IRREDUCIBLE CONSTITUENTS OF FAITHFUL INDUCED CHARACTERS

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Communication by Stephen D. Smith)

Abstract. Let $G$ be a finite group, and suppose $\chi$ is a character of $G$ obtained by inducing an irreducible character of some subgroup of $G$. If $\chi$ is faithful, we show that some irreducible constituent of $\chi$ has a solvable kernel. This yields an improved version of a theorem of Evdokimov and Ponomarenko.

1. Introduction

Suppose $G$ is a finite transitive permutation group with permutation character $\chi$, and let $d$ be the maximum of the degrees of the irreducible constituents of $\chi$. Recently, S. A. Evdokimov and I. N. Ponomarenko have shown that the (unique) largest solvable normal subgroup $S(G)$ of $G$ must have index bounded above by some function of $d$. (See [2].) Specifically, they showed that $|G : S(G)| \leq J(d)^{\log_2(d)}$, where $J(d)$ is the function associated with Jordan’s theorem on finite complex linear groups. (See Theorem 14.12 of [4].) In other words, $J(d)$ is the smallest integer with the property that every finite subgroup $F$ of $GL(d,\mathbb{C})$ has an abelian normal subgroup $A$ such that $|F : A| \leq J(d)$.

One purpose of this paper is to give an easier proof of a more general result that yields a better bound.

Theorem A. Let $\psi \in \text{Irr}(H)$, where $H \subseteq G$, and suppose that the induced character $\chi = \psi^G$ is faithful. If $d$ is the maximum of the degrees of the irreducible constituents of $\chi$, then $G$ has a solvable normal subgroup $S$ such that $|G : S| \leq J(d)$.

Note that if $\psi$ is the principal character of $H$, then $\chi$ is a faithful transitive permutation character, and we are in the situation considered by Evdokimov and Ponomarenko. We see, therefore, that Theorem A generalizes and strengthens the Evdokimov and Ponomarenko theorem, as promised.

Theorem A is an easy corollary of the following result, which we consider to be the main theorem of this paper.

Theorem B. Let $\psi \in \text{Irr}(H)$, where $H \subseteq G$, and suppose that the induced character $\chi = \psi^G$ is faithful. Then $\ker(\theta)$ is solvable for some irreducible constituent $\theta$ of $\chi$.

Received by the editors March 2, 1999.

2000 Mathematics Subject Classification. Primary 20C15.

This research was partially supported by the U.S. National Security Agency.
Proof of Theorem A. Let \( \theta \) be an irreducible constituent of \( \chi \) such that \( K = \ker(\theta) \) is solvable. Then \( G/K \) is isomorphic to a subgroup of \( GL(n, \mathbb{C}) \), where \( n = \theta(1) \leq d \), and hence \( G/K \) is isomorphic to a subgroup of \( GL(d, \mathbb{C}) \). By Jordan's theorem, there exists an abelian subgroup \( A/K < G/K \) such that \( [G : A] = [G/K : A/K] \leq J(d) \). Of course, \( A \triangleleft G \) and \( A \) is solvable, and the proof is complete.

It is possible to improve Theorem B slightly, and to obtain an upper bound on the derived length of \( \ker(\theta) \) in terms of the embedding of \( H \) in \( G \). This somewhat technical extension of Theorem B is the following.

Theorem C. In the situation of Theorem B, let \( H = H_0 < H_1 < \cdots < H_r = G \) be a saturated chain of subgroups. (In other words, for each subscript \( i \) with \( 0 \leq i < r \), assume that \( H_i \) is maximal in \( H_{i+1} \).) Then there exists an irreducible constituent \( \theta \) of \( \chi \) such that \( \ker(\theta) \) is solvable of derived length at most equal to the number of indices \( |H_{i+1} : H_i| \) that are prime powers.

