ISOMETRICAL EMBEDDINGS OF SEPARABLE BANACH SPACES INTO THE SET OF NOWHERE APPROXIMATIVELY DIFFERENTIABLE AND NOWHERE HÖLDER FUNCTIONS

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ABSTRACT. The well-known Banach-Mazur theorem says that every separable Banach space can be isometrically embedded into \( C([0,1]) \). We prove that this embedding can have the property that the image of each nonzero element is a nowhere approximatively differentiable and nowhere Hölder function. It improves a recent result of L. Rodriguez-Piazza where the images are nowhere differentiable functions.

INTRODUCTION

The well-known Banach-Mazur theorem says that every separable Banach space can be isometrically embedded into \( C([0,1]) \). We prove that this embedding can have the property that the image of each nonzero element is a nowhere approximatively differentiable and nowhere Hölder function. It improves a recent result of L. Rodriguez-Piazza [2] where the images are nowhere differentiable functions (for references about results concerning embeddings into subsets of \( C([0,1]) \) see [2]).

The basic idea of our proof is the same as in [2] but we use a more complicated construction which uses an idea of Malý and Zajíček [1].

Let \( \Delta \) be the Cantor set. It is well known that every separable Banach space is isometric to a subspace of \( C(\Delta) \) so the following theorem will be enough to obtain the announced result.

Theorem 1. There exist a closed subset \( K \) of \([0,1]\) homeomorphic to the Cantor set \( \Delta \) and a linear operator \( F : C(K) \to C([0,1]) \) such that for every \( f \in C(K) \setminus \{0\} \) we have:

(i) \( Ff(t) = f(t) \) for every \( t \in K \), so \( Ff \) is a continuous extension of \( f \) to the whole interval.

(ii) \( \| f \|_\infty = \| Ff \|_\infty \), so \( Ff \) is an isometry.

(iii) \( Ff \) is nowhere approximatively differentiable and nowhere Hölder function.

In fact we will prove a stronger result:

Proposition 2. Let \( \varphi : [0,\infty) \to [0,\infty) \) be a continuous increasing function such that \( \varphi(0) = 0 \). Then there exist a closed subset \( K \) of \([0,1]\) homeomorphic to the Cantor set \( \Delta \) and a linear operator \( F : C(K) \to C([0,1]) \) such that for every \( f \in C(K) \setminus \{0\} \) we have:

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(i) \( Ff(t) = f(t) \) for every \( t \in K \).
(ii) \( \| f \|_{\infty} = \| Ff \|_{\infty} \).
(iii) For all \( n \in \mathbb{N} \) it holds that the set \( D_z = \{ y \in [0,1] : \frac{|Ff(z) - Ff(y)|}{|z - y|} > n \} \) has the symmetric upper density 1 at \( z \) for all \( z \in (0,1) \), the set \( D_0 \) has the right upper density 1 at point 0 and the set \( D_1 \) has the left upper density 1 at point 1.

Remark. Proposition 2 not only gives us that \( Ff \) does not have finite approximate derivative but also that \( Ff \) does not have a finite one-sided preponderant derivative as well (for definition of the preponderant derivative see [3, pages 112–113]).

As a by-product of our construction we will obtain:

**Proposition 3.** There exists a nonempty perfect set \( K \subset [0,1] \) such that every continuous function on \( K \) can be extended to \([0,1]\) such that every point \( z \in [0,1] \) is a Jarník point of the extension.

Recall definitions of some notions used above. Suppose that \( f \) is measurable on \([0,1]\).

Let \( x \in [0,1] \) and \( r \in \overline{\mathbb{R}} \). We say that \( \text{ap-lim}_{y \rightarrow x} f = r \), if for each neighborhood \( U \) of \( r \)
\[
\lim_{h \rightarrow 0^+} \frac{|\{y \in [x - h, x + h] \cap [0,1] : f(y) \in U\}|}{|x - h, x + h| \cap [0,1]|} = 1,
\]
where \( |M| \) denotes the Lebesgue measure on \( \mathbb{R} \). The function \( f \) is said to be approximatively differentiable at a point \( x \in [0,1] \) if there exists \( L \in \mathbb{R} \) such that
\[
\text{ap-lim}_{y \rightarrow x} \frac{f(y) - f(x)}{y - x} = L.
\]
A point \( x \in [0,1] \) is a Jarník point of \( f \) if
\[
\text{ap-lim}_{y \rightarrow x} \frac{|f(y) - f(x)|}{|y - x|} = \infty.
\]
We say that \( f \) on \([0,1]\) is a nowhere Hölder function if for all \( x \in [0,1] \) and \( \alpha > 0 \)
\[
\sup_{y \in [0,1]} \frac{|f(x) - f(y)|}{|x - y|^\alpha} = \infty
\]
holds.

