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A NOTE ON SUMMABILITY OF MULTIPLE LAGUERRE EXPANSIONS

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ABSTRACT. A simple structure of the multiple Laguerre polynomial expansions is used to study the Cesàro summability above the critical index for the convolution type Laguerre expansions. The multiple Laguerre polynomial expansion of an ℓ^1-radial function $f_0(|\mathbf{x}|)$ is shown to be an ℓ^1-radial function that coincides with the Laguerre polynomial expansion of f_0, which allows us to settle the problem of summability below the critical index for the ℓ^1-radial functions.

INTRODUCTION

The aim of this note is to use a simple structure of the multiple Laguerre expansions to study their Cesàro summability. Let L_n^α, $\alpha > -1$, denote the Laguerre polynomial of degree n on the half line $\mathbb{R}_+ = [0, \infty)$. The polynomials L_n^α form an orthogonal system in $L^2(\mathbb{R}_+, x^\alpha e^{-x} dx)$. The multiple Laguerre polynomials are products of $L_{n_i}^{\alpha_i}$ that form an orthogonal system in $L^2(\mathbb{R}_+^d, \mathbf{x}^\alpha e^{-|\mathbf{x}|} d\mathbf{x})$, where $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $|\mathbf{x}| = x_1 + \cdots + x_d$. By considering the Laguerre functions $\mathcal{L}_n^\alpha(x) = L_n^\alpha(x) e^{-x/2} x^{-\alpha/2}$ and their other varieties, $\mathcal{L}_n^\alpha(x^2/2) x^{-\alpha}$ or $\mathcal{L}_n^\alpha(x^2)(2x)^{1/2}$, one can form orthogonal systems in $L^2(\mathbb{R}_+, dx)$ and $L^2(\mathbb{R}_+, x^{2\alpha+1} dx)$. Consequently, there are at least four types of Laguerre expansions on $\mathbb{R}_+$ studied in the literature and each has its extension in the multiple setting (see [5] for the definitions). An excellent reference on the summability of Laguerre expansion is the monograph [6] by Thangavelu; see also the references in [6] for various other studies.

Our point of view is from that of orthogonal polynomials. Thus, we study the case that the multiple Laguerre polynomials form an orthogonal basis in $L^2(\mathbb{R}_+^d, \mathbf{x}^\alpha e^{-|\mathbf{x}|} d\mathbf{x})$. Recently, we studied the summability of classical orthogonal polynomial expansions on the unit ball and on the standard simplex of $\mathbb{R}^d$ in [9], [10], and found that they are quite different from that of classical orthogonal polynomial expansions on the unit cube, and are in fact easier to deal with. The reason lies in the fact that the orthogonal structures on the ball and on the simplex are related to the orthogonal structure on the unit sphere, and the classical orthogonal polynomials are related to a special case of Dunkl's theory of spherical harmonic

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([1], [2]) for reflection invariant measures; being so, they inherit a reflection symmetry (in the special case, the reflection group being $\mathbb{Z}_2^d$), which offers easier access to the system, at least for the purpose of studying summability, than the product nature of the orthogonal structure on the cube. It is then interesting to look at the multiple Hermite and Laguerre polynomials since they both have the reflection symmetry (after composite with $\{\cdot\}^2$ in the Laguerre case) and are product in nature. The summability of the multiple Hermite expansions has been studied in [6], as well as the summability of the type IV multiple Laguerre expansions. The purpose of this note is to show that the multiple Laguerre polynomial expansions admit a simple structure that allows us to derive results on summability above the critical index from that of one variable. The expansions being considered are essentially those of type III (see [5] for the definition). The result indicates an interesting parallel between the orthogonal structure on the simplex and on $\mathbb{R}_+^d$. Moreover, we show that the Laguerre expansion of an ℓ^1-radial function $f(\mathbf{x}) = f_0(|\mathbf{x}|)$ on $\mathbb{R}^d$ is the same as the Laguerre expansion of the function f_0 on $\mathbb{R}_+$, which allows us to settle the summability below the critical index for the ℓ^1-radial functions.

