

APPROXIMATING DISCRETE VALUATION RINGS BY REGULAR LOCAL RINGS

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ABSTRACT. Let k be a field of characteristic zero and let (V, \mathfrak{n}) be a discrete rank-one valuation domain containing k with $V/\mathfrak{n} = k$. Assume that the fraction field L of V has finite transcendence degree s over k . For every positive integer $d \leq s$, we prove that V can be realized as a directed union of regular local k -subalgebras of V of dimension d .

1. INTRODUCTION

Suppose (R, \mathfrak{m}) is a local Noetherian domain with fraction field F and (V, \mathfrak{n}) is a valuation domain that *birationally dominates* R in the sense that R is a subring of V , V has fraction field F and $\mathfrak{n} \cap R = \mathfrak{m}$. Classically, in the case where R is essentially finitely generated over a field, *local uniformization of R along V* , in algebraic terms, means the existence of a regular local domain S between R and F , such that S is dominated by V and is essentially finitely generated over R . If R is a regular local ring and P is a prime ideal of R , *embedded local uniformization* translates to the existence of S as above having a prime ideal Q with $Q \cap R = P$ and S/Q a regular local domain.

The consideration of embedded local uniformization yields a representation of the valuation domain V as a directed union of regular local rings. If V/\mathfrak{n} is algebraic over R/\mathfrak{m} , then classical methods give a representation of V as a directed union of regular local rings all having dimension equal to the dimension of R [A, Lemma 12]. In this article we prove that certain rank-one discrete valuation rings (DVRs) can be represented as a directed union of regular local domains of dimension d for every positive integer d less than or equal to the dimension of R . We use a technique inspired by Nagata [N1] and developed in our earlier work for the construction of new Noetherian domains [HRW1], [HRW2], [HRW3].

The classical approach for obtaining embedded local uniformization, introduced by Zariski in the 1940's [Z1], uses local quadratic transforms of R along V . The

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first local quadratic transform of R along V is defined to be $R_1 = R[\mathbf{m}/a]_{\mathbf{m}_1}$, where $a \in \mathbf{m}$ is such that $\mathbf{m}V = aV$ and $\mathbf{m}_1 := \mathbf{n} \cap R[\mathbf{m}/a]$. R_1 is also called the *dilatation of R by the ideal \mathbf{m} along V* [N2, page 141]. More generally, if $I \subseteq \mathbf{m}$ is a nonzero ideal of R , the *dilatation of R by I along V* is the ring $R[I/a]_{\mathbf{m}_1}$, where $a \in I$ is such that $IV = aV$ and $\mathbf{m}_1 = \mathbf{n} \cap R[I/a]$. It is straightforward to check that R_1 is uniquely determined by R , V and the ideal I [N2, page 141]. For each positive integer i , we define the $(i+1)$ th *quadratic transform R_{i+1} of R along V* inductively: R_{i+1} is the first local quadratic transform of R_i along V . Then $R_{i+j} = (R_i)_j$ for all $i, j \geq 0$.

Using monoidal transforms (a special type of dilatation), S. Cutkosky in [C1] and [C2] has recently done interesting work on birational factorization and local uniformization.

1.1 Background remarks.

- (1) If R is a regular local ring, it is well known that a quadratic transform R_1 of R is again a regular local ring [N2, (38.1)].
- (2) If R is a pseudogeometric regular local domain of dimension 2 and V is a valuation domain that birationally dominates R , a classical result of Zariski and Abhyankar asserts that $V = \bigcup_{n=1}^{\infty} R_n$ [A, Lemma 12].
- (3) Suppose R is a regular local ring of dimension $d \geq 3$. For certain valuation rings V that birationally dominate R , the union $\bigcup_{n=1}^{\infty} R_n$ of the local quadratic transforms of R along V is strictly smaller than V [S, (4.13)]. In many cases Shannon in [S, (4.5), page 308] proves that V is a directed union of iterated monoidal transforms of R , where a *monoidal transform of R* is a dilatation of R by a prime ideal P for which R/P is again regular.
- (4) If (V, \mathbf{n}) is a DVR that birationally dominates a regular local ring R , then $V = \bigcup_{n=1}^{\infty} R_n$,¹ where R_n is the n th quadratic transform of R along V [A, page 336], [Z2, page 27-28].

