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SEPARATED SEQUENCES IN NONREFLEXIVE BANACH SPACES

ANDRZEJ KRYCZKA AND STANISŁAW PRUS

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Abstract. We prove that there is $c > 1$ such that the unit ball of any nonreflexive Banach space contains a c-separated sequence. The supremum of these constants c is estimated from below by $\sqrt[5]{4}$ and from above approximately by 1.71. Given any $p > 1$, we also construct a nonreflexive space so that if the convex hull of a sequence is sufficiently close to the unit sphere, then its separation constant does not exceed $2^{1/p}$.

1. Introduction

Sequences of separated elements appear quite often in the literature on Banach spaces. They are strongly related to the problem of packing balls (see [15]), average distances (see [2]) and infinite dimensional convexity (see [9]). The last notion in turn plays an important role in metric fixed point theory (see [8] and [1]).

A sequence is said to be r-separated if distances between its elements are bounded from below by r. One of the basic general results on separated sequences is due to Elton and Odell [7]. They proved that the unit ball of an arbitrary infinite dimensional Banach space contains an r-separated sequence for some $r > 1$. In this paper we show that there exists a constant $c > 1$ such that in the unit ball of any nonreflexive Banach space one can find a c-separated sequence. On the other hand for each $1 < p < \infty$ we construct a nonreflexive Banach space Y_p whose unit ball does not contain a sequence with the separation constant greater than $2^{1/p}$ such that its convex hull is arbitrarily close to the unit sphere. These results provide answers to two of the questions posed in [2]. Let us however mention that our notation does not coincide with that of [2].

2. Preliminaries

Let X be a Banach space. By the separation constant of a sequence (x_n) in X we mean the number

$$\operatorname{sep}(x_n) = \inf\{\|x_m - x_n\| : m \neq n\}.$$

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Having a bounded set $A \subset X$, we put

$$\beta(A) = \sup\{\operatorname{sep}(x_n) : \{x_n\} \subset A\}.$$

This quantity is called the separation measure of noncompactness of A (see [1]).

By B_X we denote the closed unit ball of the space X. Kottman [14] introduced the packing constant $P(\aleph_0, X)$ as the supremum of all numbers $r > 0$ such that there exists a sequence of pairwise disjoint balls of radius r contained in B_X. We have

$$P(\aleph_0, X) = \frac{\beta(B_X)}{2 + \beta(B_X)}$$

(see [14]).

Let A be a subset of X. The closed convex hull of A will be denoted by $\operatorname{co} A$. Assuming that the space X is infinite dimensional and $0 \le \epsilon < \beta(B_X)$, we set

$$\Delta(\epsilon) = \inf\{1 - \inf\{\|x\| : x \in \operatorname{co}\{x_n\}\}\}$$

where the first infimum is taken over all sequences (x_n) in B_X with $\operatorname{sep}(x_n) \ge \epsilon$. The function Δ is called the modulus of noncompact convexity of X. It is nonnegative and $\Delta(0) = 0$. Δ can be treated as an infinite dimensional counterpart of the classical modulus of convexity (see [5]). We will also consider the following coefficient:

$$S_0(X) = \sup\{\epsilon \in [0, \beta(B_X)) : \Delta(\epsilon) = 0\}.$$

We have the formula $\beta(B_{l_p}) = 2^{1/p}$ for $1 \le p < \infty$ (see [4] or [15], p. 91). This coefficient may be therefore arbitrarily close to 1. On the other hand $\beta(B_{c_0}) = 2$ and moreover $S_0(l_1) = S_0(c_0) = 2$. It follows that $\beta(B_X) = S_0(X) = 2$ for any Banach space X containing an isomorphic copy of l_1 or c_0 (see [13]).

3. Results

Theorem 1. *There exists a constant $c > 1$ such that if X is a nonreflexive Banach space, then $\beta(B_X) \ge c$.*

Proof. Let X be a nonreflexive Banach space. By a result of James [12] for every $\theta \in (0, 1)$ there are two sequences: (x_n) in B_X and (x_n^*) in B_{X^*} such that $x_k^*(x_i) = \theta$ if $k \le i$ and $x_k^*(x_i) = 0$ if $i < k$.