2. Summability of Laguerre expansions

For $\alpha > -1$, the Laguerre polynomials of type α are defined by the generating function

$$\sum_{n=0}^{\infty} L_n^\alpha(x) r^n = (1-r)^{-\alpha-1} e^{-xr/(1-r)}, \qquad |r| < 1. \tag{2.1}$$

Let $w_\alpha(x) = x^\alpha e^{-x}/\Gamma(\alpha+1)$, which is normalized to have unit integral on $\mathbb{R}_+ = [0,\infty)$. It is known that $L_n^\alpha(0) = \binom{n+\alpha}{n}$ and the polynomials

$$\tilde{L}_n^\alpha(x) = L_n^\alpha(x)/\sqrt{L_n^\alpha(0)}$$

form an orthonormal basis in $L^2(\mathbb{R}_+, w_\alpha dx)$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i > -1$, the multiple Laguerre polynomials are defined by the formula

$$L_{\mathbf{k}}^\alpha(\mathbf{x}) = L_{k_1}^{\alpha_1}(x_1) \cdots L_{k_d}^{\alpha_d}(x_d), \qquad \mathbf{k} = (k_1, \ldots, k_d).$$

We define the polynomials $\tilde{L}_{\mathbf{k}}^\alpha$ similarly. The set $\{\tilde{L}_{\mathbf{k}}^\alpha\}$ forms an orthonormal basis of $L^2(\mathbb{R}_+^d, W_\alpha d\mathbf{x})$, where

$$W_\alpha(\mathbf{x}) = w_{\alpha_1}(x_1) \cdots w_{\alpha_d}(x_d) = \mathbf{x}^\alpha e^{-|\mathbf{x}|} \Big/ \prod_{i=1}^d \Gamma(\alpha_i + 1).$$

Throughout this paper, we write $|\mathbf{x}| = x_1 + \cdots + x_d$. The degree of the multiple Laguerre polynomial $L_{\mathbf{k}}^\alpha$ is the total degree $|\mathbf{k}|$.

Associated to the family $L_{\mathbf{k}}^\alpha$, we have a Fourier expansion for each $f \in L^2(\mathbb{R}_+^d, W_\alpha d\mathbf{x})$,

$$f = \sum_{n=0}^{\infty} \sum_{|\mathbf{k}|=n} \langle f, \tilde{L}_{\mathbf{k}}^\alpha \rangle \tilde{L}_{\mathbf{k}}^\alpha,$$

where $\langle f, g \rangle$ stands for the inner product in the space $L^2(\mathbb{R}_+^d, W_\alpha d\mathbf{x})$. Let us denote the reproducing kernel of the multiple Laguerre polynomials of degree n by

$P_n(W_\alpha;\mathbf{x},\mathbf{y})$. By definition, it follows that

$$P_n(W_\alpha;\mathbf{x},\mathbf{y}) = \sum_{|\mathbf{k}|=n} \tilde{L}^\alpha_{\mathbf{k}}(\mathbf{x})\tilde{L}^\alpha_{\mathbf{k}}(\mathbf{y}).$$

For a function $f \in L^2(\mathbb{R}^d_+, W_\alpha d\mathbf{x})$, the projection of f on the space generated by the multiple Laguerre polynomials of degree n is defined by

$$P_n(W_\alpha; f, \mathbf{x}) = \sum_{|\mathbf{k}|=n} \langle f, \tilde{L}^\alpha_{\mathbf{k}}\rangle \tilde{L}^\alpha_{\mathbf{k}}(\mathbf{x}) = \int_{\mathbb{R}^d_+} f(\mathbf{y}) P_n(W_\alpha;\mathbf{x},\mathbf{y}) W_\alpha(\mathbf{y}) d\mathbf{y},$$

where the second equality follows from the definition of the kernel. We note that for the ordinary Laguerre expansion on $\mathbb{R}_+$, $P_n(w_\alpha; x, y) = \tilde{L}^\alpha_n(x)\tilde{L}^\alpha_n(y)$. We study the Cesàro (C,δ) means of the Fourier expansion, which are defined by

$$S^\delta_n(W_\alpha; f) := \binom{n+\delta}{\delta}^{-1} \sum_{j=0}^n \binom{n-j+\delta}{n-j} P_j(W_\alpha; f).$$

In particular, for $\delta = 0$, $S^\delta_n(W_\alpha; f)$ is the n-th partial sum of the Fourier expansion. Let $P^\delta_n(W_\alpha;\mathbf{x},\mathbf{y})$ denote the (C,δ) means of the kernel $P_n(W_\alpha;\mathbf{x},\mathbf{y})$; then we can write $S^\delta_n(W_\alpha; f)$ as

$$S^\delta_n(W_\alpha; f, \mathbf{x}) = \int_{\mathbb{R}^d_+} f(\mathbf{y}) P^\delta_n(W_\alpha;\mathbf{x},\mathbf{y}) W_\alpha(\mathbf{y}) d\mathbf{y}.$$

For later use, we denote the (C,δ) means of the ordinary Laguerre expansion on $\mathbb{R}_+$ by $s^\delta_n(w_\alpha; f)$.