A proof of item (4) goes as follows: a nonzero element η of V has the form $\eta = b/c$, where $b, c \in R$. If $(b, c)V = V$, then $b/c \in V$ implies $cV = V$. Since V dominates R , it follows that $cR = R$, so $b/c \in R$ in this case. If $\eta = b/c$, with $b, c \in R$ and $(b, c)V = \mathbf{n}^k$, we prove by induction that $\eta \in R_k$. We have already done the case where $k = 0$. Assume for every regular local domain (S, \mathbf{p}) birationally dominated by V , and every nonzero element $\beta/\gamma \in V$ with $\beta, \gamma \in S$, $(\beta, \gamma)V = \mathbf{n}^j$ and $0 \leq j < k$ we have $\beta/\gamma \in S_j$, where S_j is the j th iterated local quadratic transform of S along V . Suppose $\beta, \gamma \in S$, $\beta/\gamma \in V$ with $(\beta, \gamma)V = \mathbf{n}^k$. Let $S_1 = S[\mathbf{p}/a]_{\mathbf{p}_1}$, where $a \in \mathbf{p}$ is such that $\mathbf{p}V = aV$ and $\mathbf{p}_1 := \mathbf{n} \cap S[\mathbf{p}/a]$; that is, S_1 is the first local quadratic transform of S along V . Then $\beta_1 := \beta/a$ and $\gamma_1 := \gamma/a$ are in S_1 . Thus $a \in \mathbf{n}$ implies $(\beta_1, \gamma_1)V = \mathbf{n}^j$ where $0 \leq j < k$, so by induction

$$\beta/\gamma = \beta_1/\gamma_1 \in (S_1)_j = S_{j+1} \subseteq S_k,$$

which completes the proof.

In this article we prove the following theorem:

1.2 Main theorem. *Let k be a field of characteristic zero and let (V, \mathbf{n}) be a DVR containing k with $V/\mathbf{n} = k$. Assume that the fraction field L of V has finite transcendence degree s over k . Then for every integer d with $1 \leq d \leq s$, there*

¹It may happen that $R_n = V$ for some n , in which case $R_{n+i} = R_n$ for all $i \geq 0$.

exists a nested family $\{C_n^{(\alpha)} : n \in \mathbb{N}, \alpha \in \Gamma\}$ of d -dimensional regular local k -subalgebras of V such that V is the directed union of the $C_n^{(\alpha)}$ and V dominates each $C_n^{(\alpha)}$.

If the field L is finitely generated over k , then V is a countable union $\bigcup_{n=1}^{\infty} C_n$, where, for each $n \in \mathbb{N}$,

- (1) C_n is a d -dimensional regular local k -subalgebra of V ,
- (2) C_n has fraction field L ,
- (3) C_{n+1} dominates C_n and
- (4) V dominates C_n .

We have the following corollary to the main theorem.

1.3 Corollary. *Let k be a field of characteristic zero and let (R, \mathbf{m}) be a local domain essentially of finite type over k with coefficient field $k = R/\mathbf{m}$ and field of fractions L . Let (V, \mathbf{n}) be a DVR birationally dominating R with $V/\mathbf{n} = k$. For every integer d with $1 \leq d \leq s$, $s = \text{trdeg}_k(L)$, there exists a sequence of d -dimensional regular local k -subalgebras C_n of V such that $V = \bigcup_{n=1}^{\infty} C_n$, and for each $n \in \mathbb{N}$, C_{n+1} dominates C_n and V dominates C_n . Moreover C_n dominates R for all sufficiently large n .*

Proof. By Theorem 1.2, $V = \bigcup_{n=1}^{\infty} C_n$, where the C_n are regular local rings satisfying all the conditions of Corollary 1.3 except possibly that of dominating R . Since R is essentially of finite type over k and is dominated by V , it follows that R is dominated by C_n for all sufficiently large n . \square

1.4 Discussion. (1) If L/k is finitely generated of transcendence degree s , then the fact that V is a directed union of s -dimensional regular local rings follows from classical theorems of Zariski. The local uniformization theorem of Zariski [Z1] implies the existence of a regular local domain (R, \mathbf{m}) containing the field k such that V birationally dominates R . Since k is a coefficient field for V , we have

- (a) $k \hookrightarrow V \twoheadrightarrow V/\mathbf{n} \cong k$; thus k is relatively algebraically closed in L .
- (b) $R/\mathbf{m} = k$ (because V dominates R).
- (c) Every iterated local quadratic transform of R along V has dimension s .

Now by 1.1(4), V is a directed union of s -dimensional RLRs.

(2) If $d = 1$, the main theorem is trivially true by taking each $C_n = V$. Thus if L/k is finitely generated of transcendence degree $s = 2$, then the theorem is saying nothing new.

