \lor g)(x) = \text{co}(\partial(f(x) \cup g(x))).
\]
In general, we have the formulae for \((f \lor g)^*(w^*)\) given by (2.1) and (2.3), and we have the formula established by Volle in [8], Théorème 2, p. 848 that
\[
(4.2) \quad \partial(f \lor g)(x) = \text{co}(\partial(f(x) \cup g(x))) + N_{\text{dom}}(x) + N_{\text{dom}}(x),
\]
where \( N_C(x) \) stands for the normal cone to \( C \) at \( x \). However, we do not know an easy way of deducing (4.2) from (2.1) or (2.3).

**The Biconjugate of a Maximum**

It is an easy consequence of the definitions that if \( f \in PCLSC(E) \), then

\[
(4.3) \quad \left< t^*, f^{**}(t^*) \right> \leq 0 \quad \text{and} \quad w^* \in E^* \implies \left< w^*, t^* \right> \leq f^*(w^*). 
\]

**Lemma 5.** Suppose that \( f, g \in PCLSC(E) \), \( f \) and \( g \) satisfy (AB) and also that \( f^{**}(t^*) \lor g^{**}(t^*) \leq 0 \).

(a) Let \( w^* \in E^* \). Then \( \left< w^*, t^* \right> \leq (f \lor g)^*(w^*) \).

(b) \( (f \lor g)^*(t^*) \leq 0 \).

**Proof.** (a) Let \( \delta > 0 \). If \( (\rho, \sigma, u^*, v^*) \in L(w^*, \delta) \), then, using (4.3),

\[
\rho f^*(u^*) + \sigma g^*(v^*) \geq \rho(u^*, t^*) + \sigma(v^*, t^*) \\
= \left< \rho u^* + \sigma v^*, t^* \right> \\
\geq \left< w^*, t^* \right> - \delta \|t^*\|.
\]

Thus, taking the infimum over \( (\rho, \sigma, u^*, v^*) \in L(w^*, \delta) \),

\[
(f^* \land g^*)(w^*) \geq \left< w^*, t^* \right> - \delta \|t^*\|,
\]

and (a) now follows from Theorem 2 by letting \( \delta \to 0 \). (b) is immediate from (a). \( \square \)

**Theorem 6.** Suppose that \( f, g \in PCLSC(E) \), and \( f \) and \( g \) satisfy (AB). Then

\[
(f \lor g)^* = f^{**} \land g^{**} \quad \text{on } E^{**}.
\]

**Proof.** We first prove that if \( t^* \in E^* \), then

\[
(6.1) \quad (f \lor g)^*(t^*) \leq f^{**}(t^*) \lor g^{**}(t^*).
\]

Let \( \alpha := f^{**}(t^*) \lor g^{**}(t^*) \). Since (6.1) is immediate if \( \alpha = \infty \), we can and will suppose that \( \alpha \in \mathbb{R} \). Then (6.1) follows from Lemma 5(b) with \( f \) replaced by \( f - \alpha \) and \( g \) replaced by \( g - \alpha \).

Since \( f \lor g \geq f \) on \( E \), \( (f \lor g)^* \geq f^* \) on \( E^{**} \). Similarly, \( (f \lor g)^* \geq g^* \) on \( E^{**} \), and so \( (f \lor g)^* \geq f^{**} \lor g^{**} \) on \( E^{**} \). The result now follows from (6.1). \( \square \)

**Corollary 7.** Let \( g_0 \in PCLSC(E) \) and \( g_1, \ldots, g_m \in CC(E) \). Then

\[
(g_0 \lor \cdots \lor g_m)^* = g_0^{**} \lor \cdots \lor g_m^{**}.
\]

**Proof.** This is immediate from Theorem 6 and induction. \( \square \)

**Remark 8.** We now give an example showing that (0.4) can fail when (AB) is not satisfied, even if \( f \lor g \in PCLSC(E) \). (The conclusion of Theorem 2 must also fail for this example, as we shall see in Theorem 12.) Let \( E = c_0 \),

\[
C := \left\{ \{x_n\}_{n \geq 1} \in c_0 : x_1 \geq x_2 \geq x_3 \geq \ldots \geq 0 \right\},
\]

\[
D := \left\{ \{x_n\}_{n \geq 1} \in c_0 : \sum_{n=1}^{\infty} \frac{1}{2^n}(x_1 - x_{n+1}) = 0 \right\},
\]
and define $f, g \in \mathcal{PCLSC}(E)$ by $f := I_C$ and $g := I_D$. Now if $x \in C \cap D$, then
\[ \sum_{n=1}^{\infty} \frac{1}{2^n} (x_1 - x_{n+1}) = 0 \quad \text{and, for all } n \geq 1, \quad x_1 - x_{n+1} \geq 0. \]