For $d = 1$, the summability of the Cesàro means has been studied by many authors (see [3], [4], [6], [7] and the references there). As we mentioned in the introduction, there are at least 4 different types of summability of Laguerre expansions. We start with the following result on the summability at the origin, which does not depend on the types.

Theorem 2.1. *If f is continuous at the origin and if it satisfies*

$$\int_{\mathbf{x}\in\mathbb{R}^d_+, |\mathbf{x}|\ge 1} |f(\mathbf{x})\mathbf{x}^\alpha| \cdot |\mathbf{x}|^{-\delta-1/2} e^{-|\mathbf{x}|/2} d\mathbf{x} < \infty, \tag{2.2}$$

then the Cesàro (C,δ) means of the Laguerre expansion of f converges at the origin if and only if $\delta > |\alpha| + d - 1/2$.

Proof. The generating formula (2.1) of the Laguerre polynomials can be written as

$$\sum_{n=0}^\infty \tilde{L}^\alpha_n(0)\tilde{L}^\alpha_n(x) r^n = (1-r)^{-\alpha-1} e^{-xr/(1-r)}, \qquad |r| < 1.$$

Multiplying the above formula leads to a generating formula for $P_n(W_\alpha;\mathbf{x},0)$,

$$\sum_{n=0}^\infty P_n(W_\alpha;\mathbf{x},0) r^n = \sum_{n=0}^\infty \sum_{|\mathbf{k}|=n} \tilde{L}^\alpha_{\mathbf{k}}(0)\tilde{L}^\alpha_{\mathbf{k}}(\mathbf{x}) r^n = (1-r)^{-|\alpha|-d} e^{-|\mathbf{x}|r/(1-r^2)},$$

which implies readily that $P_n(W_\alpha;\mathbf{x},0) = L^{|\alpha|+d-1}_n(|\mathbf{x}|)$. Multiplying the above formula by the power expansion of $(1-r)^{-\delta-1}$, it follows that

$$P^\delta_n(W_\alpha;\mathbf{x},0) = \binom{n+\delta}{n}^{-1} L^{|\alpha|+\delta+d}_n(|\mathbf{x}|). \tag{2.3}$$

Therefore, we conclude that

$$S_n^\delta(W_\alpha; f, 0) = \frac{1}{\binom{n+\delta}{n}} \int_{\mathbb{R}_+^d} f(\mathbf{x}) L_n^{|\alpha|+d+\delta}(|\mathbf{x}|) W_\alpha(\mathbf{x}) d\mathbf{x}.$$

Let Σ^{d-1} denote the simplex $\Sigma^{d-1} = \{\mathbf{x} \in \mathbb{R}_+^d : |\mathbf{x}| = 1\}$. For $\mathbf{x} \in \mathbb{R}_+^d$, write $\mathbf{x} = r\mathbf{x}'$ such that $|\mathbf{x}'| = 1$ and $r \in \mathbb{R}_+$. Then we have

$$\int_{\mathbb{R}_+^d} f(\mathbf{x}) g(|\mathbf{x}|) d\mathbf{x} = \int_0^\infty \int_{\Sigma^{d-1}} f(r\mathbf{x}') d\mathbf{x}' r^{d-1} g(r) dr. \tag{2.4}$$

Using this formula it follows that

$$\begin{aligned} &S_n^\delta(W_\alpha; f, 0) \\ &\quad = \frac{1}{\binom{n+\delta}{n}} \int_0^\infty \Big(\int_{\Sigma^{d-1}} f(r\mathbf{y}) \mathbf{y}^\alpha d\mathbf{y} \Big) L_n^{|\alpha|+d+\delta}(r) r^{|\alpha|+d-1} e^{-r} dr / \prod_{i=1}^d \Gamma(\alpha_i + 1). \end{aligned}$$

