WEAK TYPE ESTIMATES FOR CONE MULTIPLIERS ON $H^p$ SPACES, $p < 1$

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Abstract. We consider operators $T^\delta$ associated with the Fourier multipliers

$$m^\delta(\xi', \xi_{n+1}) = \left(1 - \frac{|\xi'|^2}{\xi_{n+1}^2}\right)^\delta, \quad (\xi', \xi_{n+1}) \in \mathbb{R}^n \times \mathbb{R},$$

and show that $T^\delta$ is of weak type $(p, p)$ on $H^p(\mathbb{R}^{n+1})$, $0 < p < 1$, for the critical value $\delta = n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$.

1. Introduction

We consider the family of Fourier multipliers

$$m^\delta(\xi', \xi_{n+1}) = \left(1 - \frac{|\xi'|^2}{\xi_{n+1}^2}\right)^\delta, \quad (\xi', \xi_{n+1}) \in \mathbb{R}^n \times \mathbb{R},$$

with $m^\delta(\xi', \xi_{n+1}) = 0$ when $|\xi'| > \xi_{n+1}$. Let $\hat{f}$ be the Fourier transform of a Schwartz function $f$ on $\mathbb{R}^n \times \mathbb{R}$. Then define convolution operators $T^\delta$ by

$$\hat{T^\delta f}(\xi', \xi_{n+1}) = m^\delta(\xi', \xi_{n+1})\hat{f}(\xi', \xi_{n+1}).$$

It is conjectured that $T^\delta$ is bounded on $L^p$ and it remains open for any $n \geq 2$. No optimal $L^p$-bounds are known for $p > 1$. For partial results, see G. Mockenhaupt [3] and J. Bourgain [1]. See also Mockenhaupt, Seeger and Sogge [4] for the related results on the wave equation. When $\delta > (n - 1)/2$, it is not hard to show that $T^\delta$ is of weak type $(1, 1)$ by using Calderón-Zygmund theory.

The purpose of this paper is to prove a sharp endpoint result on $H^p(\mathbb{R}^{n+1})$, $p < 1$. This estimate implies the known result due to Stein, Taibleson and Weiss [7] that the Bochner-Riesz means of the critical index $\delta_p = n(1/p - 1/2) - 1/2$ is of weak type $(p, p)$ for functions in $H^p(\mathbb{R}^n)$ (see the Appendix). Here $H^p$ is the standard real Hardy space as defined in [6] by E. Stein.

We prove here

Theorem 1. Suppose $0 < p < 1$ and $\delta = n(1/p - 1/2) - 1/2$. Then $T^\delta$ maps $H^p(\mathbb{R}^{n+1})$ boundedly into weak-$L^p(\mathbb{R}^{n+1})$; i.e., there exists a constant $C = C(n, p)$...
such that for all \( f \in H^p(\mathbb{R}^{n+1}) \)
\[
|\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |T^\delta f(x,t)| > \alpha\}| \leq C \left( \frac{\|f\|_{H^p(\mathbb{R}^{n+1})}}{\alpha} \right)^p
\]
for all \( \alpha > 0 \), where \( |E| \) denotes the Lebesgue measure of the set \( E \subset \mathbb{R}^{n+1} \).

It is not known whether \( T^{n+1} \) is of weak type \((1,1)\) or just of weak type \((1,1)\) on functions in \( H^1 \).

With respect to the notation, we use \( C \) to denote a positive constant, whose value may be different for each occurrence.

2. Kernel estimates

Let \( \varphi, \psi \in C^\infty_0(\mathbb{R}) \) be supported in \((1/2,2)\) such that \( \sum_{k \geq 1} \varphi(2^k s) = 1 \) and \( \sum_{j = -\infty}^{\infty} \psi(2^{-j} t) = 1 \) for \( 0 < s < 1, t > 0 \). We now fix \( k \) and \( l \). We shall need pointwise estimates for the kernels of
\[
T^\delta_{k,l} f(x,t) = (2\pi)^{-(n+1)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_{k,l}(x-y,t-s)f(y,s) \, dy \, ds
\]
where
\[
G_{k,l}(x,t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi(2^k \left( 1 - \frac{|\xi|^2}{\xi_{n+1}^2} \right)) \left( 1 - \frac{|\xi|^2}{\xi_{n+1}^2} \right)^{\delta} \psi(2^{-j} \xi_{n+1})
\]
\[
\times e^{i(x,\xi') + it \xi_{n+1}} \, d\xi' \, d\xi_{n+1}. \tag{2.1}
\]

For each \( k \) and \( l \), the kernel \( G_{k,l} \) has the property
\[
G_{k,l}(\cdot,\cdot) = 2^{l(n+1)} \{ G_{0,0}(2^k,2^l) \}, \tag{2.2}
\]
and we write \( \sum_{k \geq 1} G_{k,l} = G_l = 2^{l(n+1)} G_{0,0}(2^k,2^l) \) and \( \sum_{k \geq 1} T^\delta_{k,l} = T^\delta_l \).