If we apply Proposition 2 for \( \varphi(t) = -\frac{1}{\ln t} \) we get that property (iii) of Proposition 2 clearly implies property (iii) of Theorem 1. Thus we devote the rest of the paper to the proofs of Proposition 2 and Proposition 3.

**Construction of useful sequences**

Let \( \varphi \) be as in Proposition 2. Put \( \psi = \sqrt{\varphi} \). We can clearly find a sequence \( \{a_n\}_{n=1}^\infty \) such that \( 1 > a_n > 0 \),
\[
(1) \quad \sum_{j=n+1}^\infty a_j \leq \frac{a_n}{2n},
\]
\[
(2) \quad \sum_{j=n}^\infty a_j \leq \frac{1}{2n}.
\]
Further define inductively a sequence \( \{p_n\}_{n=1}^{\infty} \) such that \( p_n > 0 \,

\begin{align*}
(3) & \quad p_1 < \frac{1}{10}, \\
(4) & \quad 2\psi((n-1)p_{n-1}) \leq \frac{a_n}{n} \text{ for } n = 2, 3, \ldots, \\
(5) & \quad 2\pi p_n \sum_{j=1}^{n-1} \frac{a_j}{p_j} \leq \frac{a_n}{n} \text{ for } n = 2, 3, \ldots, \\
(6) & \quad a_n \sqrt{n-1} p_{n-1} > 10np_n \text{ for } n = 2, 3, \ldots, \\
(7) & \quad np_n \searrow 0.
\end{align*}

Put
\[ \lambda_n = \sqrt{np_n} \quad \text{for } n \in \mathbb{N}. \]

**Construction of \( K \)**

Define \( I_{0,1} = [0,1] \) and \( \lambda_0 = 1 \). For every \( n \in \mathbb{N} \) we will choose \( 2^n \) pairwise disjoint closed intervals \( \{I_{n,j}\}_{j=1}^{2^n} \) in \([0,1]\), \( I_{n,j} = [c_{n,j}, d_{n,j}] \), such that \( |I_{n,j}| = \lambda_n \). We will also require that
\begin{align*}
(9) & \quad I_{n,2j-1} \cup I_{n,2j} \subseteq \text{Int}(I_{n-1,1}), \\
(10) & \quad \frac{c_{n,1}}{p_n} \in \mathbb{N}, \frac{c_{n,j+1} - d_{n,j}}{p_n} \in \mathbb{N} \text{ for all } j \in \{1, \ldots, 2^n - 1\}, \\
(11) & \quad c_{n,2j-1} - c_{n-1,j} \geq \frac{\lambda_{n-1}}{5}, \ c_{n,2j} - d_{n,2j-1} \geq \frac{\lambda_{n-1}}{5} \text{ and } d_{n-1,j} - d_{n,2j} \geq \frac{\lambda_{n-1}}{5}.
\end{align*}

Let us define intervals \( I_{n,j} \). Suppose that for a certain \( n \in \mathbb{N} \) we have defined all \( I_{n-1,j} \). Divide each \( I_{n-1,j} \) into five intervals of equal length \( \frac{\lambda_{n-1}}{5} \). If we choose the interval \( I_{n,2j-1} \) inside the second one, and \( I_{n,2j} \) inside the fourth one, then (9) and (11) clearly hold. Since by (6) and (8) \( p_n + \lambda_n < \frac{\lambda_{n-1}}{5} \), it is easy to see that we can choose subsequently \( I_{n,1}, I_{n,2}, \ldots, I_{n,2^n} \) such that moreover \( |I_{n,j}| = \lambda_n \) and (10) holds.

Put \( K_n = I_{n,1} \cup I_{n,2} \cup \cdots \cup I_{n,2^n} \) and \( K = \bigcap_{n \geq 1} K_n \). Clearly \( K \) is homeomorphic to the Cantor set.

**Lemma 1.** Let \( n > 2 \) and \( (z-h, z+h) \subset [0,1] \) be an interval such that \( h \leq (n-1)p_{n-1} \). Then \( (z+h, z-h) \) intersects at most two components of \( K_n \).

**Proof.** Thanks to (6) and (8) we have
\[ 2h \leq 2(n-1)p_{n-1} < \frac{1}{5} \sqrt{n-2p_{n-2}} = \frac{\lambda_{n-2}}{5}. \]

So by (11) we obtain that \((z-h, z+h)\) intersects at most one component of \( K_{n-1} \) and thus at most two components of \( K_n \). \( \square \)
CONSTRUCTION OF $T$

Construction of $T$ is analogous to the construction in Lemma 2 from Rodriguez-Piazza \[2\].

**Lemma 2.** There exists a linear $T : C(K) \to C([0, 1])$ such that for all $f \in C(K)$:

(a) $Tf(t) = f(t)$ for all $t \in K$.

