A PALEY-WIENER THEOREM
FOR THE SPHERICAL LAPLACE TRANSFORM
ON CAUSAL SYMMETRIC SPACES OF RANK 1

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(Communicated by Roe Goodman)

abstract. We formulate and prove a topological Paley-Wiener theorem for
the normalized spherical Laplace transform defined on the rank 1 causal sym-
metric spaces $\mathcal{M} = SO_o(1, n)/SO_o(1, n - 1)$, for $n \geq 2$.

introduction

The spherical Laplace transform on causal symmetric spaces was introduced
in [FHO] §8 as a generalization of the spherical Fourier transform on Riemannian
symmetric spaces defined by Helgason (see [H1] Chapter 4). Both transforms can be
expressed in terms of (integrating against) spherical functions. It was furthermore
shown in [O1] §5 that the spherical functions on the Riemannian dual of a causal
symmetric space can be written as an expansion in spherical functions on the causal
symmetric space. The inversion formula for the spherical Laplace transform easily
follows (see [O1] §6).

One of the most important results on the spherical Fourier transform is the
(topological) Paley-Wiener theorem (see [H1] Chapter 4, §7 and [H2] Chapter 3, §5
for details) generalizing the classical Paley-Wiener theorem on Euclidean spaces. In
this paper we generalize these results to the normalized spherical Laplace transform
on causal symmetric spaces $\mathcal{M}$ of rank 1, thereby partially solving an open problem
posed by the second author in [O2], §5.

The paper is divided into two sections: in the first section we recall some results
on the spherical Fourier transform on the Riemannian dual $\mathcal{M}^d$ of $\mathcal{M}$, and in the
second we consider the spherical Laplace transform defined on $\mathcal{M}$. We define the
Paley-Wiener space, the supposed image space of spherical Laplace transform, ac-
cording to the growth and symmetry conditions satisfied by the spherical functions
on $\mathcal{M}$. The Paley-Wiener theorem for the normalized spherical Laplace transform
follows by using Cauchy’s theorem to change the path of integration in the inversion
formula and from the Paley-Wiener theorem for the spherical Fourier transform on
$\mathcal{M}^d$.

Received by the editors December 9, 1998 and, in revised form, March 22, 1999.
2000 Mathematics Subject Classification. Primary 43A85, 22E30; Secondary 43A90, 33C60.
The first author was supported by a postdoc fellowship from the European Commission within
The second author was supported by LEQSF grant (1996-99)-RD-A-12.

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We have tried to keep notations and proofs to a minimum in order to make the presentation as clear as possible; we refer the reader to [FHO], [HO], [O1] and [O2] for more details on spherical functions and the spherical Laplace transform defined on causal symmetric spaces. The spherical Laplace transform in the rank 1 case can be considered as a Laplace-Jacobi transform (see [M] for a detailed analysis of the latter transform) but we note that the Paley-Wiener theorem is new even in the rank one case.

We would like to thank H. Schlichtkrull and J.M. Unterberger for helpful discussion and comments, in particular concerning Lemma 7 and its proof.

**Notation and preliminaries**

In the following we consider the causal symmetric (real hyperbolic) space $M = G/H$ with $G = SO_o(1, n)$ and $H = SO_o(1, n-1)$ and its Riemannian dual $M^d = G/K$, where $K = SO_o(n)$. Let $\mathfrak{g}$ denote the Lie algebra of $G$ and let $\mathfrak{a}$ be the abelian subalgebra of $\mathfrak{g}$ given by

$$\mathfrak{a} = \left\{ X_t = \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ t & 0 & 0 \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$  

We choose the unique positive root $\alpha \in \mathfrak{a}^*$ as $\alpha(X_t) = t$. Let $\mathfrak{a}^+ = \{ X_t \in \mathfrak{a} \mid t > 0 \}$. We identify the complex dual $\mathfrak{a}_c^*$ of $\mathfrak{a}$ with $\mathbb{C}$ via the map $z \mapsto z \alpha \in \mathfrak{a}_c$. Let $\mathfrak{n} = \mathfrak{g}_n$ and $\mathfrak{n}^* = \mathfrak{g}_{-\alpha}$ denote the positive and negative root space respectively. Let $A = \exp \mathfrak{a}$, $A^+ = \exp \mathfrak{a}^+$, $N = \exp \mathfrak{n}$ and $\mathfrak{N} = \exp \mathfrak{n}^*$, where exp is the exponential mapping from $\mathfrak{g}$ to $G$. We also consider the open semigroup $S^o = H A^+ H$ in $G$. Let finally $a_t = \exp X_t \in A$.