(3) If $s > 2$, then the classical local uniformization theorem says nothing about expressing V as a directed union of d -dimensional RLRs, where $2 \leq d \leq s - 1$. If (S, \mathbf{p}) is a local Noetherian domain containing k and birationally dominated by V with $\dim(S) = d < s$, then S does not satisfy the dimension formula. It follows that S is not essentially finitely generated over k [M1, page 119].

We use the following notation in the proof of the main theorem.

1.5 Endpieces and related localized polynomial rings. Let (R, \mathbf{m}) be a local Noetherian domain with fraction field F . Let $y \in \mathbf{m}$ be a nonzero element, let $R^* = \widehat{(R, (y))}$ be the (y) -adic completion of R . Suppose $\tau_1, \dots, \tau_s \in yR^*$ are regular elements² of R^* that are algebraically independent over K . We consider

²We say an element of a ring is a *regular element* if it is not a zero divisor.

the localized polynomial ring

$$B_0 := R[\tau_1, \dots, \tau_s]_{(\mathfrak{m}, \tau_1, \dots, \tau_s)}.$$

It is clear that B_0 is a local Noetherian domain with $\dim(B_0) = \dim(R) + s$ such that R^* dominates B_0 and B_0 dominates R .

For every $\gamma \in R^*$ and every $n > 0$, we define the n th-endpiece γ_n with respect to y of γ to be

$$(1.5.1) \quad \gamma_n := \sum_{j=n+1}^{\infty} c_j y^{j-n}, \text{ where } \gamma := \sum_{j=1}^{\infty} c_j y^j \text{ and each } c_j \in R.$$

In particular, we represent each of the τ_i by a power series expansion in y ; we use these representations to obtain for each positive integer n the n th-endpieces τ_{in} and corresponding n th-localized polynomial ring B_n . For $1 \leq i \leq s$, and $\tau_i := \sum_{j=1}^{\infty} a_{ij} y^j$, where the $a_{ij} \in R$, we have

$$\tau_{in} := \sum_{j=n+1}^{\infty} a_{ij} y^{j-n}, \quad B_n := R[\tau_{1n}, \dots, \tau_{sn}]_{(\mathfrak{m}, \tau_{1n}, \dots, \tau_{sn})}, \text{ for each } n \in \mathbb{N}.$$

The definition of B_n is independent of the representation of the τ_i as power series in y [HRW1, (2.3)]. The localized polynomial ring B_{n+1} birationally dominates B_n . We define

$$(1.5.2) \quad B := \bigcup_{n=0}^{\infty} B_n = \varinjlim B_n \quad \text{and} \quad A := F(\tau_1, \dots, \tau_s) \cap R^*.$$

It is readily seen that A birationally dominates B .

In the proof of our main theorem, we make use of the following result from previous work that we summarize as Theorem 1.6.

1.6 Theorem ([HRW1] or [HRW2]). *Suppose (R, \mathfrak{m}) is a local Noetherian domain with fraction field F , $y \in \mathfrak{m}$ is a nonzero element, and τ_1, \dots, τ_s are elements of the (y) -adic completion R^* of R that are algebraically independent over R . Then the following conditions are equivalent:*

- (1) *The ring $R^*[1/y]$ is flat over $B_0 = R[\tau_1, \dots, \tau_s]_{(\mathfrak{m}, \tau_1, \dots, \tau_s)}$.*
- (2) *The ring $R^*[1/y]$ is flat over $R[\tau_1, \dots, \tau_s]$.*
- (3) *The directed union $B := \bigcup_{n=0}^{\infty} B_n$ is Noetherian.*

Moreover, if these equivalent conditions hold, then the domain $A := F(\tau_1, \dots, \tau_s) \cap R^$ is equal to B , and thus both are Noetherian.*

2. PROOF OF THE MAIN THEOREM

Let $y \in \mathfrak{n}$ be such that $yV = \mathfrak{n}$. Then the \mathfrak{n} -adic completion \widehat{V} of V is $k[[y]]$. Hence we have

$$k[y]_{(y)} \subseteq V \subseteq k[[y]].$$

Then $V = L \cap k[[y]]$, for example, since $V \hookrightarrow k[[y]]$ is flat. Since $\text{trdeg}_{k(y)} L = s - 1$, there are $s - 1$ elements $\sigma_1, \dots, \sigma_{s-d}, \tau_1, \dots, \tau_{d-1} \in yV$ such that L is algebraic over $F := k(y, \sigma_1, \dots, \sigma_{s-d}, \tau_1, \dots, \tau_{d-1})$.