Let us put $\epsilon_1^1 = 1$, $\epsilon_1^2 = -1$, $\epsilon_i^j = (-1)^i$ if $1 \le j < i$ or $j = 2i$ and $\epsilon_i^j = (-1)^{i+1}$ if $i \le j < 2i$ where $i = 2, 3, 4$. Passing to a subsequence, we can assume that there exist positive constants a_1, a_2, a_3, a_4 for which

$$\theta a_i \le \left\| \sum_{j=1}^{2i} \epsilon_i^j x_{n_j} \right\| \le a_i$$

for any $n_1 < n_2 < \cdots < n_{2i}$ where $i = 1, 2, 3, 4$ (see [3]). We consider the following elements:

$$\begin{aligned}
x_n^0 &= x_n,\\
x_n^1 &= \frac{1}{a_1}(x_{2n-1} - x_{2n}),\\
x_n^2 &= \frac{1}{a_2}(x_1 - x_{3n-1} - x_{3n} + x_{3n+1}),\\
x_n^3 &= \frac{1}{a_3}(-x_1 - x_2 + x_{4n-1} + x_{4n} + x_{4n+1} - x_{4n+2}),\\
x_n^4 &= \frac{1}{a_4}(x_1 + x_2 + x_3 - x_{5n-1} - x_{5n} - x_{5n+1} - x_{5n+2} + x_{5n+3})
\end{aligned}$$

for $n \in \mathbb{N}$. They are contained in B_X. Moreover, $\operatorname{sep}(x_n^0) \geq \theta a_1$ and $\operatorname{sep}(x_n^i) \geq \frac{\theta a_{i+1}}{a_i}$ if $i = 1, 2, 3$. We also have the estimate

$$\|x_m^4 - x_n^4\| \geq x_{5m+3}^*(x_m^4 - x_n^4) = \frac{4\theta}{a_4}$$

whenever $m < n$. It follows that

$$\beta(B_X) \geq \max\left\{\theta a_1, \frac{\theta a_2}{a_1}, \frac{\theta a_3}{a_2}, \frac{\theta a_4}{a_3}, \frac{4\theta}{a_4}\right\} \geq \theta \sqrt[5]{4}.$$

Since $\theta \in (0, 1)$ is arbitrary, we obtain the inequality $\beta(B_X) \geq \sqrt[5]{4}$. □

Corollary 1. *There exists a constant $C > \frac{1}{3}$ such that if X is a nonreflexive Banach space, then $P(\aleph_0, X) \geq C$.*

We do not know what is the greatest possible value of the constant c in Theorem 1. From a remark made by Kottman [14] it follows that if X is James' space (see [11]), then $\beta(B_X) < 2$. We will calculate the exact value of this coefficient for a generalized James' space.

Let us first establish some notation. By an interval of natural numbers we mean a set of the form $[m, n] = \{k \in \mathbb{N} : m \leq k \leq n\}$ where $m \leq n$ are natural numbers. Having such intervals I_1, I_2, we write $I_1 \leq I_2$ if $\max I_1 \leq \min I_2$. Let now $I = [m, n]$ and $x = (x(k))$ be a sequence of real numbers. We put

$$\begin{aligned}
&\delta(I, x) = x(n) - x(m), \quad \delta_+(I, x) = \frac{1}{2}(|\delta(I, x)| + \delta(I, x)),\\
&\delta_-(I, x) = \frac{1}{2}(|\delta(I, x)| - \delta(I, x)).
\end{aligned}$$

We fix $1 < p < \infty$. Following James' construction [11], we introduce the space J_p of all real sequences $x = (x(k))$ converging to 0 and such that

$$\|x\| = \sup\left\{\left(\sum_{k=1}^{n} |\delta(I_k, x)|^p\right)^{1/p} : I_1 \leq I_2 \leq \cdots \leq I_n\right\}$$

is finite. This formula defines a norm on J_p. It is easy to see that the vectors $e_n = (0, \ldots, 0, 1, 0, \ldots)$, where 1 occupies nth place, form a basis of J_p. The basis (e_n) is not boundedly complete (see [10]) and consequently J_p is not reflexive. Having an arbitrary sequence $x = (x(k))$ of real numbers and $n \in \mathbb{N}$, we set

$$P_n x = (x(1), \ldots, x(n), 0, 0, \ldots)$$

and $R_n x = x - P_n x$. P_n restricted to J_p is a norm-one projection. Moreover, $\lim_{n\to\infty} \|R_n x\| = 0$ for every $x \in J_p$. In the sequel we will identify a number $\alpha \in \mathbb{R}$ with a constant sequence $(\alpha, \alpha, \dots)$.