Let $\eta : \mathcal{D}(M) \to \mathcal{D}(M^d)$ denote the Flensted-Jensen isomorphism between the commutative algebras of invariant differential operators on $M$ and $M^d$ respectively (mapping the Laplace-Beltrami operator $\Delta$ on $M$ onto the Laplace-Beltrami operator $\Delta^d = \eta(\Delta)$ on $M^d$). Let $\Pi(D)$ and $\Pi^d(D^d)$ denote the radial part (on $A^+$) of $D \in \mathcal{D}(M)$ and $D^d \in \mathcal{D}(M^d)$ respectively. There exists a unique map $C_c^\infty(H \backslash S^o \mathbb{R}^n/H) \ni f \mapsto f^d \in C_c^\infty(K \backslash G/K)$ such that $f_{|A^+} = f^d_{|A^+}$ and $\Pi(D) f_{|A^+} = \Pi^d(\eta(D)) f^d_{|A^+}$ (see [HO] or [O1, §4] for more details).

**The spherical Fourier transform on $M^d$**

In this section we recall some well-known definitions and results for the spherical Fourier transform on a Riemannian symmetric space (see e.g. [H1, Chapter 4] and [H2, Chapter 3]).

Let $\lambda \in \mathbb{C}$. Define the Poisson kernel for $M^d$ by

$$KAN \ni kan = x \mapsto a^{-(\lambda + \rho)} =: p^d(x),$$

where $\rho = (n-1)/2$. The spherical functions on $M^d$ can be written as

$$\psi^d_\lambda(a_t) = \int_K p^d_\lambda(a_t k) d k = 2 F_1 \left( \begin{array}{c} \frac{1}{2}(\lambda + \rho), \frac{1}{2}(-\lambda + \rho) \\ \rho + \frac{1}{2} \end{array} ; -\sinh^2 t \right),$$

for $a_t \in A$. The spherical functions are bi-$K$-invariant and $\Delta^d \psi^d_\lambda = (\lambda^2 - \rho^2) \psi^d_\lambda$ for all $\lambda \in \mathbb{C}$. They satisfy the following inequality, for all $\lambda \in \mathbb{C}$ and all $a_t \in A$;

$$|\psi^d_\lambda(a_t)| \leq c(1 + |t|) e^{|\Re \lambda| - \rho |t|},$$

for some constant $c$, and they are invariant under sign change, i.e. $\psi^d_{-\lambda} = \psi^d_\lambda$. 

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The spherical Fourier transform $\mathcal{F}$ on $\mathcal{M}^d$ is defined as
\[
\mathcal{F}(f)(\lambda) = \int_{G} f(x) \psi^d(x) dx = \int_{A^+} f(a) \psi^d(a) \delta(a) da,
\]
for $f \in C^\infty_c(K\backslash G/K)$, where $\delta(a_t) = \sinh^{2\rho} t$. Let
\[
c^d(\lambda) := \int_{\mathbb{R}} p^d(\mu) d\mu = \frac{\Gamma(2\rho)}{\Gamma(\rho)} \frac{\Gamma(\lambda)}{\Gamma(\lambda + \rho)}
\]
denote the Harish-Chandra $c$-function for $\mathcal{M}^d$. We note that $|c^d(\lambda)|^2 = c^d(\lambda)c^d(-\lambda)$ for $\lambda \in i\mathbb{R}$. The inversion formula for $\mathcal{F}$ reads (after normalizing $d\lambda$ suitably):
\[
f(x) = \int_{i\mathbb{R}} \mathcal{F}(f)(\lambda) \psi^d,\lambda(x) |c^d(\lambda)|^{-2} d\lambda,
\]
for all $f \in C^\infty_c(K\backslash G/K)$ and $x \in G$.