It follows that, for all $n \geq 1$, $x_1 - x_{n+1} = 0$, and so $x$ is a constant sequence. Since $x \in c_0$, we deduce that $x = 0$. These observations lead easily to the conclusion that $f \vee g = I_{\{0\}}$, from which $(f \vee g)^* = 0$ and $(f \vee g)^{**} = I_{\{0\}}$ (relative to $E^{**}$). In particular, if $e := (1, 1, \ldots) \in \ell^\infty = E^{**}$, then
\[ (f \vee g)^{**}(e) = \infty. \]

If $m \geq 1$, define $y^m$ and $z^m \in E$ as follows:
\[
y^m_n := \begin{cases} 
1 & \text{if } n \leq m; \\
0 & \text{otherwise}; 
\end{cases} \quad \text{and} \quad z^m_n := \begin{cases} 
1 & \text{if } n \leq m; \\
2 & \text{if } n = m + 1; \\
0 & \text{otherwise}. 
\end{cases}
\]

Then $y^m \in C$ and $z^m \in D$, from which $f(y^m) = 0$ and $g(z^m) = 0$. Using (0.1), we deduce from this that $f^{**}(y^m) = 0$ and $g^{**}(z^m) = 0$. Since $y^m \to e$ and $z^m \to e$ in the weak* topology of $E^{**}$ as $m \to \infty$, and $f^{**}$ and $g^{**}$ are weak* lower semicontinuous, it follows that $f^{**}(e) \leq 0$ and $g^{**}(e) \leq 0$, from which
\[ (f^{**} \vee g^{**})(e) \leq 0. \]

If we now combine (8.1) and (8.2), we see that (0.4) fails, as claimed.

**The preconjugate of a maximum**

**Lemma 9.** Suppose that $f, g \in \mathcal{PCLSC}(F)$ with $\text{dom } f^{**} \cap \text{dom } g^{**} \neq \emptyset$, and $\delta > 0$.

(a) Let $x^* \in F^*$. Then $f \triangledown \delta g \geq x^* - (f \vee g^*)(x^*) - \delta ||x^*||$ on $F$.

(b) $f \triangledown \delta g : F \to \mathbb{R} \cup \{\infty\}$.

(c) $f \triangledown \delta g \leq f$ on $F$ and $f \triangledown \delta g \leq g$ on $F$.

(d) $f \triangledown \delta g$ is convex.

**Proof.** (a) Since the result is trivial if $(f \vee g^*)(x^*) = \infty$, we can and will suppose that $(f \vee g^*)(x^*) \in \mathbb{R}$. Let $w \in F$ and $(\rho, \sigma, u, v)$ be an arbitrary element of $L(w, \delta)$. Then
\[
\rho f(u) + \sigma g(v) \geq \rho[u, x^*] - f(x^*) + \sigma[v, x^*] - g^*(x^*) \\
\geq (\rho \mu + \sigma v, x^*) - f^*(x^*) + g^*(x^*) \\
\geq \langle w, x^* \rangle - \delta ||x^*|| - (f^* \vee g^*)(x^*).
\]

We now obtain (a) by taking the infimum over $(\rho, \sigma, u, v) \in L(w, \delta)$.

(b) This follows from (a) by taking $x^* \in \text{dom } f^{**} \cap \text{dom } g^{**}$.

(c) We shall prove that $f \triangledown \delta g \leq f$ on $F$, the proof that $f \triangledown \delta g \leq g$ on $F$ is similar.

So let $w \in F$. We need to show that
\[ (f \triangledown \delta g)(w) \leq f(w). \]

Since this is trivial if $f(w) = \infty$, we can and will suppose that $w \in \text{dom } f$. Fix $v \in \text{dom } g$. If $\rho > 0$, $\sigma > 0$, $\rho + \sigma = 1$ and $(\rho, \sigma)$ is sufficiently close to $(1, 0)$, then $(\rho, \sigma, w, v) \in L(w, \delta)$ and so $\rho f(w) + \sigma g(v) \geq (f \triangledown \delta g)(w)$. We now obtain (9.1) by letting $(\rho, \sigma) \to (1, 0)$. 

\[ \]
(d) For \( i = 1, 2 \), let \( w_i \in E, \lambda_i > 0 \) and \( \sum_i \lambda_i = 1 \). Put \( w_3 := \sum_i \lambda_i w_i \). We shall prove that
\[
\sum_i \lambda_i (f_\delta \wedge g)(w_i) \geq (f_\delta \wedge g)(w_3),
\]
which will give the required result. To this end, let \((\rho_i, \sigma_i, u_i, v_i)\) be arbitrary elements of \( L(w_i, \delta) \). It is easy to check that
\[
\sum_i \lambda_i (\rho_i u_i + \sigma_i v_i) \in B(w_3, \delta).
\]
Put \( \rho_3 := \sum_i \lambda_i \rho_i \in (0, 1), \sigma_3 := \sum_i \lambda_i \sigma_i \in (0, 1), u_3 := \sum_i \lambda_i \rho_i u_i / \rho_3 \in F \) and \( v_3 := \sum_i \lambda_i \sigma_i v_i / \sigma_3 \in F \). Since \( \rho_3 + \sigma_3 = 1 \), it follows from these definitions that
\[
\sum_i \lambda_i \rho_i f(u_i) \geq \rho_3 f(u_3) \quad \text{and} \quad \sum_i \lambda_i \sigma_i g(v_i) \geq \sigma_3 g(v_3).
\]
Consequently,
\[
\sum_i \lambda_i [\rho_i f(u_i) + \sigma_i g(v_i)] \geq \rho_3 f(u_3) + \sigma_3 g(v_3).\]
We also derive from (9.3) that \( \rho_3 u_3 + \sigma_3 v_3 \in B(w_3, \delta) \). Combining this with (9.4), we obtain
\[
\sum_i \lambda_i [\rho_i f(u_i) + \sigma_i g(v_i)] \geq (f_\delta \wedge g)(w_3),
\]
and (9.2) now follows by taking the infima over \((\rho_i, \sigma_i, u_i, v_i)\) in \( L(w_i, \delta) \).