(b) $|Tf(t)| \leq \left(1 - \frac{1}{2^n}\right) \| f \|_\infty$ for all $t \notin K_n$.

(c) Let $n > 1$. If $I \subset [0, 1]$ is an interval such that $I \cap K_n = \emptyset$, then $Tf$ is Lipschitz on $I$ with the constant $\left(\frac{2\| f \|_\infty}{\sqrt{n} - 1}\right)$.

**Proof.** For every $n \geq 0$ and every $j \in \{1, \ldots, 2^n\}$ pick a point $x_{n,j} \in I_{n,j} \cap K$. We define $Tf(t) = f(t)$ for every $t \in K$. For every $n \geq 0$ and every $j \in \{1, \ldots, 2^n\}$, we define

$$Tf(c_{n,j}) = Tf(d_{n,j}) = f(x_{n,j}) \left(1 - \frac{1}{2^n}\right).$$

Extend $Tf$ affinely on every interval $[a, b]$ where $f$ has been defined in points $a, b$ above and $f$ has not been defined in points of interval $(a, b)$ above. Conditions (a) and (b) are clearly fulfilled. It is easy to see that $F$ is a linear operator and it is obvious that $Ff$ is a continuous function on $[0, 1]$. Now verify (c). Clearly $|f(a) - f(b)| \leq 2\| f \|_\infty$ for endpoints $a, b$ of any interval $(a, b)$ on which $f$ has been defined affinely and (11) implies that every such interval which does not intersect $K_n$ has the length at least $\frac{1}{5}\sqrt{n} - 1p_{n-1} = \frac{\lambda_{n-1}}{5}$. \qed

CONSTRUCTION OF $F$

Choose a sequence $\{y_n\}_{n=1}^\infty$ dense in $K$ and define functions $f_n$:

$$f_n(t) = 0 \text{ for } t \in K_n,$$

$$f_n(t) = a_n \sin \left(\frac{2\pi t}{p_n}\right) \text{ for } t \in [0, c_{n,1}],$$

$$f_n(t) = a_n \sin \left(\frac{2\pi (t - d_{n,j})}{p_n}\right) \text{ for } t \in [d_{n,j}, c_{n,j+1}],$$

$$f_n(t) = a_n \sin \left(\frac{2\pi (t - d_{n,2^n})}{p_n}\right) \text{ for } t \in [d_{n,2^n}, 1].$$

Notice that

$$f_n(c_{n,1}) = f_n(c_{n,j+1}) = 0$$

by (10). Put

$$Ff(t) = Tf(t) + \sum_{n=1}^\infty f(y_n) f_n(t).$$

Thanks to (2) this series converges uniformly. Thanks to (2) and Lemma 2, $F : C(K) \to C([0, 1])$ is a linear isometry so conditions (i) and (ii) from Proposition 2 are fulfilled.
Properties of \( F \)

We will need the following simple fact.

**Lemma 3.** Let \( I \) be an interval of length \( p > 0 \), \( M \subset I \), \( 0 < \alpha < 1 \) and \( \beta \in \mathbb{R} \).
For all \( x, y \in M \) let
\[
\left| \sin\left(\frac{2\pi x}{p} + \beta\right) - \sin\left(\frac{2\pi y}{p} + \beta\right) \right| \leq \alpha.
\]
Then \( |M| \leq \frac{3\alpha}{\pi} \arccos(1 - \alpha) \).

**Proof.** Lemma 3 is proved in [1, Lemma 1] in the special case \( \beta = 0 \). Applying this [1, Lemma 1] to the interval \( I^* = I - \frac{p\beta}{2\pi} \) and the set \( M^* = M - \frac{p\beta}{2\pi} \) we obtain our lemma.

Choose \( z \in [0, 1] \) and denote \( \tilde{f} = F(f) \). We shall prove that, if \( f \neq 0 \), then for \( \tilde{f} \) condition (iii) from Proposition 2 holds. It is enough to prove that
\[
(12)
\]
the set \( S = \{ x : |\tilde{f}(x) - \tilde{f}(z)| \leq \psi(|x - z|) \} \) has symmetric lower density 0 at \( Z \) for \( z \in (0, 1) \) and one-sided lower density 0 at points 0 and 1.

