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LOOP SPACES AND THE COMPRESSION THEOREM

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Abstract. For a smooth, finite-dimensional manifold M with a submanifold S we study the topology of the straight loop space $\Omega_S^{st}M$, the space of loops whose intersections with S are subject to a certain transversality condition. Our main tool is Rourke and Sanderson's compression theorem. We prove that the homotopy type of the straight loop space of a link in S^3 depends only on the number of link components.

1. Introduction and statement of results

Let M be a smooth finite-dimensional manifold, let $S \subseteq M$ be a codimension c submanifold, where $c \geq 2$, and let K be a tubular neighbourhood of S. There is a natural projection $K \to S$; each of its fibres is a c-dimensional disk normal to S. We fix a diffeomorphism between this disk bundle and an abstract disk bundle in which every fibre is equipped with the metric of a closed disk in $\mathbb{R}^c$ with diameter 1. We say a loop in M is *straight* if it has a finite number of intersections with K, and if each intersection is a straight line with constant speed in a single fibre (Figure 1(a) shows examples, where $c = 2$). The set of all based straight loops with the compact-open topology is the *straight loop space* $\Omega_K^{st}M$. If no confusion is possible, we leave out the subscript K.

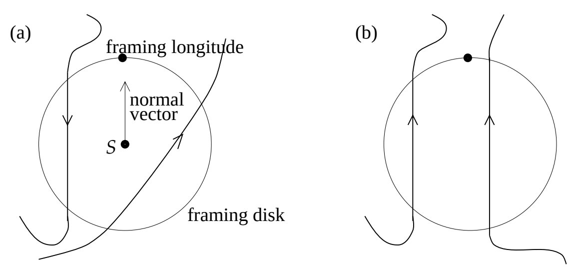

Figure 1. (a) straight loops (b) vertical loops

Straight loop spaces arise naturally as augmented quandle spaces of nontrivial irreducible links in 3-manifolds [1], [7], [6]. In these spaces, topology and geometry interact in an intricate manner. We shall see that the geometric condition that the

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loops be straight changes the topological properties of the loop space in a simple but radical way. Our aim is to prove the following theorems:

Theorem 1.1. *Let* $K_1 = K_1^1 \cup ... \cup K_1^s \subseteq M_1$ *and* $K_2 = K_2^1 \cup ... \cup K_2^s \subseteq M_2$ *be two links with the same number of link components in two 3-manifolds. Suppose there exists a homotopy equivalence* $f\colon M_1 \to M_2$ *which maps the core of* K_1^i *to a closed curve which is homotopic in* M_2 *to the core of* K_2^i*, for* $i = 1, ..., s$*. Then* $\Omega^{st}_{K_1} M_1 \simeq \Omega^{st}_{K_2} M_2$.

Theorem 1.2. *The straight loop spaces of two links in* S^3 *are homotopy equivalent if and only if the links have the same number of link components. The straight loop space of a link in* S^3 *is never homotopy equivalent to* ΩS^3.

The proofs depend crucially on Rourke and Sanderson's compression theorem [5]: Let $L \subseteq Q \times \mathbb{R}$ be a submanifold, where $dim(Q) - dim(L) \geq 1$. We think of Q as being 'horizontal' and $\mathbb{R}$ as 'vertical'. Suppose that L is equipped with a normal vector field. A subset $L_1 \subseteq L$ is called *compressible* if on every point of L_1 the normal vector points vertically upwards in the $\mathbb{R}$-direction.

Compression Theorem ([5])**.** *There exists a small isotopy of* L *and the normal vector field which makes* L *compressible. Moreover, if* $L_1 \subseteq L$ *is already compressible, then the isotopy can be assumed fixed on* L_1.

2. Example: The unknotted S^n in S^{n+2}

Let U be a tubular neighbourhood of the unknotted n-sphere in the $(n+2)$-sphere. The following result is proved in [6]. Let T be the Cartesian product of S^n with the unit interval, with both boundary components squashed to a single point; so $T = (S^n \times [0,1]) \,/\, (S^n \times \{0,1\})$. Then $\Omega^{st}_U S^{n+2}$ is weakly homotopy equivalent to $(T_\infty)^\sim$, where T_∞ denotes the free topological monoid of T (see [4]), and $^\sim$ the universal covering space. Moreover, since T is homotopy equivalent to $S^1 \vee S^{n+1}$, it follows by James's theorem ([4] or [5]) that $\Omega^{st}_U S^{n+2}$ is weakly homotopy equivalent to $(\Omega(S^2 \vee S^{n+2}))^\sim$.

