A NOTE ON HAMILTON SEQUENCES FOR EXTREMAL BELTRAMI COEFFICIENTS

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Abstract. F. P. Gardiner gave a sufficient condition for a sequence to be a Hamilton sequence for an extremal Beltrami coefficient. In this note, we shall consider the converse problem, proving that the condition is not necessary.

1. Introduction

Given a hyperbolic Riemann surface $R$ covered by the unit disk $\Delta$, we denote by $M(R)$ the unit ball of the space $L^\infty(R)$ of all essentially bounded Beltrami differentials on $R$. We also denote by $SQ(R)$ the unit sphere of the space $Q(R)$ of all integrable holomorphic quadratic differentials on $R$. Let $\Phi: M(R) \to T(R)$ denote the canonical projection from $M(R)$ to the Teichmüller space $T(R)$ of $R$.

We first recall the following

Theorem 1. Suppose $\mu \in M(R)$ is extremal and $(\phi_n)$ is a sequence in $SQ(R)$. If $\Phi(k_n|\phi_n)/\phi_n$ converges in the Teichmüller metric to $\Phi(\mu)$ for some sequence $(k_n)$, then $(\phi_n)$ is a Hamilton sequence for $\mu$.

Theorem 1 was proved by F. P. Gardiner [1] (see also [5]). In this note, we shall consider the converse problem, proving that the converse of Theorem 1 is not true. In fact, we can prove the following stronger result.

Theorem 2. Let $R$ be of conformal infinite type. Any non-zero extremal Beltrami coefficient $\mu \in M(R)$ possesses a Hamilton sequence $(\phi_n)$ such that, for any sequence $(k_n)$ in $(0, 1)$, $\Phi(k_n|\phi_n)/\phi_n$ does not converge to $\Phi(\mu)$ in the Teichmüller metric.

2. Preliminaries

In this section, we will recall some basic definitions and notations from Teichmüller theory. For more details see the book [2].

For a given $\mu \in M(R)$, denote by $f^\mu$ the quasiconformal mapping with domain $R$ and Beltrami coefficient $\mu$, which is uniquely determined up to a conformal mapping on $f^\mu(R)$. Two elements $\mu$ and $\nu$ in $M(R)$ are equivalent, which is denoted by $\mu \sim \nu$, if $f^\mu$ and $f^\nu$ are Teichmüller equivalent, meaning as usual that there...

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exists a conformal mapping $g$ from $f^\mu(R)$ onto $f^\nu(R)$ such that $f^\nu$ and $g \circ f^\mu$ are homotopic (mod $\partial R$). Then $T(R) = M(R)/\sim$ is the Teichmüller space of $R$. Recall that $\Phi : M(R) \to T(R)$ denotes the canonical projection.

For any Beltrami coefficient $\mu \in M(R)$, define

$$k_0(\mu) = \inf\{\|\nu\|_\infty : \nu \sim \mu\},$$

and set

$$K_0(\mu) = \frac{1 + k_0(\mu)}{1 - k_0(\mu)}.$$

Similarly, we define

$$h(\mu) = \inf\{\|\nu|R - E\|_\infty : \nu \sim \mu, E \subset R \text{ compact}\},$$

and let

$$H(\mu) = \frac{1 + h(\mu)}{1 - h(\mu)}.$$

Now the Teichmüller distance between points $\Phi(\mu_1)$ and $\Phi(\mu_2)$ is defined as

$$d(\Phi(\mu_1), \Phi(\mu_2)) = \frac{1}{2} \log K_0(\mu),$$

where $\mu$ is the Beltrami coefficient of the mapping $f^\mu \circ (f^\nu)^{-1}$.