Let $R > 0$. Let $C^\infty_R(K\backslash G/K) := \{ f \in C^\infty_c(K\backslash G/K) | f(a_t) = 0 \text{ for } t > R \}$ and define the Paley-Wiener space $\mathcal{H}_R(\mathbb{C})$ as the space of even holomorphic functions $g$ on $\mathbb{C}$ of exponential type $R$, i.e. satisfying the estimate
\[
\sup_{\lambda \in \mathbb{C}} e^{-R|\Re \lambda|} \left(1 + |\lambda|\right)^N |g(\lambda)| < \infty,
\]
for all $N \in \mathbb{N} \cup \{0\}$. Furthermore denote by $\mathcal{H}(\mathbb{C})$ the union of the spaces $\mathcal{H}_R(\mathbb{C})$, $R > 0$.

**Theorem 1** (The Paley-Wiener Theorem). The Fourier transform is a topological linear isomorphism from $C^\infty_c(K\backslash G/K)$ onto $\mathcal{H}(\mathbb{C})$. More precisely it is a topological linear isomorphism from $C^\infty_R(K\backslash G/K)$ onto $\mathcal{H}_R(\mathbb{C})$ for all $R > 0$.

**The spherical Laplace transform on $\mathcal{M}$**

We define spherical functions on $\mathcal{M}$ according to [O1] Definition 4.1:

**Definition 2.** An $H$-bi-invariant continuous function $\varphi : S^o \to \mathbb{C}$ is called a spherical function if there exists a character $\chi$ of $D(\mathcal{M})$ such that (in the sense of distributions) $D\varphi = \chi(D)\varphi$ for all $D \in D(\mathcal{M})$.

Define the Poisson kernel for $\mathcal{M}$ (and the open orbit $HAN$) by
\[
HAN \ni han = x \mapsto a^{-(\lambda + \rho)} =: p_\lambda(x)
\]
and $p_\lambda \equiv 0$ on $G \setminus HAN$. We can construct spherical functions $\varphi_\lambda$ as follows:
\[
\varphi_\lambda(x) := \int_H p_{-\lambda}(xh) dh,
\]
for $x \in S^o$, whenever this integral exists (see [O1] Theorem 4.10). Using the calculations in [FH0, §10] we see that the integral converges for $x \in S^o$ and $\Re \lambda < 1 - \rho$ and we get the following explicit formula for $\varphi_\lambda$:
\[
\varphi_\lambda(a_t) = c(\lambda)(2 \cosh t)^{\lambda - \rho} F_1 \left(\frac{1}{2}(-\lambda + \rho), \frac{1}{2}(-\lambda + \rho + 1); 1 - \lambda; \cosh^{-2} t\right),
\]
where
\[
c(\lambda) := \int_{\mathbb{R}} p_\lambda(\mu) d\mu = 2^{2\rho - 1} \frac{\Gamma(\lambda - \rho + 1)}{\Gamma(\lambda + 1)}.
\]
We note that $\Delta \varphi_\lambda = (\lambda^2 - \rho^2) \varphi_\lambda$ when defined.
Let \( \varphi^\circ_\lambda(x) = c(\lambda)^{-\frac{1}{2}} \varphi_\lambda(x) \). We have the following uniform growth estimate on \( \varphi^\circ_\lambda \) (due to Helgason (rank one) and Gangolli):

**Lemma 3.** Fix \( \sigma > 0 \). There exists a constant \( c_\sigma \) such that

\[
|\varphi^\circ_\lambda(a_t)| \leq c_\sigma e^{(\Re \lambda - \rho)t},
\]
for \( \Re \lambda \leq 0 \) and all \( t \in [\sigma, \infty[ \).

**Sketch of the proof.** Consider a formal power series solution to the differential equation \( \Pi(\Delta) \varphi^\circ_\lambda = \delta(t)^{-1} \frac{\partial}{\partial t} (\delta(t)^{\frac{2j+1}{2}}) = (\lambda^2 - \rho^2) \varphi^\circ_\lambda \) of the form \( c(\lambda)^{\frac{1}{2}} \sum_{n=0}^{\infty} e^{-n\Gamma_n(\lambda)} \), with \( \Gamma_n(\lambda) \) to be determined (and \( \Gamma_0 \equiv 1 \)). Substitution into the differential equation gives a recurrence formula that defines \( \Gamma_n(\lambda) \) uniquely for \( \lambda \notin \frac{1}{2} \mathbb{N} \). We can estimate \( \Gamma_n(\lambda) \) as follows: There exist constants \( c, \infty > 0 \) such that

\[
|\Gamma_n(\lambda)| < c(1 + n)^{\infty},
\]
for \( \Re \lambda \leq 0 \) and all \( n \in \mathbb{N} \). The above estimate on \( \varphi^\circ_\lambda \) follows easily. See [BS] §7-9 for a complete proof in a more general setup.

