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STRONGLY MEAGER SETS AND THEIR UNIFORMLY CONTINUOUS IMAGES

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Abstract. We prove the following theorems:

(1) Suppose that $f : 2^\omega \to 2^\omega$ is a continuous function and X is a Sierpiński set. Then
(A) for any strongly measure zero set Y, the image $f[X+Y]$ is an s_0-set,
(B) $f[X]$ is a perfectly meager set in the transitive sense.

(2) Every strongly meager set is completely Ramsey null.

This paper is a continuation of earlier works by the authors and by M. Scheepers (see [N], [NSW], [S]) in which properties (mainly, the algebraic sum) of certain singular subsets of the real line **R** and of the Cantor set 2^ω were investigated. Throughout the paper, by a set of real numbers we mean a subset of 2^ω and by "+" we denote the standard modulo 2 coordinatewise addition in 2^ω. Let us also assume that a "measure zero" (or "negligible") set always denotes a Lebesgue measure zero set. We apply the following definition of sets of real numbers.

Definition 1. An uncountable set X is said to be a Luzin (respectively, Sierpiński) set iff for each meager (respectively, measure zero) set Y, $X \cap Y$ is at most countable. We say that a set X is of strong measure zero (respectively, strongly meager) iff for each meager (respectively, measure zero) set Y, $X + Y \neq 2^\omega$.

Remark 1. It is well known (see [M] for example) that every Luzin set is strongly measure zero. Quite recently J. Pawlikowski proved that each Sierpiński set must be strongly meager as well (see [P]). Let us recall that a set X is called an s_0-set (or Marczewski set) iff for each perfect set P one can find a perfect set $Q \subseteq P$ that is disjoint from X. M. Scheepers showed in [S] that for a Sierpiński set X and a strong measure zero set Y, $X + Y$ is an s_0-set. Later, in [NSW] it was proven that this also holds when X is strongly meager. We have the following functional version of the M. Scheepers' result.

Theorem 1. *Let X be a Sierpiński set and let Y be a strong measure zero set. Assume also that $f : 2^\omega \to 2^\omega$ is a continuous function. Then the image $f[X+Y]$ is an s_0-set.*

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Proof. Let $P\subseteq 2^\omega$ be a perfect set. We can assume that $f[2^\omega]\cap P$ contains a perfect set. Otherwise, we are done. So, fix $\{R_\alpha : \alpha < \omega_1\}$, a family of pairwise disjoint perfect subsets contained in $f[2^\omega] \cap P$, and for every $\alpha < \omega_1$, put $R'_\alpha = f^{-1}[R_\alpha]$. Take $\alpha_0 < \omega_1$ such that $R'_{\alpha_0} \in \mathcal{N}$ (negligible sets). We know that Y is of strong measure zero and that R'_{α_0} is closed, so $R'_{\alpha_0} - Y$ has measure zero. From this we have that $X \cap (R'_{\alpha_0} - Y)$ (let us denote this set by X') is countable. Thus,

$$f[X + Y] = f[(X \setminus X') + Y] \cup f[X' + Y],$$

and $f[(X \setminus X') + Y]$ is disjoint from R_{α_0}. Also, since f is a uniformly continuous function, $f[X' + Y]$ is a strong measure zero set. Hence $f[X + Y]$ is disjoint from some perfect set contained in R_{α_0}. □

Definition 2. A set X is called an AFC' set (perfectly meager in the transitive sense) iff for each perfect set P there is F, an F_σ set containing X, such that for every $t \in 2^\omega$, $(F + t) \cap P$ is meager in the relative topology of P. We will say that X is a wQN-set (weakly Quasinormal set) iff for each sequence of continuous functions $f_n : X \to \mathbf{R}$, if $f_n \to 0$ (pointwise), then there is a subsequence f_{n_k} and countable family $\{X_n\}_{n\in\omega}$ such that $X = \bigcup_{n\in\omega} X_n$ and f_{n_k} converges uniformly on X_n for every $n \in \omega$.

It is easy to prove that each Sierpiński set is wQN and that for a wQN-set X and every continuous function $f : 2^\omega \to 2^\omega$, $f[X]$ is a wQN-set as well (see [BRR]). Thus, using Nowik's theorem which says that any wQN-set is an AFC' set (see [N]), we obtain the following theorem.