**Lemma 1.** Suppose \( 2^l |x| \leq 2 \) for fixed \( l \). Then for each \( k \) there is an estimate as follows: for every \( N \)
\[
|G_{k,l}(x,t)| \leq C \, 2^{l(n+1)} \, 2^{-k(\delta+1)} \min\{ 1, (2^{-k} 2^l |t|)^{-N} \}. \tag{2.3}
\]

Thus,
\[
|G_l(x,t)| \leq C \, 2^{l(n+1)} \frac{1}{(1 + 2^l |t|)^{\delta+1}}. \tag{2.4}
\]

**Proof.** In view of (2.1) and (2.2), it suffices to show (2.3) and (2.4) for \( l = 0 \). We integrate by parts \( N \) times with respect to \( \xi_{n+1} \) in (2.1). In view of the support property of \( \varphi \), (2.3) follows. Using (2.3) gives
\[
\left\{ C \sum_{|\xi| \leq 2^k} 2^{-k(\delta+1)} + C \sum_{|\xi| > 2^k} 2^{-k(\delta+1-N)} |\xi|^{-N} \right\},
\]
and thus (2.4) is established.

Now we estimate the kernel for the case \( 2^l |x| > 2 \). We let \( |x| = r \) and define \( G_{k,l}(x,t) = K_{k,l}(|x|,t) \). By Bochner’s formula and change of variables, we have
\[
K_{k,l}(r,t) = 2^{l(n+1)} (2^l r)^{-(n+2)/2} \int_0^1 J_{n-2} \left( \rho (2^l r) \xi_{n+1} \right) \varphi(2^k (1 - \rho^2)) \times (1 - \rho^2)^{\delta} \psi(\xi_{n+1}) \xi_{n+1}^{(n+2)/2} \rho^{n+2} e^{i(2^l t) \xi_{n+1}} \, d\rho \, d\xi_{n+1}. \tag{2.5}
\]
Here \( J_\mu \) is the Bessel function of order \( \mu > -\frac{1}{2} \) defined by

\[
J_\mu(t) = A_\mu t^\mu \int_{-1}^{1} e^{it\sigma} (1 - \sigma^2)^{\mu - \frac{1}{2}} d\sigma
\]

where \( A_\mu = [2^\mu \Gamma(2\mu + 1)\Gamma(\frac{1}{2})]^{-1} \).

In order to estimate the kernel (2.5), we need some properties of Bessel functions [5].

\[
\frac{d}{dt} (t^{-\mu} J_\mu(t)) = -t^{-\mu+1} J_{\mu+1}(t).
\]

(2.7)

For the following lemma, we use dyadic decompositions of Bessel functions (2.6) following the article by Muller and Seeger [5].

**Lemma 2.** Suppose that \( 2^l r > 2 \) for fixed \( l \). Then for each \( k \) there is an estimate as follows:

\[
|K_{k,l}(r,t)| \leq C 2^{l(n+1)} 2^{-k(l+1)} (2^l r)^{-2(n-1)/2} \min\{1, (2^{-k-1} 2^l r)^{-N_1}\}
\]

\[
\times \left\{ \frac{1}{(1 + 2^l |t + r|)^N} + \frac{1}{(1 + 2^l |t - r|)^N} \right\}.
\]

Moreover,

\[
|K_l(r,t)| \leq C 2^{l(n+1)} (1 + 2^l r)^{-(n+1+2\delta)/2} \left\{ \frac{1}{(1 + 2^l |t + r|)^N} + \frac{1}{(1 + 2^l |t - r|)^N} \right\}.
\]

(2.8)

(2.9)

**Proof.** Let \( \eta \in C_0^\infty(\mathbb{R}) \) be supported in \((-1/2, 2)\) and equal to 1 in \((-1/4, 1/4)\). Define \( m = 0, 1, 2, \ldots \) and

\[
\eta_{mk}(\sigma, \nu) = \begin{cases} 
\eta(2^{-k} \nu (1 - \sigma^2)) & \text{if } m = 0, \\
\eta(2^{-k-m} \nu (1 - \sigma^2)) - \eta(2^{-k-m+1} \nu (1 - \sigma^2)) & \text{if } m > 0
\end{cases}
\]

and set

\[
J_{\mu,k}^m(\rho \nu) = A_\mu(\rho \nu)^\mu \int_{-1}^{1} e^{i(\rho \nu)\sigma} (1 - \sigma^2)^{\mu - 1/2} \eta_{mk}(\sigma, \nu) d\sigma.
\]