The right-hand side is the (C, δ) means of the Laguerre expansion of a one variable function. Indeed, if we define $F(r) = \sigma_d \int_{\Sigma^{d-1}} f(r\mathbf{y}) \mathbf{y}^\alpha d\mathbf{y}$, where σ_d is a constant defined by $\sigma_d = \Gamma(|\alpha| + d) / \prod_{i=1}^d \Gamma(\alpha_i + 1)$, then we can write the above equation as

$$S_n^\delta(W_\alpha; f, 0) = s^\delta(w_{|\alpha|+d-1}; F, 0).$$

We note that $\sigma_d = 1 / \int_{\Sigma^{d-1}} \mathbf{y}^\alpha d\mathbf{y}$, so that F is just the average of f over the simplex Σ^{d-1}. This allows us to use the summability of Laguerre expansion of one variable. The desired result follows from [7, Theorem 9.1.7, p. 247]. The condition of that theorem is verified as follows:

$$\begin{aligned} \int_1^\infty |F(r)| r^{|\alpha|+d-1-\delta-1/2} e^{-r/2} dr &\le \sigma_d \int_1^\infty \int_{\Sigma^{d-1}} |f(r\mathbf{y}) \mathbf{y}^\alpha| d\mathbf{y} r^{|\alpha|+d-\delta-3/2} e^{-r/2} dr \\ &\le \sigma_d \int_{\mathbf{x} \in \mathbb{R}_+^d, |\mathbf{x}| \ge 1} |f(\mathbf{x}) \mathbf{x}^\alpha| \cdot |\mathbf{x}|^{-\delta-1/2} e^{-|\mathbf{x}|/2} d\mathbf{x}, \end{aligned}$$

so that the left-hand side is finite under (2.2), and it is evident that F is continuous at $r = 0$ if f is continuous at the origin. ∎

In [3], Görlich and Markett introduced a convolution structure for the Laguerre expansions and used it to show that the summability of the Laguerre expansion at the origin implies a summability result for functions in a proper space. The convolution structure can be extended to the multiple Laguerre expansions, so can the results on summability be extended. We start with the definition of the corresponding spaces for functions of several variables. For $1 \le p < \infty$, we denote by $L^p(\mathbb{R}_+^d, W_\alpha d\mathbf{x})$ the space of the Lebesgue integrable functions with finite L^p norm, where the norm is defined by

$$\|f\|_{p,W_\alpha} = \left(\int_{\mathbb{R}_+^d} |f(\mathbf{x}) e^{-|\mathbf{x}|/2}|^p \mathbf{x}^\alpha d\mathbf{x} \right)^{1/p}, \quad 1 \le p < \infty,$$

and for $p = \infty$, we define $L^\infty(\mathbb{R}_+^d, W_\alpha d\mathbf{x})$ as the space of continuous functions with finite norm

$$\|f\|_{\infty,W_\alpha} = \sup_{\mathbf{x} \in \mathbb{R}_+^d} |f(\mathbf{x}) e^{-|\mathbf{x}|/2}|.$$

The convolution structure of the Laguerre expansion is defined in terms of a translation operator, which is motivated by the following product formula of Laguerre polynomials:

$$L_n^\alpha(x)L_n^\alpha(y) = \frac{\Gamma(n+\alpha+1)2^\alpha}{\Gamma(n+1)\sqrt{2\pi}} \int_0^\pi L_n^\alpha(x+y+2\sqrt{xy}\cos\theta)e^{-\sqrt{xy}\cos\theta} \times j_{\alpha-\frac{1}{2}}(\sqrt{xy}\sin\theta)\sin^{2\alpha}\theta d\theta,$$

where j_μ is the Bessel function of half-integer order. The Laguerre translation operator $T_x^\alpha f$ of a function f on R_+ is defined by

$$T_x^\alpha f(y) = \frac{\Gamma(\alpha+1)2^\alpha}{\sqrt{2\pi}} \int_0^\pi f(x+y+2\sqrt{xy}\cos\theta)e^{-\sqrt{xy}\cos\theta} \times j_{\alpha-1/2}(\sqrt{xy}\sin\theta)\sin^{2\alpha}\theta d\theta.$$

The product formula implies that $L_n^\alpha(x)L_n^\alpha(y) = L_n^\alpha(0)T_x^\alpha L_n^\alpha(y)$. The convolution of f and g is defined by

$$(f * g)(x) = \int_0^\infty f(y)T_x^\alpha g(y)y^\alpha e^{-y}dy. \tag{2.5}$$

The following fundamental result is proved in [3].