Lemma 1. *Let (x_n) be a bounded sequence in J_p. For every $\gamma > 0$ there exist a subsequence (x_{n_k}), an increasing sequence (m_k) of natural numbers and a constant $\alpha \in \mathbb{R}$ such that*

$$\|x_{n_k} - (P_{m_1} x_{n_1} + P_{m_{2k}} R_{m_1} \alpha + P_{m_{2k+1}} R_{m_{2k}} x_{n_k})\| \leq \gamma$$

for every $k \in \mathbb{N}$.

Proof. Since the sequence (x_n) is bounded, passing to a subsequence, we can assume that for each $i \in \mathbb{N}$ there exists $x(i) = \lim_{n\to\infty} x_n(i)$. It is easy to see that $(x(n))$ is a Cauchy sequence, so it has a limit α. Clearly $x - \alpha \in J_p$.

Therefore, having $\gamma > 0$, we can find m_1 such that $\|R_{m_1}(x-\alpha)\| \leq \frac{1}{6}\gamma$. Next, we put $m_2 = m_1 + 1$ and choose n_1 so that $\|P_{m_2}(x_{n_1} - x)\| \leq \frac{1}{6}\gamma$. There is $m_3 > m_2$ for which $\|R_{m_3} x_{n_1}\| \leq \frac{1}{6}\gamma$.

Proceeding in this way, we obtain sequences (m_k), (n_k) such that

$$\|P_{m_{2k}}(x_{n_k} - x)\| \leq \frac{1}{6}\gamma \quad \text{and} \quad \|R_{m_{2k+1}} x_{n_k}\| \leq \frac{1}{6}\gamma$$

for every $k \in \mathbb{N}$. Then

$$\begin{aligned} \|x_{n_k} - P_{m_1} x_{n_1} - &P_{m_{2k}} R_{m_1} \alpha - P_{m_{2k+1}} R_{m_{2k}} x_{n_k}\| \\ &\leq \|P_{m_1}(x_{n_k} - x_{n_1})\| + \|P_{m_{2k}} R_{m_1}(x_{n_k} - \alpha)\| + \|R_{m_{2k+1}} x_{n_k}\| \\ &\leq \|P_{m_1}(x_{n_1} - x)\| + \|P_{m_{2k}} R_{m_1}(x - \alpha)\| + \frac{4}{6}\gamma \leq \gamma. \end{aligned}$$

□

Theorem 2. *Let $1 < p < \infty$. Then $\beta(B_{J_p}) = (1 + 2^{p-1})^{1/p}$.*

Proof. Clearly $\|2^{-1/p} e_n\| = 1$ and

$$\|2^{-1/p} e_m - 2^{-1/p} e_n\| = (1 + 2^{p-1})^{1/p}$$

for any $m \neq n$. Consequently $\beta(B_{J_p}) \geq (1 + 2^{p-1})^{1/p}$.

In order to prove the opposite inequality, we take a sequence (x_n) in B_{J_p} and $\gamma \in (0, 1)$. By Lemma 1 we obtain corresponding sequences (n_k), (m_k) and a constant α. Let us put

$$\begin{aligned} u_1 &= P_{m_1} x_{n_1} + P_{m_2} R_{m_1} \alpha + P_{m_3} R_{m_2} x_{n_1}, \\ u_2 &= P_{m_1} x_{n_1} + P_{m_4} R_{m_1} \alpha + P_{m_5} R_{m_4} x_{n_2} \end{aligned}$$

and $w = u_2 - u_1$.

There is a sequence $I_1 \leq I_2 \leq \dots \leq I_n$ of intervals of natural numbers such that

$$((1-\gamma)\|w\|)^p \leq \sum_{k=1}^{n} |\delta(I_k, w)|^p.$$

Clearly $w(i) = 0$ if $i \leq m_2$. Therefore we can assume that $m_2 \leq \min I_1$. We can also assume that there is an interval $I_s = [j_1, j_2]$ for which $j_1 \leq m_3 < j_2$.