**Lemma 10.** Suppose that \( f, g \in \mathcal{PCLSC}(F) \) and \( \text{dom } f^* \cap \text{dom } g^* \neq \emptyset \).

(a) Let \( x^* \in F^* \). Then \( x^* - f \wedge g \leq (f^* \vee g^*)(x^*) \) on \( F \).

(b) \( f \wedge g \leq f \) on \( F \) and \( f \wedge g \leq g \) on \( F \).

**Proof.** These assertions follow easily from Lemma 9 by letting \( \delta \rightarrow 0 \).

**Theorem 11.** Suppose that \( f, g \in \mathcal{PCLSC}(F) \) and \( \text{dom } f^* \cap \text{dom } g^* \neq \emptyset \). Then
\[
(\wedge_0 g) \in \mathcal{PCLSC}(F) \quad \text{and} \quad (f \wedge g)^* = f^* \vee g^* \text{ on } F^*.
\]

**Proof.** It is clear from Lemma 9(a) by letting \( \delta \rightarrow 0 \) that \( (f \wedge g): E \rightarrow \mathbb{R} \cup \{\infty\} \) and is convex. In order to show that \( f \wedge g \in \mathcal{PCLSC}(F) \), it only remains to prove that \( f \wedge g \) is lower semicontinuous on \( F \). To this end, let \( w \in F \) and \( \alpha < (f \wedge g)(w) \). We can choose \( \delta > 0 \) so that \( \alpha < (f \wedge g)(w) \). Let \( \eta := \delta / 2 \). Since
\[
x \in B(w, \eta) \implies B(x, \eta) \subset B(w, \delta),
\]
it follows by taking the appropriate infima that
\[
x \in B(w, \eta) \implies (f \wedge g)(x) \geq (f \wedge g)(w).
\]
Hence
\[
x \in B(w, \eta) \implies (f_\eta \wedge g)(x) > \alpha.
\]
This gives the required lower semicontinuity. It follows from Lemma 10(b) that
\[
(f_\delta \wedge g)^* \geq f^* \text{ on } F^* \quad \text{and} \quad (f_0 \wedge g)^* \geq g^* \text{ on } F^*,
\]
from which \( (f \wedge g)^* \geq f^* \vee g^* \text{ on } F^* \). The opposite inequality follows by taking the supremum over \( F \) in Lemma 10(a).

**Theorem 12.** Suppose that \( f, g \in \mathcal{PCLSC}(E) \) and \( \text{dom } f \cap \text{dom } g \neq \emptyset \). Then
\[
(f \vee g)^{**} = f^{**} \vee g^{**} \text{ on } E^{**} \iff (f \wedge g)^* = f^* \wedge g^* \text{ on } E^*.
\]
Proof. We first note that \( \text{dom } f^{**} \cap \text{dom } g^{**} \neq \emptyset \); hence, from Theorem 11 with \( F := E^* \) and \( f \) and \( g \) replaced by \( f^{**} \) and \( g^{**} \),
\[
(12.1) \quad f^{**} \land g^{**} = f^{**} \lor g^{**} \quad \text{on } E^{**}.
\]
It is immediate from this that
\[
(f \lor g)^{**} = f^{**} \land g^{**} \quad \text{on } E^{**} \quad \implies \quad (f \lor g)^{**} = f^{**} \lor g^{**} \quad \text{on } E^{**}.
\]
Now suppose that \((f \lor g)^{**} = f^{**} \lor g^{**}\) on \( E^{**} \). From (12.1), \((f \lor g)^{**} = (f^{*} \land g^{*})^{*}\) on \( E^{**} \), and consequently
\[
(f \lor g)^{**} = (f^{*} \land g^{*})^{**} \quad \text{on } E^{**}.
\]
Since both \((f \lor g)^{**}\) and \((f^{*} \land g^{*})^{**}\) are in \( \mathcal{PCLSC}(E^{*}) \), it follows from (0.1) (with \( E \) replaced by \( E^{*} \)) that
\[
(f \lor g)^{**} = f^{*} \land g^{*} \quad \text{on } E^{*},
\]
as required.

References


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