I thank the referee for pointing out that this space is in turn homotopy equivalent to $\Omega S^3 \times \Omega(\Omega S^2 * \Omega S^{n+2})$, where $*$ denotes the join. The proof uses a result of T. Ganea [2] which states that, for CW complexes X and Y, the homotopy theoretic fibre of the inclusion $X \vee Y \hookrightarrow X \times Y$ has the homotopy type of $\Omega X * \Omega Y$.

3. Proof of the main theorems

We need to introduce one more concept, which will enable us to apply the compression theorem to our problem. With the notation of the introduction, suppose that the normal bundle of S, the core of K, has a section. This means that S is equipped with a smooth normal vector field; equivalently, we have a copy of the core of K embedded in the boundary of K which intersects every normal disk in precisely one point. It will be useful to think of this point as being 'at the top of the normal disk'. Then we define a loop $\omega \in \Omega^{st}_K$ to be *vertical* if all its intersections with K are paths that are running vertically up, as indicated in Figure 1(b). The *vertical loop space* $\Omega^{vert}_K M \subseteq \Omega^{st}_K M$ is the space of all vertical loops.

Note that a submanifold may have several different normal vector fields. For instance, for every knot in a 3-manifold there is a whole $\mathbb{Z}$-family of possibilities. Also note that if S and M are oriented and $c = 2$, then a normal vector field is the same as a framing, i.e. a trivialisation of the normal bundle. Vertical loop spaces are much easier to understand than straight loop spaces:

Theorem 3.1. *The inclusion $\Omega_K^{vert} M \subset \Omega M$ is a weak homotopy equivalence.*

Proof. Let $p\colon \Sigma S^n \to M$ represent an element of $\pi_n(\Omega M)$. We make p transverse to K; then $p^{-1}(K)$ is a codimension c submanifold of ΣS^n with a normal vector field. The map p determines a labelling of every point of ΣS^n by a point of M. By the compression theorem there exists an isotopy of ΣS^n which is fixed on the basepoint and which carries $p^{-1}(K)$ together with its normal vector field, such that in the end of the isotopy the normal vector field points vertically upwards in the suspension direction. If we let the isotopy move points in ΣS^n together with their label, then we obtain a homotopy of the map p which makes all the loops in the image of the adjoint of p vertical. This proves that $i_*\colon \pi_n(\Omega_K^{vert} M) \to \pi_n(\Omega M)$ is surjective.

Injectivity is proved similarly, by applying the relative compression theorem to a framed submanifold of $\Sigma S^n \times [0,1]$. $\square$

Proposition 3.2. *Let $K_1 \subseteq M_1$ and $K_2 \subseteq M_2$ be tubular neighbourhoods of codimension c submanifolds ($c \geq 2$) with the same number $s \in \mathbb{N}$ of path components. Suppose that the normal bundle of $K_1 \subseteq M_1$ has a section. Suppose further that there exist maps $f\colon M_1 \to M_2$ sending K_1 diffeomorphically to K_2, and $g\colon M_2 \to M_1$ such that $g|_{K_2} = (f|_{K_1})^{-1}$. Suppose also that $g \circ f\colon M_1 \to M_1$ is homotopic to the identity by a homotopy that is fixed on K_1, and that, similarly, $f \circ g\colon M_2 \to M_2$ is homotopic to the identity relatively to K_2. Then $\Omega_{K_1}^{st} M_1$ and $\Omega_{K_2}^{st} M_2$ are weakly homotopy equivalent.*