Theorem 2. *If S is a Sierpiński set, then for every continuous function $f : 2^\omega \to 2^\omega$, we have that $f[S]$ is an AFC' set.*

We present an alternative proof of this fact with the hope that it may lead to a positive answer to Question 1 (see below).

Lemma 1 (Nowik)**.** *For each perfect set $P\subseteq 2^\omega$, there exists a continuous function $\Phi : 2^\omega \to 2^\omega$ such that for any $t \in 2^\omega$, $\Phi[P + t] = 2^\omega$.*

Proof. See [N].

Corollary 1. *Let $P\subseteq 2^\omega$ be a perfect set. Then there exists an uncountable family $\mathcal{H}$ of pairwise disjoint closed subsets of 2^ω such that for every $G \in \mathcal{H}$, $P + G = 2^\omega$.*

Proof. Let Φ be as in Lemma 1. Put $\mathcal{H} = \{\Phi^{-1}[\{h\}] : h \in 2^\omega\}$. □

Corollary 2. *For every perfect set $P\subseteq 2^\omega$, there exists an uncountable family $\mathcal{H}$ of closed, pairwise disjoint negligible sets such that for each $G \in \mathcal{H}$, $P + G = 2^\omega$.*

Proof. Obvious, since for $\mathcal{H}$ in Corollary 1 we have that $|\{G \in \mathcal{H} : G \notin \mathcal{N}\}| \leq \omega$. □

Proof of Theorem 2. Let $P\subseteq 2^\omega$ be a perfect set and let f be a continuous function. Without loss of generality we may assume that f maps 2^ω onto 2^ω. Suppose that $(P_i)_{i\in\omega}$ is an enumeration of basic clopen sets in the relative topology of P. Assume that for each $i \in \omega$, $\mathcal{H}^i$ is an uncountable family of pairwise disjoint, closed sets such that

$$\forall_{G\in\mathcal{H}^i} P_i + G = 2^\omega.$$

Let

$$\tilde{\mathcal{H}}^i = \{f^{-1}[G] : G \in \mathcal{H}^i\}.$$

We choose from every $\tilde{\mathcal{H}}^i$ a negligible set A_i. Suppose that B, a G_δ negligible set, is such that

$$\bigcup_{i<\omega} A_i \subseteq B$$

and $C = 2^\omega \setminus B$. Since S is a Sierpiński set, $S' = S \cap B$ is at most countable. We have that

$$f[C] \cap \bigcup_{i<\omega} f[A_i] = \emptyset,$$

so

$$f[S \setminus S'] \cap \bigcup_{i<\omega} f[A_i] = \emptyset.$$

It is clear that $f[C]$ is an F_σ set; thus $f[S\setminus S']$ is disjoint from some G_δ set A which contains $\bigcup_{i<\omega} f[A_i]$. Finally, for every $t \in 2^\omega$, $f[S \setminus S'] \cap (P - t)$ is disjoint from $A \cap (P - t)$. From the fact that $f[A_i] + P_i = 2^\omega$ (for every $i < \omega$) it follows that A is a dense set in $P - t$. □

Remark 2. Notice that Corollary 2 is a stronger version of the well-known Erdös-Kunen-Mauldin theorem (see [NSW]).

Definition 3. For any finite set $s \in [\omega]^{<\omega}$ and infinite $A \subseteq \omega$ with $\max(s) < \min(A)$, let $[s, A] = \{B \in [\omega]^\omega : s \subseteq B \subseteq s \cup A\}$. We say that $F \subseteq [\omega]^\omega$ is <u>a completely Ramsey null</u> (CR_0) set iff for every so-called Ellentuck basic neighbourhood $[s, A]$, there is $B \subseteq A$ infinite such that $[s, B] \cap F = \emptyset$.

Notice that the σ-ideal CR_0 is defined on subsets of the set $[\omega]^\omega$ which can be identified with a subset of 2^ω via characteristic functions. Thus, in the next part we deal with subsets of 2^ω.