We define the normalized spherical Laplace transform \( \mathcal{L}^\circ \) on \( \mathcal{M} \) as

\[
\mathcal{L}^\circ(f)(\lambda) = \int_{A^+} f(a) \varphi^\circ_\lambda(a) \delta(a) da,
\]
for \( f \in C^\infty_c(A^+) \cong C^\infty_c(H \backslash S^0 / H) \) and \( \Re \lambda < 1 - \rho \). From the explicit formula for \( \varphi^\circ_\lambda \), we see that the function \( \lambda \mapsto \mathcal{L}^\circ(f)(\lambda) \) extends to a meromorphic function on \( \mathbb{C} \) with at most poles for \( \lambda \in \mathbb{N} \).

Let \( R > r > 0 \) and define \( C^\infty_{r,R}(A^+) := \{ f \in C^\infty_c(A^+) | f(a_t) = 0 \) for \( 0 < t < r \) and \( t > R \} \). We equip \( C^\infty_{r,R}(A^+) \) with the natural Fréchet space topology and \( C^\infty_c(A^+) \) with the inductive limit topology. From Lemma 3 we get the following uniform growth estimate on the normalized spherical Laplace transform acting on \( C^\infty_{r,R}(A^+) \):

**Lemma 4.** Let \( R > r > 0 \) and let \( N \in \mathbb{N} \cup \{0\} \). There exists a constant \( c > 0 \), only depending on \( r \) and \( R \), such that, for all \( f \in C^\infty_{r,R}(A^+) \)

\[
\sup_{\Re \lambda \leq 0} e^{-r \Re \lambda}(1 + |\lambda|^2)^N \| \mathcal{L}^\circ f(\lambda) \| \leq c \sum_{n=0}^{N} \| (\Delta + \rho^2)^n f \|_{\infty} < \infty.
\]

**Proof.** Since \( \mathcal{L}^\circ((\Delta f)(\lambda)) = (\lambda^2 - \rho^2) \mathcal{L}^\circ(f)(\lambda) \) and \( \lambda^2 = \lambda^2 - \rho^2 + \rho^2 \), we easily get

\[
(1 + |\lambda|^2)^N \| \mathcal{L}^\circ(f)(\lambda) \| = \sum_{n=0}^{N} \binom{N}{n} |\lambda|^{2n} \| \mathcal{L}^\circ(f)(\lambda) \|
\]

\[
= \sum_{n=0}^{N} \binom{N}{n} \| \mathcal{L}^\circ((\Delta + \rho^2)^n f)(\lambda) \| \leq c e^{r \Re \lambda} \sum_{n=0}^{N} \binom{N}{n} \| (\Delta + \rho^2)^n f \|_{\infty},
\]
for \( \Re \lambda \leq 0 \) and all \( f \in C^\infty_{r,R}(A^+) \), where \( c > 0 \) is a constant only depending on \( r \) and \( R \).

Using the correspondence between (the radial parts of) invariant differential operators on \( \mathcal{M} \) respectively on \( \mathcal{M}_d \) (see also [O1], Theorem 5.9)), we get

\[
\psi^d_\lambda(a) = c^d(\lambda) \varphi^\circ_\lambda(a) + c^d(-\lambda) \varphi^\circ_{-\lambda}(a),
\]
for $a \in A^+$ and $\lambda \notin \mathbb{Z}\backslash\{0\}$ (or use [11] p.63, Eq.(17) & p.64, Eq.(22)) and the hypergeometric expressions for the spherical functions), which is the Harish-Chandra expansion formula for $\psi^d_i$. Let $f \in C_c^\infty(A^+)$. We see that $\mathcal{L}^\alpha f$ satisfies the following functional equation:

$$c^d(\lambda)\mathcal{L}^\alpha(f)(\lambda) + c^d(-\lambda)\mathcal{L}^\alpha(f)(-\lambda) = \mathcal{F}(f^d)(\lambda),$$

for $\lambda \notin \mathbb{Z}\backslash\{0\}$. The inversion formula for the normalized spherical Laplace transform is now an easy consequence of the inversion formula for the spherical Fourier transform (see also [XI, p. 993]):