If $k < s$, then $\delta(I_k, w) = -\delta(I_k, u_1)$ and if $k > s$, then $\delta(I_k, w) = \delta(I_k, u_2)$. It follows that

$$\sum_{k<s} |\delta(I_k, w)|^p + |u_1(j_1)|^p \le \|u_1\|^p \le (1+\gamma)^p$$

and

$$|\alpha - u_2(j_2)|^p + \sum_{k>s} |\delta(I_k, w)|^p \le (1+\gamma)^p.$$

Here we treat a sum over the empty set as 0. Consequently

$$\begin{aligned}((1-\gamma)\|w\|)^p &\le 2(1+\gamma)^p - |u_1(j_1)|^p + |u_2(j_2) - (\alpha - u_1(j_1))|^p - |\alpha - u_2(j_2)|^p \\ &\le 2(1+\gamma)^p + (1 - 2^{1-p})|u_1(j_1) + u_2(j_2) - \alpha|^p.\end{aligned}$$

But

$$2^{1-p}|2u_1(j_1) - \alpha|^p \le |u_1(j_1)|^p + |\alpha - u_1(j_1)|^p \le \|u_1\|^p \le (1+\gamma)^p$$

and similarly

$$2^{1-p}|2u_2(j_2) - \alpha|^p \le (1+\gamma)^p.$$

Hence

$$|u_1(j_1) + u_2(j_2) - \alpha| \le 2^{1-1/p}(1+\gamma)$$

which gives us the inequality $((1-\gamma)\|w\|)^p \le (1+\gamma)^p(1+2^{p-1})$. On the other hand $\|x_{n_1} - x_{n_2}\| \le \|w\| + 2\gamma$. Therefore

$$\operatorname{sep}(x_n) \le (1-\gamma)^{-1}(1+\gamma)(1+2^{p-1})^{1/p} + 2\gamma.$$

Since $\gamma > 0$ is arbitrary, we obtain $\operatorname{sep}(x_n) \le (1+2^{p-1})^{1/p}$. □

Let us notice that the minimal value of $(1+2^{p-1})^{1/p}$ is approximately equal to 1.71. It is an upper bound for the constant c in Theorem 1. Our proof of Theorem 1 gives $c = \sqrt[5]{4}$ which is about 1.32.

In [6] it was shown that if $S_0(X) < 1$, then the space X is reflexive. In contrast to Theorem 1 we will prove that 1 is the greatest lower bound of the values of $S_0(X)$ for nonreflexive spaces X.

Having $x \in J_p$, we put

$$|x|_+ = \sup\left\{\left(\sum_{k=1}^n (\delta_+(I_k, x))^p\right)^{1/p} : I_1 \le I_2 \le \cdots \le I_n\right\},$$

$$|x|_- = \sup\left\{\left(\sum_{k=1}^n (\delta_-(I_k, x))^p\right)^{1/p} : I_1 \le I_2 \le \cdots \le I_n\right\}$$

and

$$\|x\|_\pm = (|x|_+^p + |x|_-^p)^{1/p}.$$

This formula gives an equivalent norm on J_p. Namely

$$\|x\| \le \|x\|_\pm \le 2^{1/p}\|x\|$$

for every $x \in J_p$. The space J_p with the norm $\|\cdot\|_\pm$ will be denoted by Y_p. Considering the sequence $(2^{-1/p}e_n)$, we see that $\beta(B_{Y_p}) \ge (1+2^{p-1})^{1/p}$.

Theorem 3. *Let $1 < p < \infty$. Then $S_0(Y_p) = 2^{1/p}$.*

Proof. Let us consider the sequence of vectors $y_n = \sum_{k=1}^{n} e_k$. Clearly $\|y_n\|_\pm = 1$ and $\|y_m - y_n\|_\pm = 2^{1/p}$ for any $m \neq n$. It is also easy to see that $\|x\|_\pm = 1$ for every $x \in \mathrm{co}\{y_n\}$. This shows that $S_0(Y_p) \geq 2^{1/p}$.