Proof. We equip K_1 with any normal vector field (it does not matter which one is chosen). This gives rise, via the diffeomorphism $f|_{K_1}$, to a normal vector field of K_2. After a homotopy which is fixed on K_1 we have that f is transverse to K_2, and similarly we can assume that g is transverse to K_1. Now $f^{-1}(K_2)$ is a codimension c submanifold in M_1 which is equipped with a normal vector field, and which consists of K_1 and possibly some additional components $f^{-1}(K_2)\backslash K_1$. Also, $(g \circ f)^{-1}(K_1)$ consists of $f^{-1}(K_2)$ and possibly some additional components $(g \circ f)^{-1}(K_1)\backslash f^{-1}(K_2)$. So, we have $K_1 \subseteq f^{-1}(K_2) \subseteq (g \circ f)^{-1}(K_1)$. Correspondingly, we define subspaces of the straight loop space

$$\Omega^{st} M_1'' \;\subseteq\; \Omega^{st} M_1' \;\subseteq\; \Omega^{st} M_1$$

by

$$\Omega^{st} M_1' := \Omega^{st}_{f^{-1}(K_2)} M_1 \cap \Omega^{vert}_{f^{-1}(K_2)\backslash K_1} M_1,$$

$$\Omega^{st} M_1'' := \Omega^{st}_{(g\circ f)^{-1}(K_1)} M_1 \cap \Omega^{vert}_{(g\circ f)^{-1}(K_1)\backslash K_1} M_1.$$

Roughly speaking, $\Omega^{st} M_1'$ and $\Omega^{st} M_1''$ are the vertical loop spaces of $f^{-1}(K_2)$ and $(g \circ f)^{-1}(K_1)$ respectively, except that the condition on the loops is slightly relaxed: we allow intersections of loops with K_1 to be merely straight, not vertical. We observe that f and g induce maps as indicated by arrows in the following diagram; by abuse of notation we shall denote these induced maps also by f and g.

Lemma 3.3. *The inclusion $\Omega^{st} M_1' \subseteq \Omega^{st} M_1$ is a weak homotopy equivalence, i.e. it induces isomorphisms in all homotopy groups.*

$$
\begin{tikzcd}
\Omega^{st}M_1'' \arrow[ddrr] & & \Omega^{st}M_2'' \arrow[ddll] \\
\bigcap & & \bigcap \\
\Omega^{st}M_1' \arrow[ddrr] & & \Omega^{st}M_2' \arrow[ddll] \\
\bigcap & & \bigcap \\
\Omega^{st}M_1 & & \Omega^{st}M_2
\end{tikzcd}
$$

Proof of the lemma. The proof is similar to the proof of Theorem 3.1. Let $p\colon \Sigma S^n \to M_1$ represent an element $[p^{ad}]$ of $\pi_n(\Omega^{st}M_1)$ (where p^{ad} is the adjoint of p). Make p transverse to $f^{-1}(K_2)$. By the compression theorem there exists an isotopy of ΣS^n which makes $p^{-1}(f^{-1}(K_2))$ compressible. Moreover, since the isotopy can be chosen small, we can assume that it is fixed on K_1. This isotopy determines a homotopy $p_t^{ad}\colon S^n \to \Omega^{st}M_1'$ with $p_0^{ad} = p^{ad}$ and $\mathrm{im}(p_1^{ad}) \subseteq \Omega^{st}M_1'$. Injectivity is proved similarly, by applying the relative small compression theorem to a submanifold of $\Sigma(S^n \times [0,1])$. This completes the proof of the lemma. $\square$

Similarly one proves that all inclusions in the above diagram are weak homotopy equivalences.

It is now sufficient to prove that $f\colon \Omega^{st}M_1'' \to \Omega^{st}M_2'$ is a weak homotopy equivalence or, equivalently, that the following diagram commutes for all $n \in \mathbb{N}$:

$$
\begin{tikzcd}
\pi_n(\Omega^{st}M_1'') \arrow[d, "\cong"'] \arrow[dr, "f_*"] & \\
 & \pi_n(\Omega^{st}M_2') \arrow[dl, "g_*"] \\
\pi_n(\Omega^{st}M_1) &
\end{tikzcd}
$$

where the vertical arrow is induced by inclusion. Let $p\colon \Sigma S^n \to M_1$ represent an element $[p^{ad}]$ of $\pi_n(\Omega^{st}M_1'')$. We want to check that $incl_*([p^{ad}]) = g_* \circ f_*([p^{ad}]) \in \pi_n(\Omega^{st}M_1)$. By hypothesis, there exists a map $h\colon \Sigma S^n \times [0,1] \to M_1$ with $h_0 = g \circ f \circ p$, $h_1 = p$ and $h^{-1}(K_1) \supseteq h_1^{-1}(K_1) \times [0,1]$ (here h_t denotes the restriction of h to $\Sigma S^n \times \{t\}$). So we have a map $h^{ad}\colon S^n \times [0,1] \to \Omega M$ such that the restriction of h^{ad} to $S^n \times \{0,1\}$ has image in $\Omega^{st}M_1$. By the small relative compression theorem, there exists an isotopy of $\Sigma S^n \times [0,1]$, fixed on $\Sigma S^n \times \{0,1\} \cup h_1^{-1}(K_1) \times [0,1]$, which makes $h^{-1}(K_1) \backslash h_1^{-1}(K_1) \times [0,1]$ compressible. This pushes the whole homotopy h^{ad} into $\Omega^{st}M_1$. This completes the proof of Proposition 3.2. $\square$

