OPENNESS OF INDUCED PROJECTIONS

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Abstract. For continua $X$ and $Y$ it is shown that if the projection $f : X \times Y \to X$ has its induced mapping $C(f)$ open, then $X$ is $C^*$-smooth. As a corollary, a characterization of dendrites in these terms is obtained.

All spaces considered in this paper are assumed to be metric. A mapping means a continuous function. To exclude some trivial statements we assume that all considered mappings are not constant. A continuum means a compact connected space. Given a continuum $X$ with a metric $d$, we let $2^X$ denote the hyperspace of all nonempty closed subsets of $X$ equipped with the Hausdorff metric $H$ defined by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}$$

(see e.g. [6] (0.1), p. 1, and (0.12), p. 10). Further, we denote by $C(X)$ the hyperspace of all subcontinua of $X$, i.e., of all connected elements of $2^X$. The reader is referred to Nadler’s book [6] for needed information on the structure of hyperspaces.

Given a mapping $f : X \to Y$ between continua $X$ and $Y$, we consider mappings (called the induced ones)

$$2^f : 2^X \to 2^Y \quad \text{and} \quad C(f) : C(X) \to C(Y)$$

defined by

$$2^f(A) = f(A) \quad \text{for every} \ A \in 2^X \quad \text{and} \quad C(f)(A) = f(A) \quad \text{for every} \ A \in C(X).$$

A mapping $f : X \to Y$ between spaces $X$ and $Y$ is said to be open provided the image of an open subset of the domain is open in the range. The following results concerning induced mappings for the class of open mappings are known (see [1] Theorem 4.3; compare also [9] Theorem 3.2).

1. Statement. Let a surjective mapping $f : X \to Y$ between continua $X$ and $Y$ be given. Consider the following conditions:

(1.1) $f : X \to Y$ is open;
(1.2) $C(f) : C(X) \to C(Y)$ is open;
(1.3) $2^f : 2^X \to 2^Y$ is open.

Then (1.1) and (1.3) are equivalent, and each of them is implied by (1.2).
An example is known [4, Section 4] of open surjective mappings \( f : X \to Y \) between locally connected continua \( X \) and \( Y \) such that the induced mapping \( C(f) : C(X) \to C(Y) \) is not open.

A continuum, the intersection of every two subcontinua of which is connected, is said to be hereditarily unicoherent. A continuum is called a dendroid provided that it is hereditarily unicoherent and arcwise connected. Given points \( a \) and \( b \) in a dendroid \( X \), we denote by \( ab \) the (unique) arc in \( X \) joining these points.

The following result has been proved in [1, Theorem 21].

2. **Theorem.** Let \( X \) and \( Y \) be nondegenerate continua, and let \( f : X \times Y \to X \) denote the natural projection. If \( C(f) \) is open, then \( X \) is hereditarily unicoherent.

It is known that the opposite implication is not true (see [1, Example 22]). The aim of the paper is to present further results in this direction.

Given a (metric) space \( X \) we denote by \( d_X \) the metric on \( X \), and by \( B_X(p, \varepsilon) \) the (open) ball in \( X \) centered at a point \( p \in X \) and having the radius \( \varepsilon \). Given a subset \( A \subset X \), we define \( N_X(A, \varepsilon) = \bigcup \{ B_X(a, \varepsilon) : a \in A \} \) and we use the symbol \( \text{cl}_X(A) \) to denote the closure of \( A \) in \( X \). The symbol \( \mathbb{N} \) stands for the set of all positive integers.

Let \( X \) be a continuum. Define \( C^* : C(X) \to C(C(X)) \) by \( C^*(A) = C(A) \). It is known that for any continuum \( X \) the function \( C^* \) is upper semicontinuous (see [6, Theorem 15.2, p. 514]), and it is continuous on a dense \( G_\delta \) subset of \( C(X) \) (see [6, Corollary 15.3, p. 515]). A continuum \( X \) is said to be \( C^* \)-smooth at \( A \in C(X) \) provided that the function \( C^* \) is continuous at \( A \). A continuum \( X \) is said to be \( C^* \)-smooth provided that the function \( C^* \) is continuous on \( C(X) \), i.e., at each \( A \in C(X) \) (see [6, Definition 5.15, p. 517]). Each arclike continuum is \( C^* \)-smooth, (6, Theorem 15.13, p. 525]). \( C^* \)-smoothness implies hereditary unicoherence (see [2, Corollary 3.4, p. 203] and [6, Note 1, p. 530]). Thus each arcwise connected \( C^* \)-smooth continuum is a dendroid (see [6, Theorem 15.19, p. 528]). Further, a locally connected continuum is \( C^* \)-smooth if and only if it is a dendrite (see [6, Theorem 15.11, p. 522]).