Let \( M > N + N_1 + (n-1)/2 \) and set

\[
\phi_{mk}(\sigma) = \begin{cases} 
(1 - \sigma^2)^{(n-3)/2} \eta_{mk}(\sigma, \nu) & \text{if } m = 0, \\
\left( \frac{1}{i\nu} \right)^M \left( \frac{d}{d\nu} \right)^M \eta_{mk}(\sigma, \nu) (1 - \sigma^2)^{(n-3)/2} & \text{if } m > 0
\end{cases}
\]

Then

\[
J_{\mu,k}^m(\rho \nu) = A^{n/2} (\rho \nu)^{n/2} \int_{-1}^{1} e^{i(\rho \nu)\sigma} \phi_{mk}(\sigma) d\sigma
\]

by integration by parts if \( m > 0 \).

Fix \( l \) and set \( l = 0, \nu = r^{-l+1}_l \). We may decompose the kernel (2.5) as

\[
K_{k,0} = \sum_{m=0}^{\infty} K_{k,0}^m
\]
where
\[
K_{k,0}^m(r,t) = r^{-(n-2)/2} \int_{\mathbb{R}^n} \int_{0}^{1} J_{m/2}^{m/2} (\rho \nu) \varphi(2^k(1 - \rho^2)) \\
\times (1 - \rho^2)^{\delta} \psi(\xi_{n+1}) \xi_{n+1}^{(n+2)/2} \rho^{-n/2} e^{i \xi_{n+1} \cdot d \rho} d \rho d \xi_{n+1}.
\]

Formula (2.10) and straightforward computation imply that
\[
K_{k,0}^m(r,t) = A_{m/2} \int_{-1}^{1} \phi_{mkv}(\sigma) \int_{\mathbb{R}^n} \varphi(2^k(1 - \rho^2)) \\
\times (1 - \rho^2)^{\delta} \psi(\xi_{n+1}) \xi_{n+1}^{(n+2)/2} \rho^{-n/2} e^{i (t + \rho \sigma) \xi_{n+1} \cdot d \rho} d \rho d \xi_{n+1} d \sigma.
\]

We integrate by parts with respect to \( \rho \) and \( \xi_{n+1} \) in (2.11) and by Fubini’s theorem
\[
|K_{k,0}^m(r,t)| \leq C 2^{-k(n-1)/2} \int_{\mathbb{R}^n} \int_{0}^{1} \phi_{mkv}(\sigma) (1 + |\sigma \nu|)^{-N} \\
\times (1 + |t + \rho \sigma|)^{-N} \left| \frac{\partial}{\partial \rho} \right|^{N_1} \varphi(2^k(1 - \rho^2))(1 - \rho^2)^{\delta} \rho^{(n-1)/2} \\
\times \left| \frac{\partial}{\partial \xi_{n+1}} \right|^N \psi(\xi_{n+1}) \xi_{n+1}^{n} \bigg| d \rho d \sigma d \xi_{n+1}.
\]

Next note the size estimate
\[
|\phi_{mkv}(\sigma)| \leq C 2^{-mM} (2^{m+k} \nu^{-1})(n-3)/2.
\]

Moreover, \( \phi_{mkv} \) vanishes unless either \( 1 - \sigma^2 \approx 2^{m+k} \nu^{-1} \) for \( m > 0 \), or \( 1 - \sigma^2 \approx 2^k \nu^{-1} \) for \( m = 0 \). Hence if \( \sigma \) is in the support of \( \phi_{mkv} \), then either \( |\nu - \nu \sigma| \leq 2^{m+k} \) or \( |\nu + \nu \sigma| \leq 2^{m+k} \). Then using the estimates (2.13), the integrand of (2.12) is bounded by
\[
C 2^{-k} |\phi_{mkv}(\sigma)| \frac{1}{(1 + 2^{-k} |\sigma \nu|)^{N_1}} \frac{1}{(1 + |t + \rho \sigma|)^N} \\
\leq C 2^{k((n-3)/2 - \delta)} 2^{m((n-3)/2 + N_1 - M)} \nu^{-(n-3)/2} \xi_{n+1}^{-N} \\
\times \left\{ \frac{1}{(1 + 2^{-k} |\nu|)^{N_1}} \frac{1}{(1 + |t + \rho \sigma|)^N} + \frac{1}{(1 + |t - \rho \sigma|)^N} \right\}.
\]

If we integrate over the support of \( \varphi(2^k(1 - \rho^2)) \otimes \phi_{mkv} \otimes \psi \) for \( m \geq 0 \) in (2.12), we gain an additional factor of \( C 2^{m} r^{-1} \). Since \( M > N + N_1 + (n-1)/2 \), we may sum over \( m \) and the desired estimates (2.8) follow from (2.12). Hence we obtain
\[
(1 + |t - r|)^{-N} \left\{ C \sum_{r \leq r_{2^k}} 2^{-k(\delta + 1)} r^{-(n-1)/2} \\
+ C \sum_{r > r_{2^k}} 2^{-k(\delta + 1 - N_1)} r^{-(n-1)/2 - N_1} \right\},
\]

and thus (2.9) is established for \( l = 0 \). For the case \( l \neq 0 \), we use (2.2). □

