EQUILIBRIUM MEASURE OF A PRODUCT SUBSET OF $\mathbb{C}^n$

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Abstract. In this note we show that an equilibrium measure of a product of two subsets of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, is a product of their equilibrium measures. We also obtain a formula for $(dd^c V_{E,F})^n$, where $u, v$ are locally bounded plurisubharmonic functions and $2 \leq p \leq n$.

Introduction

Let $E$ be a bounded subset of $\mathbb{C}^n$. The function

$$V_E := \sup \{ u \in PSH(\mathbb{C}^n) : u|_E \leq 0, \sup_{z \in \mathbb{C}^n} (u(z) - \log^+ |z|) < \infty \}$$

is called a global extremal function (or the Siciak extremal function) of $E$. It is known that $V_E^*$, the upper regularization of $V_E$, is plurisubharmonic in $\mathbb{C}^n$ if and only if $E$ is not pluripolar. In such a case, by [BT1], $(dd^c V_E^*)^n$ is a well defined nonnegative Borel measure and it is called an equilibrium measure of $E$. We refer to [Kl] for a detailed exposition of this topic.

In this note we shall show

**Theorem 1.** Let $E$ and $F$ be nonpluripolar bounded subsets of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Then

1. $$V_{E \times F}^* = \max \{ V_E^*, V_F^* \}$$

and

2. $$E \times F \rangle$$

Note that here we treat $V_E^*$ (resp. $V_F^*$) as a function of $\mathbb{C}^{n+m}$ independent of the last $m$ (respectively first $n$) variables.

The formula (1) was proved by Siciak (see [Si]) for $E, F$ compact (see also [Ze] for a proof using the theory of the complex Monge-Ampère operator). For $n = m = 1$ the proof of (2) can be found in [BT2].

If $E \subset D$, where $D$ is a bounded domain in $\mathbb{C}^n$, then the function

$$u_{E,D} := \sup \{ v \in PSH(D) : v \leq 0, v|_E \leq -1 \}$$

is called a relative extremal function of $E$. Combining our methods of the proof of Theorem 1 with a result from [EP] we can also obtain

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Theorem 2. Let $D$ be a bounded domain in $\mathbb{C}^n$ and $G$ a bounded domain in $\mathbb{C}^m$. Then for arbitrary subsets $E \subset D$, $F \subset G$ we have
\begin{equation}
\max_{E \times F, D \times G} \{u^*_E, u^*_F\} = \max\{u^*_E, u^*_D\} \wedge \max\{u^*_F, u^*_G\}
\end{equation}
and
\begin{equation}
(dd^c u^*_E \wedge dd^c u^*_F)^{n+m} = (dd^c u^*_E)^n \wedge (dd^c u^*_F)^m.
\end{equation}

The relative Monge-Ampère capacity of $E \subset D$ is defined by
\begin{equation}
c(E, D) := \sup \left\{ \int_{E} (dd^c u)^n : u \in PSH(D), -1 \leq u \leq 0 \right\},
\end{equation}
provided that $E$ is Borel. If $E \subset D$ is arbitrary, then, as usual, we can define
\begin{equation}
c^*(E, D) := \inf_{E \subset U, \ U \text{ open}} c(U, D),
\end{equation}
\begin{equation}
c_*(E, D) := \sup_{K \subset E, \ K \text{ compact}} c(K, D).
\end{equation}
By [BT1], if $E \subset D$ and $D$ is hyperconvex (that is $(u^*_E)_* = 0$ on $\partial D$), then
\begin{equation}
c^*(E, D) = \int_{D} (dd^c u^*_E)^n.
\end{equation}
Moreover, $c^*(E, D) = c(E, D) = c_*(E, D)$ if $E$ is Borel. Theorem 2 thus gives

Theorem 3. Assume that $D$ and $G$ are bounded hyperconvex domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Then for $E \subset D$, $F \subset G$ we have
\begin{equation}
c^*(E \times F, D \times G) = c^*(E, D)c^*(F, G).
\end{equation}