3. **Lemma.** Let \( X \) be a nondegenerate continuum, and let \( \varepsilon > 0 \) be given. Then there is a finite sequence of subcontinua \( D_0 \subset D_1 \subset \ldots \subset D_m \) of \( X \) and there is an \( \varepsilon \)-net \( \{a_1, \ldots, a_m\} \) in \( X \) such that \( a_i \in D_i \setminus D_{i-1} \) for each \( i \in \{1, \ldots, m\} \).

**Proof.** Let \( \{b_1, \ldots, b_m\} \) be an \( \frac{1}{2} \)-net in \( X \). Fix a point \( b \in X \setminus \{b_1, \ldots, b_m\} \). Let \( \alpha : [0, 1] \to C(X) \) be an order arc from \( \{b\} \) to \( X \); that is, a mapping such that \( \alpha(0) = \{b\} \), \( \alpha(1) = X \) and, if \( s < t \), then \( \alpha(s) \subseteq \alpha(t) \) (for the existence of order arcs see [6, Theorem 1.8, p. 59]). Let \( t_0 > 0 \) be such that \( \alpha(t_0) \cap \{b_1, \ldots, b_m\} = \emptyset \).

Define \( D_0 = \alpha(t_0) \).

Let \( s_1 = \min\{t \in [0, 1] : \alpha(t) \cap \{b_1, \ldots, b_m\} \neq \emptyset\} \). We may assume that \( b_1 \in \alpha(s_1) \). Note that \( t_0 < s_1 \). Consider the set \( E = (X \setminus B_X(b_1, \frac{1}{2})) \cup D_0 \). Then \( E \) is a closed subset of \( X \) and \( b_1 \in X \setminus E \). Thus there exists \( t_1 \in (t_0, s_1) \) such that \( \alpha(t_1) \) is not contained in \( E \). Choose a point \( a_1 \in \alpha(t_1) \setminus E \). Observe that \( a_1 \in B_X(b_1, \frac{1}{2}) \setminus D_0 \). Define \( D_1 = \alpha(t_1) \).

Let \( s_2 = \min\{t \in [0, 1] : \alpha(t) \cap \{b_2, \ldots, b_m\} \neq \emptyset\} \). We may assume that \( b_2 \in \alpha(s_2) \). Note that \( t_1 < s_2 \). Proceeding as in the paragraph above it is possible to find a number \( t_2 \in (t_1, s_2) \) and a point \( a_2 \in \alpha(t_2) \cap B_X(b_2, \frac{1}{2}) \setminus D_1 \). Define \( D_2 = \alpha(t_2) \).

Following this procedure we can find points \( a_1, a_2, \ldots, a_m \) in \( X \) and numbers \( 0 < t_0 < t_1 < t_2 < \cdots < t_m < 1 \) such that for each \( i \in \{1, \ldots, m\} \) we have
It follows that there is a point theorem; see e.g. [5, q containing is a chain. By interiority of \( f \) we see that the continua \( D_0, D_1, \ldots, D_m \) and the points \( a_1, \ldots, a_m \) satisfy the required conditions. The proof is complete.

Let \( n \in \mathbb{N} \). A finite sequence of \( n \) sets \( L_1, \ldots, L_n \) is called a chain provided that \( L_i \cap L_j = \emptyset \) if and only if \( |i - j| \leq 1 \). Elements \( L_i \) of the chain are called its links.

**4. Theorem.** Let \( X \) and \( Y \) be nondegenerate continua, and let \( f : X \times Y \to X \) denote the natural projection. If \( C(f) \) is open, then \( X \) is \( C^* \)-smooth.

**Proof.** Assume the contrary. Let \( A = \text{Lim}(C(A_n)) \subseteq C(A) \) for a sequence of subcontinua \( A_n \) of \( X \) converging to a continuum \( A \), and take \( K \in C(A) \setminus A \). Let \( \varepsilon > 0 \) be such that \( B_{C(X)}(K, 2\varepsilon) \cap C(A_i) = \emptyset \) for each \( i \in \mathbb{N} \).

Let \( D_0, D_1, \ldots, D_m \) and \( \{a_1, \ldots, a_m\} \) be as in Lemma 3 for the continuum \( K \). Choose subcontinua \( E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m \) of \( Y \) and points \( b_i \in E_i \setminus E_{i-1} \). Fix points \( a_0 \in D_0 \) and \( b_0 \in E_0 \).

