TRACE-CLASS PERTURBATION AND STRONG CONVERGENCE: WAVE OPERATORS REVISITED

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ABSTRACT. We give a new construction of wave operators for a self-adjoint operator under trace-class perturbation. This construction requires no quantitative estimates.

The well-known theorem of Kato and Rosenblum [2], [5] asserts that if \(A\) and \(A'\) are self-adjoint operators and if \(A - A'\) belongs to \(\mathcal{C}_1\), the trace class, then the absolutely continuous parts of \(A\) and \(A'\) are unitarily equivalent. Because of the importance of the problem of trace-class perturbation, many improvements and generalizations of this theorem have appeared in the literature. [3] and [4] contain a reasonably complete account of the development between [2], [5] and the late 1970’s. In these works the unitary equivalence between \(A_{ac}\) and \(A'_{ac}\) was established through the existence of the wave operators

\[
W_{\pm}(A', A) = s-\lim_{\lambda \to \pm \infty} e^{-i\lambda A'} e^{i\lambda A} P_{ac}(A).
\]

In fact, as mentioned in [5], the original idea of using the operator \(e^{-i\lambda A'} e^{i\lambda A}\) dates back to Friedrichs [1]. The first generalization of the Kato-Rosenblum theorem to the setting of operator tuples was made by Voigt [8]. The use of the exponential function \(e^{i\lambda x}\) is fundamental to these “time-dependent” constructions of wave operators.

In [6], Voiculescu generalized wave operators to the setting of commuting tuples under perturbation by norm ideals of compact operators. He showed that, if \(T = (T_1, \ldots, T_N)\) and \(T' = (T'_1, \ldots, T'_N)\) are commuting tuples of self-adjoint operators such that \(T_j - T'_j \in \mathcal{C}(0), j = 1, \ldots, N,\) and if the norm ideal \(\mathcal{C}\) has the property

\[
\lim_{n \to \infty} n^{-1/2} \|\omega_1 \otimes \omega_1 + \cdots + \omega_n \otimes \omega_n\|_{\mathcal{C}} = 0,
\]

where \(\{\omega_n\}_{n=1}^\infty\) is any orthonormal set, then the wave operator for the perturbation problem \(T \to T'\) exists in the strong operator topology and is unique [6, Theorem 1.5]. This uniqueness is in sharp contrast with the problem of trace-class perturbation for single operator; in general, the two wave operators \(W_+\) and \(W_-\) do not necessarily coincide. While essentially a time-dependent approach, Voiculescu’s

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work shows that, for perturbations other than that by the trace class, it is possible to construct wave operators without explicitly using the exponential function \( e^{i\lambda x} \).

The treatment of the original theorem of Kato-Rosenblum in the framework of [6], however, proved to be elusive. In fact this was one of the problems Voiculescu raised during the 1983 ICM [7, page 1043]. Also, the reason for the fact that \( W_+ \) and \( W_- \) may differ or, equivalently, that the scattering operator \( S = W_+^* W_- \) is not necessarily \( P_{ac}(A) \), has never been made clear in the previous constructions of wave operators; one usually gets \( S \neq P_{ac}(A) \) from explicit computations.

The purpose of this note is to give a proof of the Kato-Rosenblum theorem within the context suggested by Voiculescu [6, 7]. Indeed, what we will prove is slightly stronger than the original version. Moreover, this new proof has the following three distinct features: (a) It identifies the cause for the exponential functions themselves. (b) Unlike previous ones, our proof involves no quantitative estimates. Indeed, our proof is surprisingly soft.

We start by recalling a few well-known facts. For the rest of the paper, \( H \) and \( M \) will denote the Hilbert transform and the multiplication by the coordinate function on \( \mathbb{R} \). That is, for \( f \in L^2(\mathbb{R}) \) or for \( f \in L^2(\mathbb{R}, \mathcal{M}) = L^2(\mathbb{R}) \otimes \mathcal{M} \), where \( \mathcal{M} \) is a Hilbert space, we write

\[
(Hf)(x) = \frac{1}{\pi} \text{p.v.} \int \frac{f(y)}{y-x} dy \quad \text{and} \quad (Mf)(x) = xf(x).
\]

Recall that \( e^{i\lambda M} H e^{-i\lambda M} = \chi_{(0,\infty)}(D - \lambda) - \chi_{(-\infty,0)}(D - \lambda) \), where \( D \) is the differential operator \((1/i)d/dx\). Hence

\[
s-\lim_{\lambda \to -\infty} e^{i\lambda M} H e^{-i\lambda M} = -1 \quad \text{and} \quad s-\lim_{\lambda \to -\infty} e^{i\lambda M} H e^{-i\lambda M} = 1.
\]

Also recall that if \( A \) and \( A' \) are self-adjoint operators, \( z \in \mathbb{C} \setminus \mathbb{R} \) and \( \lambda \in \mathbb{R} \), then

\[
(A' - z)^{-1}(e^{i\lambda A'} - e^{i\lambda A})(A - z)^{-1} = i \int_0^\lambda e^{isA'} \{(A - z)^{-1} - (A' - z)^{-1}\} e^{(\lambda - s)\lambda} ds.
\]

Therefore if \( (A - z)^{-1} - (A' - z)^{-1} \) belongs to the trace class, then so does \( (A' - z)^{-1}(e^{i\lambda A'} - e^{i\lambda A})(A - z)^{-1} \). As it turns out, these are the only properties of the exponential function which are relevant to the construction of wave operators.