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Proofs

If $\Omega$ is an open subset of $\mathbb{C}^n$ and $1 \leq p \leq n$, then by [BT1] the mapping
\begin{equation}
(u_1, \ldots, u_p) \mapsto dd^c u_1 \wedge \cdots \wedge dd^c u_p
\end{equation}
is well defined on the set $(PSH \cap L^\infty_{loc}(\Omega))^p$ and its values are nonnegative currents of bidegree $(p, p)$. Moreover, (4) is symmetric and continuous with respect to decreasing sequences. First, we shall prove

Theorem 4. Let $u, v$ be locally bounded plurisubharmonic functions. Then, if $2 \leq p \leq n$, we have
\begin{equation}
(dd^c \max\{u, v\})^p = dd^c \max\{u, v\} \wedge \sum_{k=0}^{p-1} (dd^c u)^k \wedge (dd^c v)^{p-1-k} - \sum_{k=1}^{p-1} (dd^c u)^k \wedge (dd^c v)^{p-k}.
\end{equation}

Proof. We leave it as an exercise to the reader to show that a simple inductive argument reduces the proof to the case $p = 2$. By the continuity of (4) under decreasing sequences we may also assume that $u, v$ are smooth.

Let $\chi : \mathbb{R} \to [0, +\infty)$ be smooth and such that $\chi(x) = 0$ if $x \leq -1$, $\chi(x) = x$ if $x \geq 1$ and $0 \leq \chi' \leq 1$, $\chi'' \geq 0$ everywhere. Define
\begin{equation}
\psi_j := v + \frac{1}{j} \chi(j(u - v)).
\end{equation}
Then, treating $u;v$ as functions on $D$, we conclude

\( (dd^c \max\{u,v\})^p = dd^c \max\{u,h\} \wedge (dd^c u)^p \). \hfill \Box

**Corollary 6.** Suppose $u,v$ are locally bounded plurisubharmonic functions with $(dd^c u)^p = 0$ and $(dd^c v)^q = 0$, where $1 \leq p,q \leq n$ and $p + q \leq n$. Then

\( (dd^c \max\{u,v\})^{p+q} = 0 \). \hfill \Box

The main part of the proof of (2) will be contained in

**Theorem 7.** Let $D$ be open in $\mathbb{C}^n$ and $G$ open in $\mathbb{C}^m$. Assume that $u,v$ are nonnegative plurisubharmonic functions in $D$ and $G$, respectively, such that

\[
\int_{\{u > 0\}} (dd^c u)^n = 0 \quad \text{and} \quad \int_{\{v > 0\}} (dd^c v)^m = 0. 
\]

Then, treating $u,v$ as functions on $D \times G$, we have

\( (dd^c \max\{u,v\})^{n+m} = (dd^c u)^n \wedge (dd^c v)^m. \)
Proof. Let \( w, \chi \) and \( \psi_j \) be defined in the same way as in the proof of Theorem 4. By Theorem 4 and since \((dd^c v)^{m+1} = 0, (dd^c v)^{m+1} = 0, \) we have
\[
(dd^c v)^{n+m} = dd^c w \wedge [(dd^c u)^{n-1} \wedge (dd^c v)^m + (dd^c u)^n \wedge (dd^c v)^{m-1}] - (dd^c u)^n \wedge (dd^c v)^m.
\] (8)

Using the hypothesis on \( u, v \) we may compute
\[
(dd^c \psi_j \wedge (dd^c u)^{n-1} \wedge (dd^c v)^m = [\chi'(0)(dd^c u)^n + j \chi''(j u)du \wedge d^c u \wedge (dd^c u)^{n-1}] \wedge (dd^c v)^m
\]
\[
= dd^c (\chi(j u) / j \wedge (dd^c u)^{n-1} \wedge (dd^c v)^m).
\]
(9)

Since \( \chi(j u) / j \downarrow u \) as \( j \uparrow \infty \), it follows that
\[
(dd^c w \wedge (dd^c u)^{n-1} \wedge (dd^c v)^m = (dd^c u)^n \wedge (dd^c v)^m
\]
and, similarly,
\[
(dd^c w \wedge (dd^c u)^n \wedge (dd^c v)^m = (dd^c u)^n \wedge (dd^c v)^m.
\]

This, together with (8), finishes the proof. \( \square \)

For the proof of Theorem 1 we need a lemma which is an extension of a result from [Sa].