In Section 4 we will need estimates for the derivatives of the kernels. When \( |x| \leq 2 \), straightforward computations in (2.14) give us Lemma 3(a). If \( |x| = r > 2 \), we use (2.7) to show Lemma 3(b).
Lemma 3. (a) Suppose that |x| ≤ 2. Then for every N,
\[ |G_{k,0}(x, t)| \leq C 2^{-k(\delta + 1)} \min\{1, (2^{-k}|t|)^{-N}\}. \]
Moreover,
\[ |G_0^{(\gamma)}(x, t)| \leq C \sum_{|t| \leq 2^k} 2^{-k(\delta + 1)} + C \sum_{|t| > 2^k} 2^{-k(\delta + 1 - N)|t|^{-N}} \]
\[ \leq C \frac{1}{(1 + |t|)^{\delta + 1}} \text{ for } \gamma \in \mathbb{N}^{n+1} \text{ and } |\gamma| = 1, 2, 3, \ldots. \]
(b) Suppose that r > 2. Then
\[ |K_{k,0}^{(\gamma)}(r, t)| \leq C 2^{-k(\delta + 1)} r^{-(n-1)/2} \min\{1, (2^{-k}r)^{-N_1}\} (1 + |t - r|)^{-N}. \]
Moreover,
\[ |K_0^{(\gamma)}(r, t)| \leq (1 + |t - r|)^{-N} \left\{ C \sum_{r \leq 2^k} 2^{-k(\delta + 1)} r^{-(n-1)/2} \right. \]
\[ + C \sum_{r > 2^k} 2^{-k(\delta + 1 - N_1)} r^{-(n-1)/2 - N_1} \right\} \]
\[ \leq C (1 + r)^{-(n+1+2\delta)/2} \left\{ \frac{1}{(1 + |t + r|)^N} + \frac{1}{(1 + |t - r|)^N} \right\} \]
for \( \gamma \in \mathbb{N}^2 \) and \( |\gamma| = 1, 2, 3, \ldots. \)

3. THE ATOMIC DECOMPOSITION OF \( H^p \) AND PRELIMINARY LEMMATA

Definition 1. Let \( 0 < p \leq 1 \) and \( d \) be an integer that satisfies \( d \geq (n+1)(1/p - 1) \). Let \( Q \) be a cube in \( \mathbb{R}^{n+1} \). We say that \( a \) is a \( (p, d) \)-atom associated with \( Q \) if \( a \) is supported on \( Q \subset \mathbb{R}^{n+1} \) and satisfies
\[ |a(x)| \leq |Q|^{-1/p} \quad \text{almost everywhere} \]
\[ \int_{\mathbb{R}^{n+1}} a(x) x^\beta \, dx = 0 \]
where \( \beta = (\beta_1, \beta_2, \ldots, \beta_{n+1}) \) is an \( (n+1) \)-tuple of non-negative integers satisfying \( |\beta| = \beta_1 + \beta_2 + \cdots + \beta_{n+1} \leq d \), and \( x^\beta = x^{\beta_1} x^{\beta_2} \cdots x^{\beta_{n+1}} \).

If \( \{a_i\} \) is a collection of \( (p, d) \)-atoms and \( \{\lambda_i\} \) is a sequence of complex numbers with \( \sum_i |\lambda_i|^p < \infty \), then the series
\[ f = \sum_{i=1}^\infty \lambda_i a_i \]
converges in the sense of distributions, and its sum belongs to \( H^p \) with \( ||f||_{H^p} \leq C (\sum_i |\lambda_i|^p)^{1/p} \) and the converse inequality also holds (see \([9]\)).

The following lemma is due to Stein, Taibleson and Weiss \([7]\). It is a version of the triangle inequality on \( L^{p,\infty} \), \( 0 < p < 1 \).