We will show the opposite inequality. To this end we take $\epsilon \in (2^{1/p}, \beta(B_{Y_p}))$ and choose $\gamma > 0$ so that

$$\gamma \leq \frac{1}{3}(8^{-p}(\epsilon^p - 2))^p.$$

Now let (x_n) be a sequence in B_{Y_p} with $\mathrm{sep}(x_n) \geq \epsilon$. From Lemma 1 we obtain sequences (n_k), (m_k) and a constant α such that

$$\|x_{n_k} - u_k\|_\pm \leq \frac{\gamma}{p2^p}$$

for every $k \in \mathbb{N}$, where $u_k = P_{m_1}x_{n_1} + P_{m_{2k}}R_{m_1}\alpha + P_{m_{2k+1}}R_{m_{2k}}x_{n_k}$. We set $u = P_{m_1}x_{n_1} + P_{m_2}R_{m_1}\alpha$ and $z_N = \frac{1}{N}\sum_{k=1}^{N} u_k$ where $N \in \mathbb{N}$. Clearly it suffices to consider the case when $\alpha \geq 0$.

We will show that

$$\left\|\frac{1}{N}\sum_{k=1}^{N} x_{n_k}\right\|_\pm \leq \|u\|_\pm + 2\gamma \tag{1}$$

for some $N \in \mathbb{N}$. It is easy to see that

$$\left\|\frac{1}{N}\sum_{k=1}^{N} P_{m_{2k+1}}R_{m_{2k}}x_{n_k}\right\|^p \leq \frac{2^{p-1}}{N^p}\sum_{k=1}^{N}\|P_{m_{2k+1}}R_{m_{2k}}x_{n_k}\|^p \leq \frac{2^{2p-1}}{N^{p-1}}$$

for every $N \in \mathbb{N}$. Therefore we can find N such that

$$\left\|\frac{1}{N}\sum_{k=1}^{N} P_{m_{2k+1}}R_{m_{2k}}x_{n_k}\right\|_\pm \leq \gamma.$$

But one can easily check that

$$\left\|\frac{1}{N}\sum_{k=1}^{N}(P_{m_1}x_{n_1} + P_{m_{2k}}R_{m_1}\alpha)\right\|_\pm = \|u\|_\pm.$$

Hence

$$\left\|\frac{1}{N}\sum_{k=1}^{N} x_{n_k}\right\|_\pm \leq \|z_N\|_\pm + \gamma \leq \|u\|_\pm + 2\gamma.$$

Our next aim is to prove that

$$\|u\|_\pm^p < 1 - \frac{1}{3}(8^{-p}(\epsilon^p - 2))^p. \tag{2}$$

There are sequences $T_1^1 \leq T_2^1 \leq \cdots \leq T_r^1$ and $T_1^2 \leq T_2^2 \leq \cdots \leq T_s^2$ of intervals for which

$$(1-\gamma)|u|_+^p \leq \sum_{k=1}^{r}(\delta_+(T_k^1, u))^p, \quad (1-\gamma)|u|_-^p \leq \sum_{k=1}^{s}(\delta_-(T_k^2, u))^p.$$

Since $\alpha \geq 0$, we can assume that $\max T_r^1 \leq m_2$ and $\max T_s^2 = m_2 + 1$. As in the proof of Theorem 2 we put $w = u_2 - u_1$ and find sequences $I_1^1 \leq I_2^1 \leq \cdots \leq I_n^1$ and

$I_1^2 \le I_2^2 \le \cdots \le I_m^2$ of intervals such that $m_2 \le \min I_1^j$, $j = 1, 2$,

$$(1-\gamma)|w|_+^p \le \sum_{k=1}^{n} (\delta_+(I_k^1, w))^p, \quad (1-\gamma)|w|_-^p \le \sum_{k=1}^{m} (\delta_-(I_k^2, w))^p.$$

We can assume that there are s_1, s_2 for which $k_j = \min I_{s_j}^j \le m_3 < \max I_{s_j}^j = l_j$, $j = 1, 2$. Let us put $K_1 = [k_1, m_3 + 1]$, $K_2 = [m_3 + 1, l_2]$. We have

$$\begin{aligned}\sum_{k<s_1} (\delta_+(I_k^1, w))^p &+ (\delta_+(K_1, w))^p + \sum_{k<s_2} (\delta_-(I_k^2, w))^p \\ &= \sum_{k<s_1} (\delta_-(I_k^1, u_1))^p + (\delta_+(K_1, w))^p + \sum_{k<s_2} (\delta_+(I_k^2, u_1))^p \\ &\le \|u_1\|_\pm^p < 1 + \gamma.\end{aligned}$$