Our next aim is to simplify the hypothesis of Proposition 3.2. So, let $K_1 \subseteq M_1$ and $K_2 \subseteq M_2$ be tubular neighbourhoods of codimension c submanifolds ($c \geq 2$) with the same number $s \in \mathbb{N}$ of path components. Suppose that the normal bundle of $K_1 \subseteq M_1$ has a section.

Condition 1. There exist homotopy inverse maps $f\colon M_1 \to M_2$ and $g\colon M_2 \to M_1$ such that f sends K_1 homeomorphically to K_2 and $g|_{K_2} = (f|_{K_1})^{-1}$. Moreover, $g \circ f \simeq id_{M_1}$ by a homotopy which is fixed on K_1, and $f \circ g \simeq id_{M_2}$ by a homotopy which is fixed on K_2.

Condition 2. There exist homotopy inverse maps $f'\colon M_1 \to M_2$ and $g'\colon M_2 \to M_1$ such that f', restricted to K_1, is homotopic in M_2 to a diffeomorphism $K_1 \to K_2$.

Proposition 3.4. *Conditions 1 and 2 are equivalent.*

Proof. Note that Condition 1 is just the hypothesis of Proposition 3.2. Clearly Condition 1 implies Condition 2, with $f' := f$ and $g' := g$.

So, we suppose that Condition 2 is satisfied; then we can assume that f', restricted to K_1, is a diffeomorphism $K_1 \to K_2$. Let $\Lambda_j \supseteq K_j$ $(j = 1, 2)$ be larger tubular neighbourhoods, consisting of normal disks of diameter say 2. Then $\Lambda_j - K_j$ is an annulus-bundle over the core of K_j. We denote the set of points with distance t from the core of K_j $(t \in [1, 2],\ j = 1, 2)$ by $A_j^{(t)}$; so $A_j^{(1)} = \partial K_j$ and $A_j^{(2)} = \partial\Lambda_j$. Let $F_t\colon M \to M$ be a homotopy with $F_0 = g' \circ f'$ and $F_1 = id_M$. We now define $f := f'$, and let g be the following map which is homotopic to g':

$$g|_{M_2-\Lambda_2} := g', \quad g|_{K_2} := (f|_{K_1})^{-1},$$

and for points in $\Lambda_j - K_j$ we define:

$$g(x) := F_t \circ f^{-1}(x) \ \text{ if } \ x \in A_2^{(2-t)} \ \ (\text{where } t \in [0, 1]).$$

We have that $g \circ f \simeq id_{M_1}$ by a homotopy which is fixed on K_1. We now claim that it follows that $f \circ g \simeq id_{M_2}$ by a homotopy which is fixed on K_2.

To prove this claim, consider the group of all maps $\mu\colon M_2 \to M_2$ which are homotopic to the identity and restrict to the identity on K_2, under the equivalence relation that $\mu \sim \nu$ if μ is homotopic to ν by a homotopy which fixes K_2. In order to prove that this is a group we construct an inverse to a given element $[\mu]$, where $\mu\colon M_2 \to M_2$ with $\mu|_{K_2} = id$. After a homotopy we can assume that $\mu|_{\Lambda_2} = id$. Let $\mu_t\colon M_2 \to M_2$ be a homotopy with $\mu_0 = \mu$ and $\mu_1 = id_{M_2}$. Then we define $\nu\colon M_2 \to M_2$ by

$$\nu|_{M_2-\Lambda_2} = id,\ \nu|_{K_2} = id \text{ , and for } x \in A_2^{(2-t)}: \quad \nu(x) = \mu_t(x).$$

Then $\nu \circ \mu$ is homotopic to the identity map by a homotopy that is fixed on K_2. Thus $[\nu]$ is an inverse; the other group axioms are clear.