We say that $\mu \in M(R)$ is extremal if $\|\mu\|_\infty = k_0(\mu)$. It is well known (see [3], [4], [6] or Chapter 6 in [2]) that $\mu$ is extremal iff $\mu$ satisfies the Hamilton-Krushkal condition, that is, there exists a sequence $(\phi_n)$ in $SQ(R)$ such that

$$\lim_{n \to \infty} \Re \int_R \mu \phi_n = \|\mu\|_\infty.$$

Such a sequence $(\phi_n)$ is called a Hamilton sequence for $\mu$. It is called degenerate if $\phi_n \to 0$ locally uniformly in $\bar{R}$.

We also need two fundamental Reich-Strebel inequalities (see [6] or Chapter 6 in [2]). They are

1. \[ \frac{1}{K_0(\mu)} \leq \int_R |\phi| \frac{|1 - \mu \frac{\phi}{|\phi|}|^2}{1 - |\mu|^2} \quad \text{for all } \phi \in SQ(R) \]

and

2. \[ K_0(\mu) \leq \int_R |\phi_0| \frac{|1 + \mu \frac{\phi_0}{|\phi_0|}|^2}{1 - |\mu|^2}, \]

where $\phi_0 \in SQ(R)$ satisfies $k_0(\mu)|\phi_0|/\phi_0 \sim \mu$.

3. **Proof of Theorem 2**

Let $\mu$ be an extremal Beltrami coefficient with $k = \|\mu\|_\infty > 0$. If $\Phi(\mu)$ is a Strebel point (for the definition of Strebel point, see, for example, [2]), then there exists some $\phi$ in $SQ(R)$ such that $\mu = k|\phi|/\phi$. By the density of non-Strebel differentials (see [5]), there exists a sequence $(\phi_n)$ in $SQ(R)$ such that $\|\phi_n - \phi\| \to 0$ as $n \to \infty$, but each point $\Phi(k_n|\phi_n|/\phi_n)$ is not a Strebel point for $k_n \in (0, 1)$. Since $\lim_{n \to \infty} \|\phi_n - \phi\| = 0$, it is obvious that $(\phi_n)$ is a Hamilton sequence for $\mu$. On the other hand, since $\Phi(\mu)$ is a Strebel point, by the openness of Strebel points, $\Phi(k_n|\phi_n|/\phi_n)$ cannot converge in the Teichmüller metric to $\Phi(\mu)$ for any sequence $(k_n)$ in $(0, 1)$. 
Now let \( \mu \) be an extremal Beltrami coefficient with \( k = \| \mu \|_{\infty} > 0 \) such that \( \Phi(\mu) \) is not a Strebel point, that is, \( \mu \) possesses a degenerate Hamilton sequence. Let \( (\phi_n) \) in \( SQ(R) \) be such a sequence; then

\[
(3) \quad \lim_{n \to \infty} \Re \iint_{R} \mu \phi_n = k.
\]

Choose a sequence \( \{E_n\} \) of compact subsets of \( R \) such that

\[
(4) \quad \iint_{E_n} |\phi_n| = 1 + o(1) \quad \text{as} \quad n \to \infty.
\]

Define \( \mu_n = \mu \chi_{E_n} \), where \( \chi \) denotes the characteristic function of a set.

By the fundamental inequality (1), noting (3) and (4), we get, as \( n \to \infty \), that

\[
\frac{1}{K_0(\mu_n)} \leq \iint_{R} \frac{|1 - \mu_n \bar{\phi_n}|^2}{|1 - |\mu_n|^2|} \phi_n \\overline{\phi_n} + \iint_{R - E_n} |\phi_n| = \iint_{E_n} \frac{|1 - \mu \bar{\phi_n}|^2}{|1 - |\mu|^2|} \phi_n \\overline{\phi_n} + \iint_{R - E_n} |\phi_n| = \iint_{R} \frac{|1 - \mu \bar{\phi_n}|^2}{|1 - |\mu|^2|} \phi_n \\overline{\phi_n} + o(1)
\]

\[
= \iint_{E_n} \frac{|1 - \mu \bar{\phi_n}|^2}{|1 - |\mu|^2|} \phi_n \\overline{\phi_n} + o(1)
\]

\[
= \iint_{E_n} \frac{|1 - \mu \bar{\phi_n}|^2}{|1 - |\mu|^2|} \phi_n \\overline{\phi_n} + o(1)
\]

which implies

\[
(5) \quad K_0(\mu_n) = \frac{1 + k}{1 - k} + o(1) \quad \text{as} \quad n \to \infty.
\]