Set $K := k(y, \sigma_1, \dots, \sigma_{s-d})$ and $R := V \cap K$. Then R is a DVR and the (y) -adic completion of R is $R^* = k[[y]]$. In the notation of 1.5, $B_0 := R[\tau_1, \dots, \tau_{d-1}]_{(y, \tau_1, \dots, \tau_{d-1})}$ is a d -dimensional regular local ring and $V_0 := V \cap F$ is a DVR that birationally

dominates B_0 and has y -adic completion $\widehat{V}_0 = k[[y]]$. The following diagram displays these domains:

$$\begin{array}{ccccccccc} k & \xrightarrow{\subseteq} & K & \xrightarrow{\subseteq} & F & \xlongequal{\quad} & F & \xrightarrow{\subseteq} & L := \mathcal{Q}(V) \\ \parallel \uparrow & & \cup \uparrow & & \cup \uparrow & & \cup \uparrow & & \cup \uparrow \\ k & \xrightarrow{\subseteq} & R := V \cap K & \xrightarrow{\subseteq} & B_0 & \xrightarrow{\subseteq} & V_0 := V \cap F & \xrightarrow{\subseteq} & V \end{array}$$

Let τ_{in} be the n th-endpiece of τ_i for each i ; since $V_0 = F \cap k[[y]]$, the $\tau_{in} \in V_0$ and $B_n = R[\tau_{1n}, \dots, \tau_{d-1,n}]_{(y, \tau_{1n}, \dots, \tau_{d-1,n})}$ is, for each $n \in \mathbb{N}$, the first quadratic transform of B_{n-1} along V_0 .

Now R_y^* is a field and thus is flat as an $R[\tau_1, \dots, \tau_{d-1}]$ -module. By Theorem 1.6, $V_0 = \bigcup_{n=1}^{\infty} B_n$. An alternate way to justify this description of V_0 is to use the statement 1.1(4) where the R of 1.1(4) is B_0 , the V of 1.1(4) is V_0 , and each $R_n = B_n$. We have

$$V_0 \xrightarrow{\text{alg}} V \longrightarrow k[[y]],$$

where V_0 and V are DVRs of characteristic zero having completion $k[[y]]$. Since V_0 and V are excellent, their Henselizations V_0^h and V^h are the set of elements of $k[[y]]$ algebraic over V_0 or V [N2, (44.3)]. Thus $V_0^h = V^h$ and V is a directed union of étale extensions of V_0 [R, (1.3)].

2.1 Proposition. *Let B_n^h denote the Henselization of B_n . Then $\bigcup_{n=1}^{\infty} B_n^h = V_0^h = V^h$.*

Proof. The ring $C := \bigcup B_n^h$ is Henselian and contains V_0 , so $V_0^h = V^h \subseteq C$. Moreover, the inclusion map $V \rightarrow C = \bigcup B_n^h$ extends to a map $V^h \xrightarrow{\sigma} C = \bigcup B_n^h$.

On the other hand, the maps $B_n \rightarrow V$ extend to maps $B_n^h \rightarrow V^h$ yielding a map $\rho : C \rightarrow V^h$ with $\sigma\rho = 1_C$, and $\rho\sigma = 1_{V^h}$. Thus $\bigcup_{n=1}^{\infty} B_n^h = V^h$. \square

Continuation of the proof of the main theorem. First assume L is finitely generated over k . It follows that L is finite algebraic over F . Since $\bigcup_{n=1}^{\infty} B_n^h = V^h$, we have $\bigcup_{n=1}^{\infty} \mathcal{Q}(B_n^h) = \mathcal{Q}(V^h)$ and $L \subseteq \mathcal{Q}(V^h)$. Since L/F is finite algebraic, $L \subseteq \mathcal{Q}(B_n^h)$ for all sufficiently large n . By relabeling, we may assume $L \subseteq \mathcal{Q}(B_n^h)$ for all n . Let $C_n := B_n^h \cap L$. Since B_n is a regular local ring, C_n is a regular local ring with $C_n^h = B_n^h$ [R, (1.3)].

Claim. For every n , C_{n+1} dominates C_n and V dominates C_n . Also $\bigcup_{n=1}^{\infty} C_n = V$.