A sequence of Borel functions \( \{\varphi_n\} \) on \( \mathbb{R} \) is said to be of class \( \Omega_+ \) if

(i) \( |\varphi_n(t)| = 1 \) for all \( t \in \mathbb{R} \) and \( n \in \mathbb{N} \).

(ii) If \( A \) and \( A' \) are self-adjoint operators such that \( (A - z)^{-1} - (A' - z)^{-1} \in \mathcal{C}_1 \) for some \( z \in \mathbb{C} \setminus \mathbb{R} \), then \( (A' - z)^{-1}(\varphi_n(A') - \varphi_n(A))(A - z)^{-1} \in \mathcal{C}_1 \) for every \( n \in \mathbb{N} \).

(iii\(^+\)) \( s\text{-lim}_{n \to \infty} \varphi_n(M)H\varphi_n^*(M) = 1 \).

A sequence of Borel functions \( \{\varphi_n\} \) on \( \mathbb{R} \) is said to be of class \( \Omega_- \) if it satisfies (i), (ii) and

(iii\(^-\)) \( s\text{-lim}_{n \to \infty} \varphi_n(M)H\varphi_n^*(M) = 1 \).

It is elementary that (ii) and (i) imply

(II) If \( A \) and \( A' \) are self-adjoint operators such that \( (A - z)^{-1} - (A' - z)^{-1} \in \mathcal{C}_1 \) for some \( z \in \mathbb{C} \setminus \mathbb{R} \), then \( (\varphi_n(A') - \varphi_n(A))(A - z)^{-2} \in \mathcal{C}_1 \) for every \( n \in \mathbb{N} \).

Because \( \|(H \pm 1)\varphi_n^*(M)\| = \|\varphi_n(M)(H \pm 1)\varphi_n^*(M)\| \), (iii\(^+\)) and (iii\(^-\)) respectively imply

(III\(^+\)) \( s\text{-lim}_{n \to \infty} (H + 1)\varphi_n(M) = 0 \).

(III\(^-\)) \( s\text{-lim}_{n \to \infty} (H - 1)\varphi_n(M) = 0 \).
Thus, if \( \{\lambda_n\} \) are positive numbers such that \( \lim_{n \to \infty} \lambda_n = \infty \), then the sequence \( \{\exp(i\lambda_n x)\} \) (resp. \( \{\exp(-i\lambda_n x)\} \)) is of class \( \Omega_+ \) (resp. \( \Omega_- \)). As we will see, the dichotomy between (iii\(^+\)) and (iii\(^-\)) is the cause for \( W_+ \not= W_- \).

**Theorem.** Let \( A \) and \( A' \) be self-adjoint operators on a Hilbert space \( \mathcal{H} \) such that

\[
(A-z)^{-1} - (A'-z)^{-1} \in \mathcal{C}_1 \quad \text{for some } z \in \mathbb{C} \setminus \mathbb{R}.
\]

Then there are partial isometries \( W_+(A', A) \) and \( W_-(A', A) \) such that

\[
\begin{align*}
\text{(i)} & \quad s- \lim_{n \to \infty} \varphi_n^*(A') \varphi_n(A) P_{ac}(A) = W_+(A', A) \quad \text{and} \\
\text{(ii)} & \quad s- \lim_{n \to \infty} \psi_n^*(A') \psi_n(A) P_{ac}(A) = W_-(A', A),
\end{align*}
\]

where \( \{\varphi_n\} \) is any sequence of class \( \Omega_+ \) and \( \{\psi_n\} \) any sequence of class \( \Omega_- \).

**Proof.** We only need to establish the strong convergence; that the limits \( W_+(A', A) \) and \( W_-(A', A) \) are independent of the choices of \( \{\varphi_n\} \) and \( \{\psi_n\} \) follows from an observation borrowed from [6]: If one mixes two sequences of a given class, then one obtains a new sequence of the same class. Moreover, we will only consider the case of \( \Omega_+ \); the case of \( \Omega_- \) differs only in one detail, which will be pointed out in due course.

We may assume that \( \mathcal{H} = (\bigoplus_{j \in J} L^2(\Delta_j)) \oplus \mathcal{H}_s \), where each summand is invariant under \( A, A|\mathcal{H}_s \) is purely singular, and \( A(\bigoplus_{j \in J} L^2(\Delta_j)) = M \), the multiplication by the coordinate function. Here, each \( \Delta_j \) is a Borel set in \( \mathbb{R} \) and, as usual, \( L^2(\Delta_j) = \chi_{\Delta_j} L^2(\mathbb{R}) \). Define \( W_n = \varphi_n^*(A') \varphi_n(A) \) and \( T_{k,n} = W_n^* W_n - 1 \) for \( n, k \in \mathbb{N} \).