**Theorem 5** (The Inversion Formula). Let $f \in C_c^\infty(A^+)$. Then

$$f(a) = 2 \int_{\mathbb{R}} \mathcal{L}^\alpha(f)(\lambda)\psi^d(a)\mathcal{F}(f^d)^{-1}d\lambda,$$

for all $a \in A^+$.

All the above suggests the following definition of the Paley-Wiener space, the supposed image space of the normalized spherical Laplace transform:

**Definition 6.** Let $R > r > 0$. We define the Paley-Wiener space $PW_{r,R}(\mathbb{C})$ as the space of meromorphic functions $g$ on $\mathbb{C}$, with at most poles for $\lambda \in \mathbb{N}$, such that

(i)  
$$\sup_{\lambda \in \mathbb{N}} e^{-rRe\lambda}(1 + |\lambda|)^N |g(\lambda)| < \infty,$$

for all $N \in \mathbb{N} \cup \{0\}$, and

(ii) the $c^d$-weighted average $\text{P}^av g(\lambda) := c^d(\lambda)g(\lambda) + c^d(-\lambda)g(-\lambda)$ extends to a function in $\mathcal{H}_R(\mathbb{C})$.

Furthermore denote by $PW(\mathbb{C})$ the union of the spaces $PW_{r,R}(\mathbb{C})$ over all $R > r > 0$.

We define a Fréchet space topology on $PW_{r,R}(\mathbb{C})$ by means of the seminorms

$$\sigma_{r,N}(g) = \sup_{\lambda \leq 0} e^{-rRe\lambda}(1 + |\lambda|)^N |g(\lambda)|$$

and

$$\sigma_{R,N}(g) = \sup_{\lambda \in \mathbb{C}} e^{-R|Re\lambda|(1 + |\lambda|)^N |\text{P}^av g(\lambda)|}.$$

We furthermore equip the space $PW(\mathbb{C})$ with the inductive limit topology.

We remark that $\mathcal{P}^av \mathcal{L}^\alpha$ acts injectively on $C_c^\infty(A^+)$, since $\mathcal{P}^av \mathcal{L}^\alpha(f) = \mathcal{F}(f^d) = 0$ implies $f = f^d = 0$ on $A^+$ for any $f \in C_c^\infty(A^+)$, by the injectivity of the spherical Fourier transform is injective. But we will need that $\mathcal{P}^av$ is injective on $PW(\mathbb{C})$.

The following lemma and its proof was communicated to us by H. Schlichtkrull.

**Lemma 7.** Let $g$ be a meromorphic function on $\mathbb{C}$ that satisfies item (i) of Definition 6. Assume that $\mathcal{P}^av g = 0$. Then $g = 0$.

**Proof.** Let $g^f(\lambda) = g(\lambda)/c^d(-\lambda)$. Then $\mathcal{P}^av g(\lambda) = 2c^d(\lambda)c^d(-\lambda)\text{avg}^f(\lambda)$, where

$$\text{avg}^f(\lambda) := \frac{1}{2}[g^f(\lambda) + g^f(-\lambda)].$$
is the average of $g^t$ over the Weyl group $\pm 1$. It follows from the assumption $P^av g = 0$ that $av g^t = 0$. Let

$$\gamma(s) = \int_{\mathbb{R}} g^t(i\lambda)e^{-is\lambda} \, d\lambda, \quad s \in \mathbb{R},$$

denote the Euclidean Fourier transform of $g^t(i \cdot)$. It follows from (i) and [H1 Proposition IV.7.2] that (i) is satisfied by $g^1$ as well. In particular, $g^1(i \cdot) \in L^1(\mathbb{R})$. The condition (i) implies that $g$ is holomorphic in an open set containing $\{ z \in \mathbb{C} \mid \text{Re} z \leq 0 \}$. Moreover, the standard argument with Cauchy’s theorem shows that $\gamma$ is supported on $[r, \infty]$. On the other hand, the average $av g$ is the Laplace transform of $av g^1(i \cdot)$, which vanishes; hence $av \gamma$ vanishes as well. The support condition now implies that $\gamma = 0$. Since the Euclidean Fourier transform is injective on $L^1(\mathbb{R})$, we conclude that $g^1$, and therefore also $g$, vanishes.