Theorem 3. *For any* $[s, A]$*, where* $A \in [\omega]^\omega$, $\max s < \min A$*, there exists a negligible set* H *(even "small" in the sense of T. Bartoszyński) such that*

$$\forall_{t \in 2^\omega} \exists_{B \in [A]^\omega} [s, B] \subseteq H + t.$$

Proof. Consider a partition of ω into finite disjoint intervals, say $(I_n)_{n<\omega}$, which satisfies the following conditions:

1.
$$\forall_{n<\omega} \frac{\ln|A \cap I_n| + 1}{|A \cap I_n|} \leq \frac{1}{2^n},$$

2.
$$\max(s) < \min(I_0).$$

By Lorentz's theorem (see for example [NSW]), we can find $H_n \subseteq 2^{I_n}$ with the properties:

1.
$$|H_n| \leq \frac{\ln|A \cap I_n| + 1}{|A \cap I_n|} \cdot 2^{|I_n|},$$

2.
$$H_n + \{e^n_a : a \in A \cap I_n\} = 2^{I_n},$$

where e_a^n is an element of 2^{I_n} defined by the following condition:

$$e_a^n(b) = \begin{cases} 0 & \text{if} \quad a \neq b, \\ 1 & \text{if} \quad a = b. \end{cases}$$

It is clear that the set

$$H = \{x \in 2^\omega : \exists_n^\infty x|I_n \in H_n\}$$

is negligible; moreover, it is "small" (see [BJ] for the definition of a "small" set). Let us fix $t \in 2^\omega$. For every $n \in \omega$, there exists $a_n \in A$ such that

$$(t|I_n) + e_{a_n}^n \in H_n.$$

Put $B = \{a_n\}_{n<\omega}$. It is sufficient to show that $[s, B] \subseteq H + t$. So, let $C \in [\omega]^\omega \subseteq 2^\omega$ satisfy $s \subseteq C \subseteq s \cup B$. We have that

$$\exists_n^\infty C|I_n = e_{a_n}^n.$$

Thus,

$$C \in \{x : \exists_n^\infty (x+t)|I_n \in H_n\} = H + t.$$

□

Theorem 4. *Every strongly meager set is a completely Ramsey null set.*

Proof. Immediately follows from Theorem 3. □

In the proof of Theorem 1 we used an observation that for a strong measure zero set $X \subseteq 2^\omega$ and for every continuous function $f : 2^\omega \to 2^\omega$, the image $f[X]$ is also strongly measure zero. It is due to Rothberger (see [M]) that (assuming CH) there exist a set X of strong measure zero and a continuous function $f : X \to 2^\omega$ such that $f[X] = 2^\omega$. Also, (assuming CH) one can find a strongly meager set X and a continuous function $f : X \to 2^\omega$ such that $f[X] = 2^\omega$. It is a natural guess that for a strongly meager set X, and for every continuous function $f : 2^\omega \to 2^\omega$, we have that $f[X]$ is also strongly meager. However, it is not even known if for such X and f, the image $f[X]$ has to be an s_0-set.

Question 1. Is it true that for a strongly meager set X and for every continuous $f : 2^\omega \to 2^\omega$, $f[X]$ has the Marczewski property s_0?

Question 2. Is it possible to find for each continuous function $f : 2^\omega \to 2^\omega$ a negligible set H such that

$$\forall_{t \in 2^\omega} \exists_{P \in Perf} f^{-1}[P] \subseteq H + t?$$

We have the following simple observation.

Observation 1. *A positive answer to Question 2 yields the answer "yes" to Question 1.*

Proof. Assume that X is strongly meager and $f : 2^\omega \to 2^\omega$ is a continuous function. Let $P \subseteq 2^\omega$ be a perfect set. Fix a homeomorphism $h : P \to 2^\omega$ and a retraction $g : 2^\omega \to P$. Consider $\phi = h \circ g \circ f$. There is a negligible set $H \subseteq 2^\omega$ such that

$$\forall_{t \in 2^\omega} \exists_{Q \subseteq 2^\omega, Q \text{ perfect}} \phi^{-1}[Q] \subseteq H + t.$$

Take $t_0 \in 2^\omega$ such that $(H + t_0) \cap X = \emptyset$. We have that for some perfect set $Q' \subseteq P$, $f^{-1}[Q'] \cap X = \emptyset$. This implies that $f[X] \cap Q' = \emptyset$. □