Theorem 2.2. *Let $\alpha \geq 0$ and $1 \leq p \leq \infty$. Then for $f \in L^p(\mathbb{R}_+, w_\alpha dx)$ and $g \in L^1(\mathbb{R}_+, w_\alpha dx)$,*

$$\|f * g\|_{p,w_\alpha} \leq \|f\|_{p,w_\alpha}\|g\|_{1,w_\alpha}. \tag{2.6}$$

By the definition of the (C,δ) means and the product formula, it follows that

$$s_n^\delta(w_\alpha; f, x) = f * P_n^\delta(w_\alpha, x, 0).$$

Hence, the previous theorem allows us to derive the general summability result from the summability at the origin. The analogy structure holds for the multiple expansions, which leads to the following result.

Theorem 2.3. *Let $\alpha_i \geq 0$, $1 \leq i \leq d$, and $1 \leq p \leq \infty$. The Cesàro (C,δ) means of the Laguerre expansion is uniformly bounded as an operator on $L(\mathbb{R}_+^d, W_\alpha d\mathbf{x})$ if $\delta > |\alpha| + d - 1/2$. Moreover, for $p = 1$ and ∞, the (C,δ) means converge in the $L^p(\mathbb{R}_+^d, W_\alpha d\mathbf{x})$ norm if and only if $\delta > |\alpha| + d - 1/2$.*

Proof. The product nature of the multiple Laguerre polynomials yields that

$$P_n^\delta(W_\alpha; \mathbf{x}, \mathbf{y}) = T_{x_1}^{\alpha_1} \cdots T_{x_d}^{\alpha_d} P_n^\delta(W_\alpha; 0, \mathbf{y}),$$

where T_{x_i} acts on the variable y_i. Therefore, it follows that

$$S_n^\delta(W_\alpha; f, \mathbf{x}) = \int_{\mathbb{R}_+^d} f(\mathbf{y})T_{x_1}^{\alpha_1} \cdots T_{x_d}^{\alpha_d} P_n^\delta(W_\alpha; 0, \mathbf{y})W_\alpha(\mathbf{y})d\mathbf{y}$$

which can be written as a d-fold convolution using the definition (2.5) in an obvious way. Therefore, if we apply the inequality d times, we end up with

$$\|S_n^\delta(W_\alpha; f)\|_{p,W_\alpha} \leq \|P_n^\delta(W_\alpha; 0, \cdot)\|_{1,W_\alpha}\|f\|_{p,W_\alpha}.$$

To estimate the norm of $P_n^\delta(W_\alpha, 0, \mathbf{x})$, we use the formula (2.3) and (2.4) to write

$$\|P_n^\delta(W_\alpha;0,\cdot)\|_{1,W_\alpha} = \binom{n+\delta}{\delta}^{-1} \int_{\Sigma^{d-1}} y^\alpha d\mathbf{y} \\ \times \int_0^\infty |L_n^{|\alpha|+\delta+d}(r)| r^{|\alpha|+d-1} e^{-r/2} dr \Big/ \prod_{i=1}^d \Gamma(\alpha_i+1).$$

The first integral is a constant independent of n, the second integral is $\mathcal{O}(n^{-\delta})$ when $\delta > |\alpha| + d - 1/2$, and at least $\mathcal{O}(n^{-\delta}\log n)$ when $\delta \le |\alpha| + d - 1/2$ (see [4, Lemma 1] or [6, Lemma 1.5.4]). Since $\binom{n+\delta}{\delta} = n^\delta(1+\mathcal{O}(n^{-1}))/\Gamma(\delta+1)$, the integral is bounded if and only if $\delta > |\alpha| + d - 1/2$. This proves the boundedness of the operator. In fact, we can also use the result of Theorem 2.1, since the condition (2.2) is satisfied for $f \in L^p(\mathbb{R}_+^d, W_\alpha)$, $1 \le p \le \infty$. The convergence part follows from the usual density argument. $\square$

The index $|\alpha| + d - 1/2$ is the *critical index* of this type of Laguerre expansion. For $d = 1$, the above theorem is due to Görlich and Markett in [3]. The Laguerre expansion of this type is essentially the type III as defined in [5], also called the convolution type in [6]. The above theorem answers the question of Cesàro (C, δ) summability for δ above the critical index. In the following we will derive necessary conditions for the summability below the critical index. The fact that the kernel of the (C, δ) means $S_n^\delta(W_\alpha; f)$ is a function of $|\mathbf{x}|$ suggests that we examine the Laguerre expansion of an ℓ^1-radial function $f(\mathbf{x}) = f_0(|\mathbf{x}|)$, where f_0 is defined on $\mathbb{R}_+$.