Note that the sequence \( \{a_0\} \times E_0, D_1 \times \{b_0\}, \{a_1\} \times E_1, D_2 \times \{b_1\}, \{a_2\} \times E_2, \ldots, D_m \times \{b_{m-1}\}, \{a_m\} \times E_m \) is a chain. Let \( P \) be the union of the chain, i.e.,

\[
P = (A \times \{b_m\}) \cup (D_1 \times \{b_0\}) \cup (D_2 \times \{b_1\}) \cup \cdots \cup (D_m \times \{b_{m-1}\})
\]

\[
\cup (\{a_0\} \times E_0) \cup (\{a_1\} \times E_1) \cup \cdots \cup (\{a_m\} \times E_m),
\]

and note that \( P \) is a subcontinuum of \( X \times Y \) and that \( C(f)(P) = A \).

Choose a number \( \eta \) with \( 0 < \eta < \varepsilon \) satisfying the two conditions

\[
N_X(D_i, \eta) \cap B_X(a_j, \eta) = \emptyset \quad \text{for} \quad 0 \leq i < j,
\]

\[
N_Y(E_i, \eta) \cap B_Y(b_j, \eta) = \emptyset \quad \text{for} \quad 0 \leq i < j.
\]

It follows that the sequence

(4.1)

\[
N_{X \times Y}(\{a_0\} \times E_0, \eta), \quad N_{X \times Y}(D_1 \times \{b_0\}, \eta), \quad N_{X \times Y}(\{a_1\} \times E_1, \eta),
\]

\[
N_{X \times Y}(D_2 \times \{b_1\}, \eta), \quad N_{X \times Y}(\{a_2\} \times E_2, \eta), \ldots,
\]

\[
N_{X \times Y}(D_m \times \{b_{m-1}\}, \eta), \quad N_{X \times Y}(\{a_m\} \times E_m, \eta), \quad N_{X \times Y}(A \times \{b_m\}, \eta)
\]

is a chain. By interiority of \( C(f) \) at \( P \) there is a \( \delta > 0 \) such that \( B_{C(X)}(f(P), \delta) \subset C(f)(B_{C(X \times Y)}(P, \eta)) \). Let \( k \in \mathbb{N} \) be such that \( H(A_k, A) < \delta \). Then there is a continuum \( Q \subset X \times Y \) such that \( H(P, Q) < \eta \) and \( f(Q) = A_k \). Take a point \( q \in Q \) such that \( d_{X \times Y}(q, (a_0, b_0)) < \eta \). Let \( L \) be the component of \( Q \setminus N_{X \times Y}(A \times \{b_m\}, \eta) \) containing \( q \). By the Janiszewski theorem (known also as the boundary bumping theorem; see e.g. [3] 547, III, Theorems 1 and 2, p. 172) and compare [6] 20.1-20.3, p. 625-626) there is a point \( r \in L \cap \text{cl}N_{X \times Y}(A \times \{b_m\}, \eta) \). Thus \( L \) intersects the first and the closure of the last link of the chain (4.1), and it is contained in the union of all links of this chain. Consequently, \( L \) intersects each intermediate link of the chain.

Let \( q_i \in L \cap N_{X \times Y}(\{a_i\} \times E_i, \eta) \). Note that \( d_X(f(q_i), a_i) < \eta \). Thus we have

\[
f(L) \subset N_X(K, \eta) \subset N_X(K, 2\varepsilon)
\]

\[
K \subset N_X(\{a_1, \ldots, a_m\}, \varepsilon) \subset N_X(\{f(q_1), \ldots, f(q_m)\}, \varepsilon + \eta)
\]

\[
\subset N_X(f(L), \varepsilon + \eta) \subset N_X(f(L), 2\varepsilon).
\]

It follows that \( H(K, f(L)) < 2\varepsilon \) and \( f(L) \in C(A_k) \), contrary to the definition of \( \varepsilon \). The proof is then complete. \( \square \)
5. **Corollary.** For a fixed continuum $X$ let $f_Y : X \times Y \to X$ denote the natural projection. The following conditions are equivalent for a locally connected continuum $X$:

(5.1) $X$ is a dendrite;
(5.2) for each continuum $Y$ the induced mapping $C(f_Y)$ is open;
(5.3) the induced mapping $C(f_{[0,1]})$ is open;
(5.4) there exists a continuum $Y$ such that the induced mapping $C(f_Y)$ is open.

**Proof.** The implication from (5.1) to (5.2) is known from [1, Corollary 35]. The implications (5.2) \implies (5.3) \implies (5.4) are obvious. Finally (5.4) implies that $X$ is $C^*$-smooth according to Theorem 4, which for locally connected continua is equivalent to be a dendrite (see [6, Theorem 15.11, p. 522]).

The following problem seems to be interesting.

6. **Problem.** Characterize the continua $X$ for which the converse implication to that of Theorem 4 is true, i.e., the continua $X$ such that the induced mapping $C(f_Y) : C(X \times Y) \to C(X)$ is open for each continuum $Y$. In particular, is $C^*$-smoothness of $X$ sufficient?

**References**