Theorem 5. *For every continuous $f : 2^\omega \to 2^\omega$, there is H, a closed nowhere dense set, such that*

$$\forall_{t\in 2^\omega} \exists_{P\in Perf} f^{-1}[P] \subseteq H + t.$$

Proof. Let $n \in \omega$. Choose $N_n \in \omega$ such that for every $s \in 2^{N_n}$, one can find $t_s^{(n)} \in 2^n$ satisfying

$$f[C_s] \subseteq C_{t_s^{(n)}}, \tag{1}$$

where for $t \in 2^{<\omega}$, $C_t = \{x \in 2^\omega : x| \text{ length } (t) = t\}$.

We put $n_0 = 1$, $k_0 = N_{n_0}$ and choose n_1 to get the inequality $2^{n_1 - n_0} > 2^{k_0} + 1$, $k_1 = N_{n_1}$. In general, we choose n_l in such a way that the inequality $2^{n_l - n_{l-1}} > 2^{k_{l-1}} + 1$, $k_l = N_{n_l}$, holds.

We define

$$H_l = \{s \in 2^{[k_{l-1}, k_l)} : s \not\equiv 0 | [k_{l-1}, k_l)\},$$

and we put

$$H = \{x \in 2^\omega : \forall_{l>0} x | [k_{l-1}, k_l) \in H_l\}.$$

One can check that $H \in \mathcal{MGR}$ (meager sets). In fact, H is a closed, nowhere dense set. Consider $t \in 2^\omega$. We have that

$$\begin{aligned} &f\big[2^\omega \setminus \{x \in 2^\omega : (x+t)|[k_{l-1}, k_l) \not\equiv 0|[k_{l-1}, k_l)\}\big] \\ &\subseteq f[\{x \in 2^\omega : x|[k_{l-1}, k_l) = t|[k_{l-1}, k_l)\}] \\ &\subseteq \bigcup_{s \in 2^{k_l}, s|[k_{l-1}, k_l) = t|[k_{l-1}, k_l)} f[C_s] \\ &\subseteq \bigcup_{s \in 2^{k_l}, s|[k_{l-1}, k_l) = t|[k_{l-1}, k_l)} C_{t_s^{(n_l)}}. \end{aligned}$$

The last inclusion follows from (1).

Since $2^{k_{l-1}} + 1 < 2^{n_l - n_{l-1}}$, one can find $x_l^{(0)}, x_l^{(1)} \in 2^{[n_{l-1}; n_l)}$, $x_l^{(0)} \neq x_l^{(1)}$, such that

$$\{x \in 2^\omega : x|[n_{l-1}, n_l) = x_l^{(r)}\}$$
$$\cap\, f\big[2^\omega \setminus \{x \in 2^\omega : (x+t)|[k_{l-1}, k_l) \not\equiv 0|[k_{l-1}, k_l)\}\big] = \emptyset, \tag{2}$$

for $r \in 2$.

We put

$$P = \{x \in 2^\omega : \forall_{l>0} x|[n_{l-1}, n_l) \in \{x_l^{(0)}, x_l^{(1)}\}\}.$$

Clearly, P is a perfect subset of 2^ω. We must check that

$$f^{-1}[P] \subseteq H + t.$$

It suffices to show that if $l < \omega$ and $z \in 2^\omega$ are such that

$$f(z)|([n_{l-1}, n_l)) \in \{x_l^{(0)}, x_l^{(1)}\},$$

then

$$(z+t)|[k_{l-1}, k_l) \not\equiv 0|[k_{l-1}, k_l).$$

So, take $z \in 2^\omega$ satisfying

$$(z+t)|[k_{l-1}, k_l) \equiv 0|[k_{l-1}, k_l).$$

This means that

$$z \in 2^\omega \setminus \{x \in 2^\omega : (x+t)|[k_{l-1}, k_l) \not\equiv 0|[k_{l-1}, k_l)\}.$$

Thus, $f(z)|[n_{l-1}, n_l) \notin \{x_l^{(0)}, x_l^{(1)}\}$ by (2). $\square$

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