In this group we have $f \circ g = f \circ (g \circ f) \circ g = (f \circ g)^2$, so $f \circ g = 1$. This proves the claim, which implies that Condition 1 is satisfied. □

Condition 2 is easier to check than Condition 1, but in general still entails a complicated obstruction theory. However, for 1-dimensional links K_j in closed 3-manifolds M_j $(j = 1, 2)$ satisfying the hypotheses of Theorem 1.1 we observe that Condition 2 is satisfied. So, Theorem 1.1 follows from Proposition 3.2.

Open Question. In the statements of Propositions 3.2 and 3.4, can we drop the hypothesis that the normal bundles of the submanifolds have sections?

Proposition 3.5. *Let $K \subseteq S^3$ be a link with s components. Then $\pi_2(\Omega_K^{st} S^3) \cong \mathbb{Z}^{s+1}$.*

Proof. We shall construct a homomorphism $\phi\colon \pi_2(\Omega_K^{st} S^3) \to \mathbb{Z}^{k+1}$. Let $p\colon \Sigma S^2 \to S^3$ be a map representing an element of $\pi_2(\Omega_K^{st} S^3)$. Then the degree of p is an integer number, so we have a surjective homomorphism $\pi_2(\Omega_K^{st} S^3) \to \mathbb{Z}$. After making p transverse to K, $p^{-1}(K)$ is a link K' in ΣS^2, disjoint from the basepoint of ΣS^2, which projects to a number of *immersed* curves in $S^2 \backslash *_{S^2}$ under the natural projection $\Sigma S^2 \backslash *_{\Sigma S^2} \to S^2 \backslash *_{S^2}$. Thus, by the blackboard framing convention, we can assign a framing number $fr(K'_j) \in \mathbb{Z}$ to each component K'_j of K' (i.e. $fr(K'_j)$ is the linking number of K'_j with the framing curve whose two-dimensional projection is parallel to the projection of K'_j). Let $K_1, ..., K_s$ be the link components of K. We define s integer numbers by

$$f_i := \sum_{p(K'_j) \subset K_i} fr(K'_j) \in \mathbb{Z} \quad (i = 1, ..., s).$$

A homotopy of maps $S^2 \to \Omega^{st}_K S^3$ gives rise to an embedded oriented surface in $[0,1] \times \Sigma S^2$ disjoint from $[0,1] \times *_{\Sigma S^2}$ which projects to an *immersed* surface in $[0,1] \times (S^2 \backslash *_{S^2})$. Note that preimages in $[0,1] \times \Sigma S^2$ of different components of K are disjoint, so that we can think of s separate cobordisms, of the links $p^{-1}(K_i)$ $(i = 1, ..., s)$. It is well-known that the sum of the framing numbers of links is invariant under such cobordisms (in the language of [1], the diagram in $S^2 \backslash *_{S^2}$ is only modified by births, deaths, and bridge moves). Therefore the homomorphisms $f_i \colon \pi_2(\Omega^{st}_K) \to \mathbb{Z}$ $(i = 1, ..., s)$ are well-defined. We can now define $\phi \colon \pi_2(\Omega^{st}_K S^3) \to \mathbb{Z} \oplus \mathbb{Z}^s$ by $\phi = deg \times f_1 \times \ldots \times f_s$.

In order to prove surjectivity of ϕ it is sufficient to construct, for any $(i_1, \ldots, i_s) \in \mathbb{Z}^s$, a map $p \colon \Sigma S^2 \to S^3$ representing an element of $\pi_2(\Omega^{st}_K S^3)$ with $\phi(p) = (1, i_1, \ldots, i_s)$. We do this by isotoping the link K in $S^3 \cong \Sigma S^2$ such that in its two-dimensional projection the jth component has framing number i_j (by performing some Reidemeister 1 moves), and then letting p be the identity map.