Similarly

$$\sum_{k>s_1} (\delta_+(I_k^1, w))^p + (\delta_-(K_2, w))^p + \sum_{k>s_2} (\delta_-(I_k^2, w))^p < 1 + \gamma.$$

Hence

$$\begin{aligned}(1-\gamma)\|w\|_\pm^p \le 2(1+\gamma) &+ (\delta_+(I_{s_1}^1, w))^p - (\delta_+(K_1, w))^p \\ &+ (\delta_-(I_{s_2}^2, w))^p - (\delta_-(K_2, w))^p.\end{aligned}$$

But

$$\|w\|_\pm^p \ge \left(\epsilon - \frac{\gamma}{p2^{p-1}}\right)^p > \epsilon^p - \gamma.$$

It follows that

$$\frac{1}{2}(\epsilon^p - 2) < (\delta_+(I_{s_1}^1, w))^p - (\delta_+(K_1, w))^p + (\delta_-(I_{s_2}^2, w))^p - (\delta_-(K_2, w))^p.$$

We have two cases.

I. $\frac{1}{4}(\epsilon^p - 2) < (\delta_+(I_{s_1}^1, w))^p - (\delta_+(K_1, w))^p$.

By the mean value theorem

$$\begin{aligned}(\delta_+(I_{s_1}^1, w))^p - (\delta_+(K_1, w))^p &\le p\|w\|_\pm^{p-1}(\delta_+(I_{s_1}^1, w) - \delta_+(K_1, w)) \\ &\le p4^{p-1}\frac{1}{2}(|w(l_1) - \alpha| + w(l_1) - \alpha) \\ &= p4^{p-1}(u_2(l_1) - \alpha).\end{aligned}$$

Therefore $8^{-p}(\epsilon^p - 2) < u_2(l_1) - \alpha$, so we obtain

$$\begin{aligned}\|u_2\|_\pm^p &\ge \sum_{k=1}^{r} (\delta_+(T_k^1, u))^p + (u_2(l_1) - \alpha)^p + \sum_{k=1}^{s} (\delta_-(T_k^2, u))^p \\ &> (1-\gamma)\|u\|_\pm^p + (8^{-p}(\epsilon^p - 2))^p.\end{aligned}$$

Consequently

$$\|u\|_\pm^p < (1-\gamma)^{-1}(1 + \gamma - (8^{-p}(\epsilon^p - 2))^p)$$

which gives (2).

II. $\frac{1}{4}(\epsilon^p - 2) < (\delta_-(I^2_{s_2}, w))^p - (\delta_-(K_2, w))^p$.

Similarly as in the previous case we see that

$$\begin{aligned}(\delta_-(I^2_{s_2}, w))^p &- (\delta_-(K_2, w))^p \\ &\le p\|w\|_{\pm}^{p-1}(\delta_-(I^2_{s_2}, w) - \delta_-(K_2, w)) \\ &\le p4^{p-1}(-u_1(k_2)).\end{aligned}$$

Hence $8^{-p}(\epsilon^p - 2) < -u_1(k_2)$. But

$$\sum_{k=1}^{r}(\delta_+(T^1_k, u))^p + \sum_{k=1}^{s}(\delta_-(T^2_k, u))^p + (-u_1(k_2))^p \le \|u_1\|_{\pm}^p.$$

Therefore

$$(1-\gamma)\|u\|_{\pm}^p + (8^{-p}(\epsilon^p - 2))^p < 1 + \gamma$$

which implies (2).

From (1) and (2) we see that

$$\inf\{\|x\| : x \in \operatorname{co}\{x_n\}\} < \left(1 - \frac{1}{3}(8^{-p}(\epsilon^p - 2))^p\right)^{1/p} + 2\gamma.$$

Since $\gamma > 0$ can be arbitrarily small, we obtain

$$\Delta(\epsilon) \ge 1 - \left(1 - \frac{1}{3}(8^{-p}(\epsilon^p - 2))^p\right)^{1/p}$$

where Δ is the modulus of noncompact convexity of Y_p. This shows that $S_0(Y_p) \le 2^{1/p}$. □

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Department of Mathematics, M. Curie-Skłodowska University, 20-031 Lublin, Poland
E-mail address: akryczka@golem.umcs.lublin.pl

Department of Mathematics, M. Curie-Skłodowska University, 20-031 Lublin, Poland
E-mail address: bsprus@golem.umcs.lublin.pl