Lemma 4. Suppose \( 0 < p < 1 \) and \( \{f_i\} \) is a sequence of measurable functions such that
\[ |\{x : |f_i(x)| > \alpha > 0\}| \leq \alpha^{-p} \]
for \( i = 1, 2, 3, \cdots \). If \( \sum_{i=1}^{\infty} |\lambda_i|^p \leq 1 \), then

\[
(3.3) \quad \left| \left\{ x : \sum_{i=1}^{\infty} \lambda_i f_i(x) > \alpha \right\} \right| \leq \frac{2 - p}{2 - p} \alpha^{-p}.
\]

**Proof.** See (12).

**Lemma 5.** For given \( \beta > 0 \), assume that \( 0 < p < 1 \). Suppose that \( \{g_l\} \) is a sequence of measurable functions such that

\[
(3.4) \quad |\{ x : |g_l(x)| > \alpha \}| \leq A^p 2^{-\beta p} \alpha^{-p}
\]

for \( l \geq 0 \) and all \( \alpha > 0 \). Then

\[
\left| \left\{ x : \sum_{l \geq 0} |g_l(x)| > \alpha \right\} \right| \leq C \alpha^{-p}.
\]

**Proof.** From (3.4), we have

\[
\left| \left\{ x : |g_l(x)| \cdot \left( \frac{\beta}{A} \right) > \alpha \right\} \right| \leq A^p 2^{-\beta p} (A 2^{-\beta p})^{-p} = \alpha^{-p}.
\]

Moreover, \( \sum_{l \geq 0} A^p 2^{-\beta p} = A^p \beta \), where \( A^p = \frac{1}{1 - 2^{-\beta p}} \).

By Lemma 4 we then obtain

\[
\left| \left\{ x : \sum_{l \geq 0} g_l(x) > \alpha \right\} \right| \leq \left| \left\{ x : \sum_{l \geq 0} A 2^{-\beta p} \frac{|g_l(x)|}{A 2^{-\beta p}} > \frac{\alpha}{A 2^{-\beta p}} \right\} \right|
\]

\[
\leq \left( \frac{2 - p}{1 - p} \right) \left( \frac{\alpha}{A 2^{-\beta p}} \right)^{-p} = C \alpha^{-p}.
\]

The following technical estimates will be used in Section 4.

**Lemma 6.** Suppose \( 0 < p < 1 \) and \( \delta = n(1/p - 1/2) - 1/2 \). Suppose

\[
\int \int \{ x | > 2^{1-l}, \left| t \right| > 2^{1-l} : 2^h x |^{-\delta} X_{\left[ \left| x \right| - 2^{1-l} \right]} > \alpha/16C \} dx dt
\]

\[
+ \int \int \{ x | > 2^{1-l}, \left| t \right| > 2^{1-l} : 2^h x |^{-\delta} X_{\left[ \left| t \right| - 2^{1-l} \right]} > \alpha/16C \} dx dt
\]

\[
+ \int \int \{ x | > 2^{1-l}, \left| t \right| \leq 2^{1-l} : 2^h x |^{-\delta} X_{\left[ \left| x \right| - 2^{1-l} \right]} > \alpha/16C \} dx dt
\]

\[
+ \int \int \{ x | \leq 2^{1-l}, \left| t \right| > 2^{1-l} : 2^h x |^{-\delta} X_{\left[ \left| x \right| - 2^{1-l} \right]} > \alpha/16C \} dx dt
\]

\[
\leq C 2^{\delta} \alpha^{-p}.
\]

Then

(i) if \( a = n + 1 - n/p, b = n + 1 - n/p - N \) and \( c = n - \delta \), then \( d = (n + 1)(p - 1) \),

(ii) if \( a = n + N + 2 - n/p, b = n + 2 - n/p \) and \( c = n + N + 1 - \delta \), then \( d = (n + N + 2)p - (n + 1) \).

**Proof.** Applying Fubini’s theorem to the first integral and Chebyshev’s inequality to the third and last integrals yields the desired estimates.
4. Weak type estimates

We will show that $T_\delta f$ satisfies the uniform weak type estimates \((5.2)\) when $f$ is a \((p, N)\)-atom \((N \geq (n + 1)(1/p - 1))\), and prove Theorem \((\text{III})\).

**Proposition 1.** Suppose $f$ is a \((p, N)\)-atom \((N \geq (n + 1)(1/p - 1))\) on $\mathbb{R}^{n+1}$ and $\delta = n(1/p - 1/2) - 1/2$. Then there exists a constant $C = C(n, p)$ such that

$$
\text{(4.1)} \quad \left| \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : |T_\delta f(x, t)| > \alpha \} \right| \leq C \alpha^{-p}
$$

for all $\alpha > 0$.

**Proof.** Since $T_\delta$ is translation invariant, we can assume that $f$ is supported in a cube $Q$ of diameter $2R$ centered at the origin. We observe

$$
\left| \{(x, t) : |T_\delta f(x, t)| > \alpha \} \right| 
\leq \left| \{(x, t) \in Q^* : |T_\delta f(x, t)| > \alpha/2 \} \right| 
+ \left| \{(x, t) \in (Q^*)^c : |T_\delta f(x, t)| > \alpha/2 \} \right|
$$

where $Q^*$ is the cube concentric with $Q$ and with sides of twice the length, and we will show that each term is bounded by $C \alpha^{-p}$.