To see injectivity, we let $p \colon \Sigma S^2 \to S^3$ be a representative of an element of $\ker \phi$. We note that after a homotopy of p we can assume that K' has at most one component K'_i such that $p(K'_i) \subset K_i$ for each $i = 1, ..., s$. Since $\phi(p) = 0_{\mathbb{Z}^{1+s}}$, every component of K' in the diagram has 0-framing. Now equip every single link component of K with the 0-framing. Then the image of the adjoint of p lies in the *vertical* loop space, so p represents an element of $\pi_2(\Omega^{vert}_K S^3)$ with respect to this framing of K. Since $\phi(p) = 0$, $p \colon S^3 \to S^3$ has degree 0. By Theorem 3.2 we have $\Omega^{vert}_K S^3 \simeq \Omega S^3$, and therefore p represents the trivial element of $\pi_2(\Omega^{vert}_K S^3)$. But $\Omega^{vert}_K S^3$ is a subset of $\Omega^{st}_K S^3$, and it follows that p represents the trivial element of $\pi_2(\Omega^{st}_K S^3)$. Therefore ϕ is an isomorphism. $\square$

Now Theorem 1.2 is an immediate consequence of Theorem 1.1 and Proposition 3.5.

4. Applications to rack and quandle spaces

In [7] the nonaugmented rack and quandle spaces of links in 3-manifolds, considered only as topological spaces (disregarding their cubical structure), were examined. It turned out that the weak homotopy type of these spaces is a very insensitive invariant of the link and the 3-manifold. By combining the results presented here with those of [7] we can now prove that the same is true for the *augmented* rack and quandle spaces.

We call a link $K \subseteq M$ *irreducible* if $M - K$ is an irreducible 3-manifold [3]. An irreducible framed link is called *trivial* if its framing longitude bounds an embedded disk in $M - K$. In contrast to standard notation we say an unframed link is *trivial* if some noncontractible path in ∂L (for instance a meridian) bounds a disk in $M - K$.

Theorem 4.1. *The augmented rack space of an irreducible nontrivial framed link in a 3-manifold M is weakly homotopy equivalent to ΩM.*

Proof. It was proved in [6] that, under the hypotheses of the theorem, the augmented rack space is weakly homotopy equivalent to $\Omega^{vert} M$ (see also [7] and [1]). The result now follows from Theorem 3.1. □

Theorem 4.2. *The augmented quandle spaces of two irreducible nontrivial unframed links in S^3 are homotopy equivalent if and only if they have the same number of link components. The augmented quandle space of a nontrivial knot in S^3 is homotopy equivalent to $(\Omega(S^2 \vee S^3))^{\sim}$, where $^{\sim}$ denotes the universal covering space.*

Proof. According to [6], the augmented quandle space is weakly homotopy equivalent to the straight loop space. The first part follows immediately from Theorem 1.2. Now let $U \in S^3$ be the unknot. For the second part we note that for a nontrivial knot K in S^3 with augmented fundamental quandle $\partial \colon Q \to G$ we have $B_G Q \simeq \Omega^{st}_K S^3 \simeq \Omega^{st}_U S^3 \simeq (\Omega(S^2 \vee S^3))^{\sim}$. Here, the first equivalence follows from the first part, the second equivalence follows from Theorem 1.2 and the third from the example in section 2. □

Theorem 4.3. *Let $K_1 \subseteq M_1$ and $K_2 \subseteq M_2$ be two irreducible, nontrivial (unframed) links in 3-manifolds. Suppose in addition that the hypotheses of Theorem 1.1 are satisfied. Then their augmented quandle spaces are homotopy equivalent.*

Proof. This now follows from Theorem 1.1. □

Let $\mathcal{F}(n+1, X)$ be the group of cobordism classes of framed codimension 2 links in $\mathbb{R}^{n+1}$ with a homomorphism of their fundamental racks to X. If X is a quandle, then this is isomorphic to the group of cobordism classes of framed codimension 2 links in $\mathbb{R}^{n+1}$ with a homomorphism of their fundamental *quandles* to X. It was proved in [1] that $\mathcal{F}(n+1, X)$ is naturally isomorphic to $\pi_n(BX)$. Since for $n \geq 2$ we have $\pi_n(BX) \cong \pi_n(B_G X)$, we obtain immediately results about the groups $\mathcal{F}(n+1, X)$, for instance

Theorem 4.4. *Let X and Q be the rack and quandle respectively of a nontrivial knot in S^3. Then for $n \geq 2$ we have $\mathcal{F}(n+1, X) \cong \pi_n(\Omega S^3) \cong \pi_{n+1}(S^3)$ and $\mathcal{F}(n+1, Q) \cong \pi_n(\Omega(S^2 \vee S^3)) \cong \pi_{n+1}(S^2 \vee S^3)$.*

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