**Theorem 8 (The Paley-Wiener Theorem).** The normalized spherical Laplace transform $L^a$ is a topological linear isomorphism from $C^\infty_c(A^+)$ onto $PW(C)$. More precisely it is a topological linear isomorphism from $C^\infty_{r,R}(A^+)$ onto $PW_{r,R}(C)$ for all $R > r > 0$.

**Proof.** By the Paley-Wiener theorem for the spherical Fourier transform, Lemma 4 and the open mapping theorem for Fréchet spaces, it only remains to show that the normalized spherical Laplace transform maps $C^\infty_{r,R}(A^+)$ onto $PW_{r,R}(C)$ for all $R > r > 0$.

Consider the wave packet $Ig \in C^\infty(A^+)$ of $g \in PW_{r,R}(C)$ defined by

$$Ig(a) = 2\int_{\mathbb{R}} g(\lambda)\psi^d_\lambda(a)\psi^d(-\lambda)^{-1} \, d\lambda,$$

for $a \in A^+$. By Cauchy’s theorem we get, for $0 < t < r$ and $\mu < 0$,

$$Ig(a_t) = 2\int_{\mathbb{R}} g(\lambda)\psi^d_\lambda(a_t)\psi^d(-\lambda)^{-1} \, d\lambda$$

$$= 2\int_{\mathbb{R}} g(\lambda + t)\psi^d_\lambda(a_t)\psi^d(-\lambda)^{-1} \, d\lambda$$

$$\to 0 \quad \text{for} \quad \mu \to -\infty,$$

since $|\psi^d_\lambda(a_t)\psi^d(-\lambda)^{-1}| \leq c(1 + |\lambda|)^r(1 + t)e^{(|\text{Re}\lambda| - \rho)t}$ for $\text{Re}\lambda \geq 0$, for some constant $c > 0$. An easy calculation shows that

$$Ig(a_t) = 2\int_{\mathbb{R}} g(\lambda)\psi^d_\lambda(a_t)\psi^d(-\lambda)^{-1} \, d\lambda$$

$$= \sum_{w = \pm 1} \int_{\mathbb{R}} g(w\lambda)\psi^d_{-w\lambda}(a_t)\psi^d(-w\lambda)^{-1} \, d\lambda$$

$$= \sum_{w = \pm 1} \int_{\mathbb{R}} g(w\lambda)\psi^d_{w\lambda}(a_t)\psi^d_{-w\lambda}(-\lambda)^{-1} \, d\lambda$$

$$= \int_{\mathbb{R}} \left( \sum_{w = \pm 1} c^d(w\lambda)g(w\lambda) \psi^d_{w\lambda}(a_t) \right) \psi^d_{-\lambda}(a_t) |c^d(\lambda)|^{-2} \, d\lambda$$

$$= \int_{\mathbb{R}} g^a(\lambda)\psi^d_{-\lambda}(a_t) |c^d(\lambda)|^{-2} \, d\lambda,$$
which we recognize as the inverse Fourier transform of $P^\text{av} g \in \mathcal{H}_R(\mathbb{C})$; whence $Lg(a_t) = 0$ for all $t > R$ by the Paley-Wiener theorem for the spherical Fourier transform on $M^d$.

Since $P^\text{av} L^o f = F f^d$ for all $f \in C_c^\infty(A^\times)$, the above also yields

$$P^\text{av} L^o I g = F(I g)^d = P^\text{av} g,$$

for all $g \in PW(\mathbb{C})$; hence Lemma 7 implies that $L^o I g = g$ for all $g \in PW(\mathbb{C})$ and we conclude that $L^o$ maps $C_c^\infty(A^\times)$ onto $PW(\mathbb{C})$.

References