Consider $(x, t) \in Q^*$. Suppose $0 < p < 2$ and $p/2 + 1/q = 1$. By Hölder’s inequality and the Plancherel theorem, we have

$$
\int\int_{Q^*} |T_\delta f(x, t)|^p \, dx \, dt \leq \left( \int\int_{\mathbb{R}^{n+1}} |T_\delta f|^2 \, dx \, dt \right)^{p/2} |Q^*|^{1/q} \leq C.
$$

Hence for all $\alpha > 0$,

$$
\text{(4.2)} \quad \left| \{(x, t) \in Q^* : |T_\delta f(x, t)| > \alpha/2 \} \right| \leq C \alpha^{-p}.
$$

Next, to establish the following estimate

$$
\text{(4.3)} \quad \left| \{(x, t) \in (Q^*)^c : |T_\delta f(x, t)| > \alpha/2 \} \right| \leq C \alpha^{-p}
$$

for all $\alpha > 0$, we first assume that $f$ is supported in the cube $I$ of diameter 1 centered at the origin. We now consider the case $(x, t) \in (I^*)^c$ and claim that

$$
\text{(4.4)} \quad |T_\delta f(x, t)| \leq C \left\{ 2^{la} \frac{|x|}{n-p} \chi_{\{|x|>2^{-l-1}, |t|>2^{-l+1}, \|t\| \leq 2^{l-1}\}} + 2^{lb} \frac{|x|}{n-p} \chi_{\{|x|>2^{-l-1}, |t|>2^{l-1}\}} + 2^{lc} \frac{|t|}{n-p} \chi_{\{|x| \leq 2^{l-1}, |t|>2^{l-1}\}} \right\}
$$

where

(i) $a = n + 1 - n/p$, $b = n + 1 - n/p - N$ and $c = n - \delta$ for $l \geq 0$, 
(ii) $a = n + N + 2 - n/p$, $b = n + 2 - n/p$ and $c = n + N + 1 - \delta$ for $l < 0$.

Fix $l \geq 0$. Since $f$ is supported in the cube $I$ of diameter 1, by \((2.3)\) we have

$$
|T_\delta f(x, t)| \leq 2^{l(n+1)} \int_{I^*} |f(y, s)| |K_0(2^l|x-y|, 2^l(t-s))| dy \, ds.
$$
Consider the case $2^l |x| > 2$, $2^l |t| > 2$ and $2^l |t| - |x| \leq 1$. Then by Lemma 2 we have

$$|T^l f(x, t)| \leq C 2^{l(n+1-n/p)} |x|^{-n/p} \chi(|t| - |x| \leq 2^{l-1}).$$

If $2^l |x| > 2$ and $2^l |t| - |x| > 1$, then by Lemma 2 we have

$$|T^l f(x, t)| \leq C 2^{l(n+1-n/p-N)} \left\{ |x|^{-n/p-N} \chi_{|t| \leq 2^{l-1}} + |x|^{-n/p} |t|^{-N} \chi_{|t| > 2^{l-1}} \right\}.$$

Finally, when $2^l |x| \leq 2$, $2^l |t| > 2$, we use Lemma 1 and thus

$$|T^l f(x, t)| \leq C 2^{l(n-\delta)} |t|^{-(\delta+1)} \chi_{|x| \leq 2^{l-1}, |t| > 2^{l-1}}.$$

Combining the estimates (4.5)–(4.7), we obtain (4.3) for $l \geq 0$, and by Lemma 6

$$\int_{\{|x| > 2^{l-1}, |t| > 2^{l-1} : |T^l f(x, t)| > \alpha/4\}} dx dt \leq C 2^{l(n+1)(p-1)} \alpha^{-p}.$$

Now applying Lemma 2 with $\beta p = (n+1)(1-p)$, we obtain

$$\left\{ (x, t) \in (I^*)^c : \sum_{l \geq 0} |T^l f(x, t)| > \alpha/4 \right\} \leq C \alpha^{-p}.$$

We now fix $l < 0$. Let $P_{k,l,x}(y, s)$ denote the $N$-th order Taylor polynomial of the function $(y, s) \rightarrow K_{k,l}((x-y), (l-s))$ expanded about the origin, the center of the cube. Now $P_{k,l,x} = 2^{l(n+1)} P_{k,0,x}(2^l \cdot, 2^l \cdot)$ for fixed $k$ and $l$. Then using the moment conditions on $f$,

$$T^l_{k,l} f(x, t) = \int_{I^*} f(y, s) 2^{l(n+1)} K_{k,0}(2^l |x-y|, 2^l (t-s)) dy ds$$

$$\quad \quad \quad \quad \quad \quad \quad = \int_{I^*} f(y, s) 2^{l(n+1)} [K_{k,0}(2^l |x-y|, 2^l (t-s)) - P_{k,0,x}(2^l |y|, 2^l s)] dy ds.$$