Pick a \( j_0 \in J \) and let \( \xi \in L^2(\Delta_{j_0}) \) be a bounded function whose support is contained in a finite interval \( I \). Define the operator \( M_\xi \) on \( \mathcal{H} \) by the formula \( M_\xi f = \xi f_{j_0} \) for \( f = (\bigoplus_{j \in J} f_j) \oplus h \), where \( h \in \mathcal{H}_s \) and \( f_j \in L^2(\Delta_j), j \in J \). Define

\[
Y_\xi = \pi i (A-z)M_\xi H + 1)M_\xi^*(A-z).
\]

For the case of \( \Omega_- \), replace the operator \( H+1 \) above by \( H-1 \). Because supp \( \xi \subset I \), \( Y_\xi \) is a bounded operator. Denote \( K = (A-z)^{-1} - (A'-z)^{-1} \). We have

\[
[T_{k,n}, (A-z)^{-1}] = \varphi_n^*(A') K \varphi_n(A) - \varphi_k^*(A) K \varphi_k(A') W_n \in \mathcal{C}_1 \quad \text{and, therefore,}
\]

\[
|\text{tr}[(T_{k,n}, (A-z)^{-1})Y_\xi]| \leq \|K \varphi_n(A)Y_\xi\|_1 + \|Y_\xi \varphi_k(A)K\|_1.
\]

It follows from (III\(^+\)), the identity \( M_\xi^* \varphi_n(A) = \varphi_n^*(A) M_\xi^* \), and the assumptions on \( \xi \) that s-lim\( \limits_{n \to \infty} Y_\xi^* \varphi_n^*(A) = 0 \) s-lim\( \limits_{k \to \infty} Y_\xi^* \varphi_k^*(A) \), which leads to

\[
\|Y_\xi \varphi_k(A)K\|_1 \to 0 \quad \text{and} \quad \|K \varphi_n(A)Y_\xi\|_1 = \|Y_\xi \varphi_n^*(A)K^*\|_1 \to 0 \quad \text{as } \min\{k,n\} \to \infty.
\]

That is,

\[
\lim_{\min\{k,n\} \to \infty} \text{tr}[(T_{k,n}, (A-z)^{-1})Y_\xi] = 0.
\]

By (ii), \( (W_n^* - 1)(A-z)^{-2} \in \mathcal{C}_1 \) and \( (W_n^* - 1)(A-z)^{-2} \in \mathcal{C}_1 \). Since \( T_{k,n} = (W_n^* - 1)(W_n^* - 1) + (W_n^* - 1) \), we have \( T_{k,n} = (A-z)^{-2} \in \mathcal{C}_1 \). Since \( (A-z)^2 \xi = 0 \) is bounded, \( T_{k,n} Y_\xi = (T_{k,n}(A-z)^{-2}) (A-z)^2 Y_\xi \in \mathcal{C}_1 \). Thus, tr\( \text{tr}[(A-z)^{-1} T_{k,n} Y_\xi] = \text{tr}(T_{k,n} Y_\xi) = \text{tr}(T_{k,n} Y_\xi) = \text{tr}(T_{k,n} Y_\xi) = (T_{k,n}, (A-z)^{-1}) Y_\xi \). Hence

\[
\text{tr}[(T_{k,n}, (A-z)^{-1})Y_\xi] = \text{tr}(T_{k,n} (A-z)^{-1}, Y_\xi) = \text{tr}(T_{k,n} Y_\xi) = (T_{k,n}, (A-z)^{-1}).
\]
Because \((W_k - W_n)^* (W_k - W_n) = -T_{k,n} - T_{n,k}\), it follows from (1) and (2) that
\[
\lim_{\min\{k,n\} \to \infty} \|(W_k - W_n)\xi\|^2 = \lim_{\min\{k,n\} \to \infty} \langle (T_{k,n} + T_{n,k})\xi, \xi \rangle = 0.
\]
Since the linear span of such \(\xi\)'s is dense in \(\bigoplus_{j \in J} L^2(\Delta_j)\), this completes the proof. \(\square\)

**Remark.** To deduce that \((A' - w)^{-1} W_+ (A', A) = W_+ (A', A) (A - w)^{-1}\) for \(w \in \mathbb{C} \setminus \mathbb{R}\), which is necessary for establishing the unitary equivalence of \(A_{ac}\) and \(A'_{ac}\), one needs one sequence of class \(\Omega_+\) which has the additional property

\[(3) \quad \text{w-} \lim_{n \to \infty} \varphi_n(A) P_{ac}(A) = 0.\]

For example, use the sequence \(\{\exp(inx)\}\). But, nowhere in our proof did we need anything like (3) for the strong convergence of \(\{\varphi^*_n(A') \varphi_n(A) P_{ac}(A)\}\). In fact it is not even clear that \(\Omega_+\) would imply (3), although it is difficult to imagine that (iii+) and (3) are completely unrelated.

**References**


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