As Evdokimov and Ponomarenko pointed out, an important special case of Theorem A is where \( H = 1 \) and \( \chi \) is the regular character of \( G \). In that case, of course, every member of \( \text{Irr}(G) \) is a constituent of \( \chi \), and thus \( d = b(G) \), the maximum of the degrees of the irreducible characters of \( G \). While Theorem A only guarantees the existence of a solvable normal subgroup of bounded index in \( G \), an old theorem of D. S. Passman and the author [3] shows that \( G \) actually has an abelian normal subgroup of bounded index. It is unclear, however, whether or not Theorem A can be improved to yield an abelian normal subgroup of bounded index in the general case. (As an alternative to [3], see Theorem 12.23 of [4] for a proof of the existence of a not necessarily normal abelian subgroup of index at most \( (b(G))! \) This, of course, yields an normal abelian subgroup of index at most \( ((b(G))!)^2! \).

What happens in Theorem B if \( \chi \) is the regular character? The conclusion is that for any finite group \( G \), there exists \( \theta \in \text{Irr}(G) \) such that \( \ker(\theta) \) is solvable. But a better result is known. In fact, there exists \( \theta \in \text{Irr}(G) \) such that \( \ker(\theta) \) is nilpotent. (See [4] or Theorem 12.20 of [4] for this result of D. Broune.) We have been unable to decide, however, whether or not in the general situation of Theorem B, there must exist an irreducible constituent of \( \chi \) having a nilpotent kernel.

2. Proofs

We begin with a lemma.

(2.1) Lemma. Let \( M \) be a maximal subgroup of \( G \), and suppose that \( \alpha \in \text{Irr}(M) \) is faithful, but that no irreducible constituent of \( \alpha^G \) is faithful. Then \( |G : M| \) is a prime power and the kernels of all irreducible constituents of \( \alpha^G \) are abelian.

Proof. Let \( \beta \in \text{Irr}(G) \) be a constituent of \( \alpha^G \). Write \( K = \ker(\beta) \), and note that since \( \alpha \) is a constituent of \( \beta_M \), we have \( K \cap M \subseteq \ker(\alpha) = 1 \). But \( K > 1 \) by assumption, and thus \( K \nsubseteq M \), and we conclude from the maximality of \( M \) that \( KM = G \). Also, since \( K \) is a normal complement for the maximal subgroup \( M \), it follows that \( K \) is minimal normal in \( G \), and thus \( K \) is abelian if and only if it is a \( p \)-group for some prime \( p \). Furthermore, \( |K| = |G : M| \), and thus \( K \) is abelian if and only if \( |G : M| \) is a prime power.

Assuming, now, that \( |G : M| \) is not a prime power, we work to derive a contradiction. Since \( G = MK = M\ker(\beta) \), we see that \( \beta_M \) is irreducible. Thus \( \beta_M = \alpha \), and we have \( [\alpha^G, \beta] = [\alpha, \beta_M] = 1 \). Since \( \beta \) was an arbitrary irreducible constituent
of \( \alpha^G \), we see that \( \alpha^G \) is a sum of distinct extensions of \( \alpha \), and the number of these is \( \alpha^G(1)/\alpha(1) = |G : M| \). Also, we know that the kernels of these extensions are normal complements for \( M \) in \( G \).

We claim that the kernels of the \(|G : M| \) extensions of \( \alpha \) to \( G \) are distinct. To see why this is so, observe that, since \( KM = G \), we have

\[
[(\alpha^G)_K, 1_K] = [(\alpha_K \cap M)^K, 1_K] = [\alpha_K \cap M, 1_K] \leq \alpha(1) = \beta(1) = [\beta_K, 1_K].
\]

It follows that \( K \) is not the kernel of any irreducible constituent of \( \alpha^G \) other than \( \beta \).

We now know that \( M \) has \(|G : M| \) distinct normal complements in \( G \), and each of these is a nonabelian minimal normal subgroup. The product \( U \) of these complements is therefore direct, and we see that \(|U| = |G : M|^{|G : M|} \) and that \( Z(U) = 1 \). Also, we can write \( U = K \times C \), where \( K \) is a normal complement for \( M \) and \( C \triangleleft G \) with \(|C| = |G : M|^{|G : M|^{-1}} \). Because \(|G : M| \) is not a prime power, we have \(|G : M| > 2 \), and thus \(|C| > |G : M| \). It follows that \( C \cap M \subseteq 1 \).