Proof of the claim. Since B_{n+1} dominates B_n , we see that B_{n+1}^h dominates B_n^h and hence $C_{n+1} = B_{n+1}^h \cap L$ dominates $C_n = B_n^h \cap L$. Since $C_n = B_n^h \cap L \subseteq V^h \cap L = V$, it follows that V dominates C_n and $V_0 \subseteq \bigcup_{n=1}^{\infty} C_n \subseteq V$. Since V birationally dominates $\bigcup_{n=1}^{\infty} C_n$, it suffices to show that $\bigcup_{n=1}^{\infty} C_n$ is a DVR.

But by the same argument as before, $\bigcup_{n=1}^{\infty} C_n^h = (\bigcup_{n=1}^{\infty} C_n)^h = V^h$. This shows that $\bigcup_{n=1}^{\infty} C_n$ is a DVR, and therefore $\bigcup_{n=1}^{\infty} C_n = V$. Thus in the case where L/k is finitely generated we have completed the proof of the main theorem including the ‘‘Moreover’’ statement.

2.2 Remark. An alternate approach to the definition of C_n is as follows. Since V is a directed union of étale extensions of V_0 and $\mathcal{Q}(V) = L$ is finite algebraic over $\mathcal{Q}(V_0) = F$, V is étale over V_0 and therefore $V = V_0[\theta] = V_0[X]/(f(X))$, where $f(X)$ is a monic polynomial such that $f(\theta) = 0$ and $f'(\theta)$ is a unit of V . Let B'_n denote

the integral closure of B_n in L and let $C_n = (B'_n)_{(\mathfrak{n} \cap B'_n)}$. Since $\bigcup_{n=1}^{\infty} B_n = V_0$, it follows that $\bigcup_{n=1}^{\infty} C_n = V$. Moreover, for all sufficiently large n , $f(X) \in B_n[X]$ and $f'(\theta)$ is a unit of C_n . Therefore C_n is a regular local ring for all sufficiently large n [N2, (38.6)]. As we note in Remark 2.3 below, this allows us to deduce a version of the main theorem also in the case where k has characteristic $p > 0$, provided the field F can be chosen so that L/F is separable.

Completion of the proof of the main theorem. If L is not finitely generated over k , we choose a nested family of fields L_α , with $\alpha \in \Gamma$, such that

- (1) $F \subseteq L_\alpha$, for all α .
- (2) L_α is finite algebraic over F .
- (3) $\bigcup_{\alpha \in \Gamma} L_\alpha = L$.

The rings $V_\alpha = L_\alpha \cap V$ are DVRs with $\bigcup_{\alpha \in \Gamma} V_\alpha = V$ and $V_\alpha^h = V^h$, since $V_0 \subseteq V_\alpha$, for each $\alpha \in \Gamma$.

As above, $\bigcup_{n=1}^{\infty} B_n^h = V^h$, $\bigcup_{n=1}^{\infty} \mathcal{Q}(B_n^h) = \mathcal{Q}(V^h)$ and $L \subseteq \mathcal{Q}(V^h)$. Thus we see that for each $\alpha \in \Gamma$, there is an $n_\alpha \in \mathbb{N}$ such that $L_\alpha \subseteq \mathcal{Q}(B_n^h)$ for all $n \geq n_\alpha$.

Put $C_n^{(\alpha)} = L_\alpha \cap B_n^h$ for each $n \geq n_\alpha$. Then $V_\alpha = \bigcup_{n=n_\alpha}^{\infty} C_n^{(\alpha)}$ and V_α birationally dominates $C_n^{(\alpha)}$. Hence

$$V = \bigcup_{\alpha \in \Gamma, n \geq n_\alpha} C_n^{(\alpha)}.$$

This completes the proof of the main theorem. \square

2.3 Remark. If the characteristic of k is $p > 0$, then the Henselization V_0^h of $V_0 = F \cap k[[y]]$ may not equal the Henselization V^h of $V = L \cap k[[y]]$, because the algebraic field extension L/F may not be separable. But in the case where L/F is separable algebraic, the fact that the DVRs V and V_0 have the same completion implies that V is a directed union of étale extensions of V_0 (see, for example, [AH, Theorem 2.7]). Therefore in the case where L/F is separable algebraic, V is a directed union of regular local rings of dimension d .

Thus for a local domain (R, \mathfrak{m}) essentially of finite type over a field k of characteristic $p > 0$, a result analogous to Corollary 1.3 is true provided there exists a subfield F of L such that F is purely transcendental over k , L/F is separable algebraic, and F contains a generator for the maximal ideal of V .

In characteristic $p > 0$, with V excellent and the extension separable, the ring V_0 need not be excellent (see for example [HRS, (3.3) and (3.4)]).

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