A straightforward calculation shows that the absolute value of the last term is dominated by

$$C \int_{I^*} |f(y, s)| 2^{l(n+1)} \sum_{|\gamma| = N+1} |K^{(\gamma)}_{k,0}(2^l |x|, 2^l t)| 2^l (y, s)|^{N+1} dy ds.$$

We apply Lemma 3 and the same arguments used for (4.5)–(4.7) and thus obtain the bounds (1.4) for $l < 0$. Moreover, by Lemma 6 we see that

$$\int_{\{|x| > 2^{l-1}, |t| > 2^{l-1} : |T^l f(x, t)| > \alpha/4\}} dx dt \leq C 2^{l(n+N+2p-(n+1))} \alpha^{-p}.$$

Since $(n+1) < (n+N+2)p$, by Lemma 5 with $\beta p = (n+N+2)p - (n+1)$, we obtain

$$\left\{ (x, t) \in (I^*)^c : \sum_{l \leq 0} |T^l f(x, t)| > \alpha/4 \right\} \leq C \alpha^{-p}.$$

Putting together the estimates (4.8) and (4.9), we obtain (4.4) for the cube $I$.

Suppose now that $f$ is a $(p, N)$-atom $(N \geq (n+1)(1/p-1))$, supported in a cube $Q$ of diameter $2^R$ centered at $(x_Q, t_Q)$. By translation invariance we can assume
(x_Q, t_Q) = (0, 0). Let h(x, t) = 2^{R(n+1)/p} f(2^R x, 2^R t). Then h is an atom supported in the cube I centered (0, 0). But this implies
\[ T_I^g f(x, t) = \int_{I} \int_{I} 2^{-R(n+1)/p} h \left( \frac{(x-y)}{2^R}, \frac{(t-s)}{2^R} \right) G_I(y, s) \, dy \, ds \]
\[ = 2^{-R(n+1)/p} 2^{R(n+1)} (G_I(2^R, 2^R) * h) \left( \frac{x}{2^R}, \frac{t}{2^R} \right) \]
\[ = 2^{-R(n+1)/p} (G_I * h) \left( \frac{x}{2^R}, \frac{t}{2^R} \right). \]

If we repeat the same arguments used for (4.8) and (4.9), we get (4.10). This proves Proposition 1.

We now proceed with the proof of Theorem 1.

Proof. If \( f = \sum_{i=1}^{\infty} \lambda_i f_i \in H^p(R^n) \), \( T_I^g f \) is well defined since each \( (T_I^g f_i)(x, t) \) is the convolution of the atom \( f_i \) with an integrable function \( G_I \). Using the weak type \((p, p)\) estimates of \( T_I^g \) for each \( I \) and by Lemma 5 we obtain that \( T_I^g f \) satisfies a uniform weak type estimate when \( f_i \) is a \((p, N)\)-atom \((N \geq (n+1)(1/p - 1))\) in Proposition 1. Since \( |T_I^g f(x, t)| \leq \sum_{i=1}^{\infty} |\lambda_i||T_I^g f_i(x, t)| \) and \( \sum_{i=1}^{\infty} |\lambda_i|^p < \infty \), Theorem 1 is a consequence of Lemma 4.

**Appendix**

Let \( 0 < p < 1 \). Denote the quasi-norm (sup_{\alpha>0} \alpha^p \{ x \in R^n : |g(x)| > \alpha \} )^{1/p} of \( g \) in \( L^{p,\infty} \) by \( ||g||_{L^{p,\infty}} \). Let \( T_n f = m \ast f \). We define the class of Fourier multipliers \( M(H^p, L^{p,\infty})(R^n) \) to be the set of all bounded measurable functions \( m \) so that for all \( f \in C_c^\infty(R^n) \cap H^p(R^n) \),
\[ ||T_m f||_{L^{p,\infty}} \leq C ||f||_{H^p}. \]

The best constant \( C \) is the quasi-norm of the operator \( T_m \), and we write \( ||m||_M \) for this quantity.

**Lemma 7.** Suppose \( f \) and \( f_e \) are measurable functions on \( R^n \) and \( f_e \to f \) almost everywhere. Assume that \( ||f_e||_{L^{p,\infty}} \leq M^{1/p} \) for some \( M > 0 \) and for all \( \epsilon > 0 \). Let \( \alpha > 0 \) be fixed. Then \( \alpha^{p'} \{ x \in R^n : |f(x)| > \alpha \} \leq M \).