Theorem 2.4. *Let f_0 be a function defined on $\mathbb{R}_+$ and define $f(\mathbf{x}) = f_0(|\mathbf{x}|)$ on $\mathbb{R}_+^d$. Then*

$$S_n^\delta(W_\alpha; f, \mathbf{x}) = s_n^\delta(w_{|\alpha|+d-1}; f_0, |x|).$$

In particular, for $1 \le p \le \infty$, the Cesàro means $S_n^\delta(W_\alpha; f)$ are uniformly bounded on $L^p(\mathbb{R}^d, W_\alpha d\mathbf{x})$ if and only if p lies in the interval

$$\frac{4|\alpha|+4d}{2|\alpha|+2d+1+2\delta} < p < \frac{4|\alpha|+4d}{2|\alpha|+2d-1-2\delta}. \tag{2.7}$$

Proof. In order to prove this result we introduce another orthonormal basis for the space of polynomials with respect to the inner product of $L^2(\mathbb{R}_+^d, W_\alpha d\mathbf{x})$. For $\mathbf{x} \in \mathbb{R}_+^d$, we write $\mathbf{x} = r\mathbf{x}'$ as before, where $r \in \mathbb{R}_+$ and $\mathbf{x}' \in \Sigma^{d-1}$. We need a basis of homogeneous orthonormal polynomials for the inner product

$$\langle f, g\rangle_\Sigma = \int_{\Sigma^{d-1}} f(\mathbf{y}) g(\mathbf{y}) \mathbf{y}^\alpha d\mathbf{y}.$$

The basis can be obtained from the classical orthogonal polynomials with respect to the weight function $x_1^{\alpha_1} \cdots x_d^{\alpha_d}(1-|\mathbf{x}|)^{\alpha_{d+1}}$ on the simplex $\{\mathbf{x} \in \mathbb{R}^{d-1} : x_1 \ge 0, \ldots, x_{d-1} \ge 0, 1 - x_1 - \cdots - x_{d-1} \ge 0\}$. Such a basis is indeed homogeneous; we showed in [8] that it can be obtained from a $\mathbb{Z}_2^d$ invariant basis of homogeneous orthonormal polynomials with respect to $\prod x_i^{2\alpha_i+1} d\omega$ on the surface of the unit sphere S^{d-1} on $\mathbb{R}^d$ by a simple mapping $(x_1^2, \ldots, x_d^2) \in S^{d-1} \mapsto (y_1, \ldots, y_d) \in \Sigma^{d-1}$. Let us denote by $\{R_\mathbf{k}^n\}$, where $\mathbf{k} \in \mathbb{N}_0^d$ and $|\mathbf{k}| = n$, such a basis of homogeneous orthonormal polynomials with respect to the inner product $\langle f, g\rangle_\Sigma$. The total degree of $R_{\mathbf{k}}^n$ is n. Then it is easy to verify, using (2.4), that the polynomials

$$P_{m,\mathbf{k}}^n(\mathbf{x}) = \tilde{L}_m^{2n-2m+|\alpha|+d-1}(r) r^{n-m} R_{\mathbf{k}}^{n-m}(\mathbf{x}'), \qquad \mathbf{x} = r\mathbf{x}',$$

where $|\mathbf{k}| = n$, $0 \le m \le n$ and $n \in \mathbb{N}_0$, form an orthonormal basis for the space of polynomials with respect to $W_\alpha d\mathbf{x}$. In particular, the reproducing kernel $P_n(W_\alpha; \mathbf{x}, \mathbf{y})$ can be written in terms of this basis as

$$P_n(W_\alpha; \mathbf{x}, \mathbf{y}) = \sum_{m=0}^{n} \sum_{|\mathbf{k}|=n-m} P_{m,\mathbf{k}}^n(\mathbf{x}) P_{m,\mathbf{k}}^n(\mathbf{y}),$$

from which we derive, using (2.4) and the orthogonality of $R_{\mathbf{k}}^m$, that