Choose arbitrary \( 1 > \delta > 0 \) and put \( M = \{ n : |f(y_n)| > \delta \| f \|_{\infty} \} \). If \( f \neq 0 \), then \( \text{card}(M) = \infty \). Choose an arbitrary \( 0 < h < p_1 \). Thanks to (7) there is a unique \( n = n(h) \) such that \( np_n < h \leq (n - 1)p_{n-1} \). First prove that
\[
(13)
\]
if \( n \in M \) is big enough and \( I \subset (z - h, z + h) \setminus K_n \) is an interval of length \( p_n \), then
\[
\frac{|I \cap S|}{|I|} < \arccos\left(1 - \frac{C}{n}\right),
\]
where \( C = 4 \max\left(\frac{1}{\delta}, \frac{1}{\delta \| f \|_{\infty}}\right) \). Choose \( x, y \in S \cap I \). The definition of \( S \) and (4) give
\[
|\tilde{f}(x) - \tilde{f}(y)| \leq |\tilde{f}(x) - \tilde{f}(z)| + |\tilde{f}(y) - \tilde{f}(z)| \leq \psi(|x - z|) + \psi(|y - z|)
\leq 2\psi((n - 1)p_{n-1}) \leq \frac{a_n}{n}.
\]

Put \( s_n(x) = \sum_{j=1}^{n} f(y_j) f_j(x) \) and \( r_n(x) = \sum_{j=n+1}^{\infty} f(y_j) f_j(x) \). From (1) and (4) we have
\[
|s_{n-1}(x) - s_{n-1}(y)| \leq |x - y| \sup_{t \in [x, y]} |(s'_{n-1})(t)|
\leq 2\pi|x - y| \| f \|_{\infty} \sum_{i=1}^{n-1} \frac{a_i}{p_i} \leq 2\pi p_n \| f \|_{\infty} \sum_{i=1}^{n-1} \frac{a_i}{p_i} \leq \frac{a_n}{n} \| f \|_{\infty}
\text{and}
\]
\[
|r_n(x) - r_n(y)| \leq 2\| f \|_{\infty} \sum_{j=n+1}^{\infty} a_j \leq \frac{a_n}{n} \| f \|_{\infty}.
\]
Now from Lemma 2 (c) and (6) we obtain

\[ |T f(x) - T f(y)| \leq |x - y| \frac{2 \| f \|_\infty}{\sqrt[6]{n - 1}} \leq p_n \frac{2 \| f \|_\infty}{\sqrt[6]{n - 1}} \leq a_n \frac{\| f \|_\infty}{n} \cdot \]

Since \( f(y_n) f_n = \tilde{f} - r_n - s_{n-1} - T f \), we obtain

\[
\left| \sin \left( \frac{2\pi x}{p_n} + \beta \right) - \sin \left( \frac{2\pi y}{p_n} + \beta \right) \right| = \frac{1}{a_n|f(y_n)|} |f(y_n) f_n(x) - f(y_n) f_n(y)| \\
\leq \frac{1}{a_n|f(y_n)|} (|\tilde{f}(x) - \tilde{f}(y)| + |s_{n-1}(x) - s_{n-1}(y)|) \\
+ |r_n(x) - r_n(y)| + |T f(x) - T f(y)|) \\
\leq \frac{1}{\delta \| f \|_\infty} + \frac{3}{\delta \| f \|_\infty} n \leq \max \left( \frac{1}{\delta \| f \|_\infty}, \frac{1}{\delta \| f \|_\infty} \right) = \frac{C}{n}. \]

Thus Lemma 3 gives that

\[ |I \cap S| \leq \frac{3p_n}{\pi} \arccos \left( 1 - \frac{C}{n} \right) \quad \text{whenever} \quad \frac{C}{n} < 1. \]

So (13) really holds for \( n \) big enough.

Choose arbitrary \( z \in [0, 1) \) and \( h < 1 - z \). From Lemma 1 we get that \((z, z + h)\) intersects at most two components of \( K_n(h) \) and \( h > n(h)p_n \) so \( \lim_{h \to 0^+} n(h) = \infty \), we obtain

\[ |S \cap (z, z + h)| = 0. \]

This all together gives us

\[ \lim_{n(h) \in M} \frac{|S \cap (z, z + h)|}{n(h)} = 0. \]

Analogously for \( z \in (0, 1] \) it holds that

\[ \lim_{n(h) \in M} \frac{|S \cap (z - h, z)|}{n(h)} = 0. \]

Since \( \text{card}(M) = \infty \), we obtain that (12) holds and we are done.

**Proof of Proposition 3.** If \( f(t) \neq 0 \) for all \( t \in K \), then there exists \( 1 > \delta > 0 \) such that \( M = \mathbb{N} \). In this case our construction gives us that \( \lim_{h \to 0^+} \frac{|S \cap (z, z + h)|}{h} = 0 \) and \( \lim_{h \to 0^+} \frac{|S \cap (z - h, z)|}{h} = 0 \). So every point \( z \in [0, 1] \) is a Jarník point of \( F f \).

For a given continuous function \( f \) on \( K \) find \( c \in \mathbb{R} \) such that \( f(t) + c \neq 0 \) for all \( t \in K \). Then \( F(f + c) - c \) is the desired extension of \( f \).
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REFERENCES


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