Proof. Suppose \( \alpha^{p'} \{ x : |f(x)| > \alpha \} > M \). Then from the right continuity of \( \{ x : |f(x)| > \alpha \} \), there exists \( \beta > \alpha \) such that \( \alpha^{p'} \{ x : |f(x)| > \beta \} > M \). Define \( E = \{ x : |f(x)| > \beta \} \). We know \( |E| < \infty \). Then by Egoroff’s theorem, for every \( \eta > 0 \), there exists \( F \subset E \) such that \( |E \setminus F| < \eta \) and \( f_e \to f \) uniformly on \( F \). We choose \( \epsilon \) small so that \( |f_e(x) - f(x)| < (\beta - \alpha)/2 \) for \( x \in F \). Moreover, on \( F \) the inequality \( |f(x)| > \beta \) implies that \( |f_e(x)| \geq |f(x)| - |f_e(x) - f(x)| > \beta - (\beta - \alpha)/2 > \alpha \).

From this, \( \{ x : |f(x)| > \beta \} \cap F \subset \{ x : |f_e(x)| > \alpha \} \) and \( \alpha^{p'} \{ x : |f(x)| > \beta \} \cap F \leq \alpha^{p'} \{ x : |f_e(x)| > \alpha \} \). Since \( \{ x : |f(x)| > \beta \} = (\{ x : |f(x)| > \beta \} \cap F) \cup (\{ x : |f(x)| > \beta \} \cap (E \setminus F)) \),
\[ \alpha^{p'} \{ x : |f(x)| > \beta \} \leq \alpha^{p'} \{ x : |f(x)| > \beta \} \cap F \]
\[ \leq \alpha^{p'} \{ x : |f_e(x)| > \alpha \} \leq M. \]

When \( \eta \) is sufficiently small, this is a contradiction.
The following is based on de Leeuw’s restriction theorem \[2\].

**Theorem 2.** Let \( m(\xi', \xi'') \) be contained in the class \( \mathcal{M}(H^p, L^{p, \infty})(\mathbb{R}^{k+1}) \) and be continuous. Then \( m_{\mathcal{C}}(\xi') = m(\xi', \xi'') \) is contained in the class \( \mathcal{M}(H^p, L^{p, \infty})(\mathbb{R}^k) \) and the multiplier norm of \( m_{\mathcal{C}} \) does not exceed that of \( m \).

**Proof.** Let \( f_1 \in C_0^\infty(\mathbb{R}^k) \cap H^p(\mathbb{R}^k) \), \( f_{2, \epsilon} \in C_0^\infty(\mathbb{R}^k) \) with \( \widehat{f_{2, \epsilon}}(\xi'') = \epsilon^{(1/p-1)} \phi(\xi'' - a) \), where \( \phi \) is supported in \( B(0, 1) \) (the unit ball about the origin). Define \( f_\epsilon(x', x'') = (f_1 \otimes f_{2, \epsilon})(x', x'') \). From this \( \|f_\epsilon\|_{H^p} \leq A_\phi \|f_1\|_{H^p} \). Since we have

\[
T_m(f_1 \otimes f_{2, \epsilon})(x', x'') = \frac{1}{(2\pi)^{k+1}} \int_{\xi'} \int_{\xi''} m(\xi', \xi'') \widehat{f_1}(\xi') \epsilon^{(1/p-1)} \phi(\xi'' - \epsilon) e^{i(x', \xi') + i(x'', \xi'')} \, d\xi' \, d\xi''
\]

where \( m'(\xi', \xi'') = m(\xi', \xi'' + a) \),

\[
||T_{m'}(f_1 \otimes \hat{\phi})||_{L^{p, \infty}} \leq A_\phi \|m\|_\mathcal{M} \|f_1\|_{H^p}.
\]

Then for all \( \alpha > 0 \), we have

\[
\alpha^p \{ (x', x'') : ||T_{m'}(f_1 \otimes \hat{\phi})(x', x'') || > \alpha \} \leq A_\phi^p \|m\|_\mathcal{M} \|f_1\|_{H^p}.
\]

and by the Lebesgue Dominated Convergence Theorem, \( T_{m'}(f_1 \otimes \hat{\phi}) \) converges to \( (T_{m_a} f_1) \otimes \phi \) as \( \epsilon \to 0 \) where \( m_a(\xi') = m(\xi', a) \). So from Lemma 7 we have

\[
||(T_{m_a} f_1) \otimes \hat{\phi}||_{L^{p, \infty}} \leq A_\phi \|m\|_\mathcal{M} \|f_1||_{H^p}.
\]

Hence \( m_a \) is contained in the class \( \mathcal{M}(H^p, L^{p, \infty})(\mathbb{R}^k) \) and thus

\[
||m_a||_{\mathcal{M}} \leq ||m||_{\mathcal{M}}.
\]

\[\square\]

**Remark 1.** Even if we replace the continuity assumption of \( m \) by almost everywhere conditions, Theorem 2 still holds.

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