**Lemma 8.** Let \( E, F, D, G \) be as in Theorem 2. For \( \varepsilon > 0 \) set
\[
E_\varepsilon := \{ V_E^* < \varepsilon \}, \quad F_\varepsilon := \{ V_F^* < \varepsilon \},
\]
\[
E_\varepsilon := \{ u_{E,D}^* < -1 + \varepsilon \}, \quad F_\varepsilon := \{ u_{F,G}^* < -1 + \varepsilon \}.
\]

Then
\[
(9) \quad V_{E_\varepsilon}^* \uparrow V_E^*, \quad V_{F_\varepsilon}^* \uparrow V_F^*, \quad V_{E_\varepsilon \times F_\varepsilon}^* \uparrow V_{E \times F}^*,
\]
\[
(10) \quad u_{E_\varepsilon,D}^* \uparrow u_{E,D}^*, \quad u_{F_\varepsilon,D}^* \uparrow u_{F,D}^*, \quad u_{E_\varepsilon \times F_\varepsilon,D \times G}^* \uparrow u_{E \times F,D \times G}^*,
\]
as \( \varepsilon \downarrow 0 \), and every convergence is uniform.

**Proof.** The set \( E \setminus E_\varepsilon = E \cap \{ V_E^* \geq \varepsilon \} \) is pluripolar by Bedford-Taylor’s theorem on negligible sets (see [BL]). It follows that
\[
V_{E_\varepsilon}^* - \varepsilon \leq V_{E_\varepsilon}^* \leq V_{E_\varepsilon}^* \leq V^*_E
\]
which gives the first two convergences of (9). In order to show the third one, observe that
\[
(11) \quad \max\{ V_E, V_F \} \leq V_{E \times F} \leq V_E + V_F.
\]

Indeed, the first inequality in (11) follows easily from the definition of extremal function. Fixing one of the variables \( (z, w) \in C^n \times C^m \), we see that the second inequality in (11) is satisfied, first on the cross \( (E \times C^m) \cup (C^n \times F) \), and then everywhere.

By (11) \( V_{E \times F} \leq 2\varepsilon \) on \( E_\varepsilon \times F_\varepsilon \). On the other hand, by (11) the set \( (E \times F) \setminus (E_\varepsilon \times F_\varepsilon) \) is contained in \( (E \times F) \cap \{ V_{E \times F}^* \geq \varepsilon \} \) and is thus pluripolar. Therefore
\[
V_{E \times F}^* - 2\varepsilon \leq V_{E_\varepsilon \times F_\varepsilon}^* \leq V_{E \times F}^* V_{E_\varepsilon \times F_\varepsilon}^* \leq V_{E \times F}^*
\]
and this gives (9).
Similarly as in (11) we can show
\[
\max\{u_E, D, u_F, G\} \leq u_{E \times F, D \times G} \leq -u_{E, D} u_{F, G}.
\]
Now the proof of (10) is parallel to that of (9).

**Proof of Theorem 1.** If \( E, F \) are compact and \( L \)-regular (that is, \( V_E \) and \( V_F \) are continuous), then (1) was shown in [Si] and (2) follows immediately from Theorem 7. For \( E, F \) open we can find sequences of compact, \( L \)-regular sets with \( E_j \uparrow E \) and \( F_j \uparrow F \). Then \( V_{E_j} \downarrow V_E, V_{F_j} \downarrow V_F \) and \( V_{E_j \times F_j} \downarrow V_{E \times F} \) as \( j \uparrow \infty \). This gives (1) and (2) for open sets. The general case can now be deduced from Lemma 8.

**Proof of Theorem 2.** The proof of (3) for open subsets can be found in [EP]. Now the proof is the same as the proof of Theorem 1.

**Remark.** Although (3) is stated in [EP] for arbitrary subsets \( E, F \), the way from open subsets to the general case is not so straightforward as the authors claim—one needs Lemma 8.

**References**


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