$$\begin{aligned}
&\int_{\mathbb{R}_+^d} f_0(|\mathbf{y}|) P_n(W_\alpha; \mathbf{x}, \mathbf{y}) \mathbf{y}^\alpha e^{-\mathbf{y}} d\mathbf{y} \\
&\quad = \int_0^\infty f(r) r^{|\alpha|+d-1} e^{-r} \int_{\Sigma^{d-1}} P_n(W_\alpha; \mathbf{x}, r\mathbf{y}) \mathbf{y}^\alpha d\mathbf{y} dr \\
&\quad = \sigma_d^{-1} \int_0^\infty f(r) r^{|\alpha|+d-1} e^{-r} \tilde{L}_n^{|\alpha|+d-1}(r) dr \; \tilde{L}_n^{|\alpha|+d-1}(|\mathbf{x}|).
\end{aligned}$$

Taking into account the normalization constant, we see that the above formula can be written as $P_n(W_\alpha; f, \mathbf{x}) = P_n(w_{|\alpha|+d-1}; f_0, |\mathbf{x}|)$. The desired result follows upon taking the Cesàro means. Finally, the uniform boundedness of the Cesàro means follows from (2.4) and the corresponding result of one variable ([6, p. 145] or [4]). $\square$

An immediate consequence of this theorem is the following necessary condition for the Cesàro summability below the critical index.

Corollary 2.5. *Let $0 < \delta \le |\alpha| + d - 1/2$. The Cesàro means of the Laguerre expansion is uniformly bounded on $L^p(\mathbb{R}_+, W_\alpha dx)$ only if p lies in the interval (2.7).*

For $d = 1$, the condition (2.7) is also sufficient as established in [4] and [6]. We naturally conjecture that the condition is also sufficient for $d > 1$. Theorem 2.4 says that the conjecture is true for the ℓ^1-radial functions for $d > 1$. We do not know how to prove it in the general case. There are two proofs in the case of $d = 1$. The one in [6] uses a restriction theorem and Fefferman-Stein interpolation. The other one in [4] is based on a sharp estimate of the norm of the (C, δ) means, which essentially comes down to a sharp estimate of the norm of the $(C, 0)$ means by the use of an interpolation theorem of Askey-Hirschman. The norm of the $(C, 0)$ means is usually estimated with the help of the Christoffel-Darboux formula, which provides a compact formula for the kernel. For $d > 1$, however, such a compact formula does not seem to be available. Both proofs involve technique estimates that seem to be difficult to establish in higher dimensions.

Let us mention that there is a significant difference between the Laguerre expansions of type III and type IV. In the latter case, the complete study of the summability for $1 < p < \infty$ is conducted in [6] using Kanjin's theorem of transplantation.

References

1. C. Dunkl, *Reflection groups and orthogonal polynomials on the sphere*, Math. Z. **197** (1988), 33-60. MR **89b:**42016
2. C. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311** (1989), 167-183. MR **90k:**33027
3. E. Görlich and C. Markett, *A convolution structure for Laguerre series*, Indag. Math. **44** (1982), 161-171. MR **83g:**42015
4. C. Markett, *Mean Cesàro summability of Laguerre expansions and norm estimates with shift parameter*, Analysis Math. **8** (1982), 19-37. MR **83j:**40004
5. S. Thangavelu, *Transplantation, summability and multipliers for multiple Laguerre expansions*, Tôhoku Math. J. **44** (1992), 279-298. MR **93h:**42021
6. S. Thangavelu, *Lectures on Hermite and Laguerre expansions*, Princeton University Press, Princeton, NJ, 1993. MR **94i:**42001
7. G. Szegő, *Orthogonal polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ. vol.23, Providence, RI, 1975. MR **51:**8724
8. Y. Xu, *Orthogonal polynomials and cubature formulae on spheres and on simplices*, Method Appl. of Analysis **5** (1998), 169-184. CMP 98:16
9. Y. Xu, *Summability of Fourier orthogonal series for Jacobi weights on the simplex in* $\mathbb{R}^d$, Proc. Amer. Math. Soc. **126** (1998), 3027-3036. MR **99i:**42039
10. Y. Xu, *Summability of Fourier orthogonal series for Jacobi weight on a ball in* $\mathbb{R}^d$, Trans. Amer. Math. Soc. **351** (1999), 2439-2458. MR **99i:**42040

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