Now \( K \) normalizes \( C \cap M \) since \( K \subseteq C_G(C) \). Also, \( C \cap M \triangleleft M \) because \( C \triangleleft G \), and it follows that \( C \cap M \triangleleft KM = G \). But each of the normal complements to \( M \) intersects \( C \cap M \) trivially, and thus the normal subgroup \( C \cap M \) centralizes each of them. It follows that \( 1 < C \cap M \subseteq Z(U) = 1 \), and this contradiction completes the proof.

Proofs of Theorems B and C. If \( H = G \), we see that \( \psi = \chi \) is faithful, and thus \( 1 = \ker(\psi) \) has derived length 0, as desired. We can thus assume that \( H < G \), and we work by induction on \(|G : H| \). Let \( H \subseteq M \), where \( M \) is maximal in \( G \), and let \( r \) be the number of indices that are prime powers in a saturated chain of subgroups running from \( H \) to \( M \). Our task is to find an irreducible constituent \( \theta \) of \( \psi^G \) such that \( \ker(\theta) \) is solvable of derived length at most \( r \) if \(|G : M| \) is not a prime power, and of derived length at most \( r + 1 \) if \(|G : M| \) is a prime power.

Let \( \eta = \psi^M \), and write \( N = \ker(\eta) \subseteq \ker(\psi) \subseteq H \). By the inductive hypothesis applied in the group \( M/N \), we can choose an irreducible constituent \( \alpha \) of \( \eta \), with \( \ker(\alpha) = L \), and such that \( L/N \) is solvable with derived length at most \( r \). Thus \( L^{(r)} \subseteq N \), and if \( K \) is a normal subgroup of \( G \) contained in \( L \), then \( K^{(r)} \subseteq N \subseteq \ker(\psi) \). Also \( K^{(r)} \triangleleft G \), and it follows that \( K^{(r)} \subseteq \ker(\psi^G) = \ker(\chi) = 1 \). In other words, every normal subgroup of \( G \) contained in \( L \) is solvable with derived length at most \( r \).

Now let \( \theta \) be any irreducible constituent of \( \alpha^G \), and write \( U = \ker(\theta) \triangleleft G \). Then \( U \cap M \subseteq \ker(\theta_M) \subseteq \ker(\alpha) = L \). If \( U \subseteq M \), therefore, we have \( U \subseteq L \), and since \( U \triangleleft G \), we see by the result of the previous paragraph that \( U \) is solvable with derived length at most \( r \). There is nothing further to prove in this case, and so we can assume \( U \not\subseteq M \), and that this holds for every choice of the irreducible constituent \( \theta \) of \( \alpha^G \). In particular, since \( U \triangleleft G \) and \( M \) is maximal in \( G \), we have \( UM = G \). But \( U = \ker(\theta) \), and it follows that \( \theta_M \) is irreducible, and thus each of the irreducible constituents of \( \alpha^G \) is an extension of \( \alpha \). We can now apply Garrison’s lemma, which is Lemma 12.47 of [4], and we deduce that \( V(\alpha) \triangleleft G \). (Recall that, by definition, \( V(\alpha) \) is the subgroup of \( G \) generated by all elements \( m \in M \) such that \( \alpha(m) \neq 0 \).) Now

\[
U \cap V(\alpha) \subseteq U \cap M = \ker(\theta_M) = \ker(\alpha) \subseteq U \cap V(\alpha),
\]

and so \( L = \ker(\alpha) = U \cap V(\alpha) \triangleleft G \).
It follows that $L$ is solvable of derived length at most $r$, and that $\alpha$ is a faithful character of the maximal subgroup $M/L$ of $G/L$. We know that none of the irreducible constituents $\theta$ of $\alpha^G$ has kernel contained in $M$, and thus when viewed as a character of $G/L$, none of them is faithful. It follows by Lemma 2.1 that $|G:M|$ is a prime power and that $U/L$ is abelian. This shows that $\ker(\theta) = U$ is solvable of derived length at most $r + 1$, and the proof is complete.

References