Noting that the boundary dilatation \( H(\mu_n) = 1 \), we conclude by Strebel’s Frame Mapping Criterion (see Chapter 6 in [2]) that \( \Phi(\mu_n) \) is a Strebel point when \( n \) is sufficiently large, so there is a \( \psi_n \) in \( SQ(R) \) such that \( \mu_n \approx k_0(\mu_n) ||\psi_n||/\psi_n \). By the fundamental inequality (2), we get

\[
K_0(\mu_n) \leq \iint_{R} |\psi_n| \frac{|1 + \mu_n \bar{\psi_n}|^2}{|1 - |\mu_n|^2|} \frac{\psi_n}{|\psi_n|} \\overline{\psi_n} + \iint_{R - E_n} |\psi_n| \\overline{\psi_n}
\]

\[
= \iint_{E_n} |\psi_n| \frac{|1 + \mu \bar{\psi_n}|^2}{|1 - |\mu|^2|} \phi_n \\overline{\phi_n} + \iint_{R - E_n} |\psi_n| \\overline{\psi_n}
\]

\[
\leq \frac{1 + k}{1 - k} \iint_{E_n} |\psi_n| \\overline{\psi_n} + \iint_{R - E_n} |\psi_n| \\overline{\psi_n},
\]

from which along with (5) it follows that

\[
(7) \quad \iint_{E_n} |\psi_n| = 1 + o(1) \quad \text{as} \quad n \to \infty.
\]
Consequently, by (6), (7) we have, as \( n \to \infty \), that

\[
K_0(\mu_n) \leq \iint_{E_n} |\psi_n| \left| \frac{1 + \mu \frac{\psi_n}{|\psi_n|}}{1 - |\mu|^2} \right|^2 + o(1)
\]

\[
\leq \iint_{E_n} |\psi_n| \left| \frac{1 + \mu \frac{\psi_n}{|\psi_n|}}{1 - |\mu|^2} \right|^2 + \int_{R - E_n} |\psi_n| \left| \frac{1 + \mu \frac{\psi_n}{|\psi_n|}}{1 - |\mu|^2} \right|^2 + o(1)
\]

\[
= \iint_{R} |\psi_n| \left| \frac{1 + \mu \frac{\psi_n}{|\psi_n|}}{1 - |\mu|^2} \right|^2 + o(1),
\]

which, by (5), forces that

\[
\lim_{n \to \infty} \Re \iint_R \mu \psi_n = k,
\]

that is, \((\psi_n)\) is a Hamilton sequence for \( \mu \).

Now we prove that, for any sequence \((k_n)\) in \((0, 1)\), \( \Phi(k_n|\psi_n|/\psi_n) \) does not converge to \( \Phi(\mu) \) in the Teichmüller metric.

In fact, by definition, the Beltrami coefficient \( \nu_n \) of the mapping \( f^{\mu_n} \circ (f^{\mu})^{-1} \) is \( \nu \chi^{f^{\mu}(R - E_n)} \), where \( \nu \) is the Beltrami coefficient of the inverse mapping \( (f^{\mu})^{-1} \), that is

\[
\nu = -\frac{\partial f^{\mu}}{\partial f^{\mu}} \circ (f^{\mu})^{-1}.
\]

Since \( \mu \) is extremal which possesses a degenerate Hamilton sequence, so does \( \nu \).

Noting that \( f^{\mu}(E_n) \) is compact in \( f^{\mu}(R) \), we conclude that \( \nu_n = \nu \chi^{f^{\mu}(R - E_n)} \) is also extremal. Consequently,

\[
d(\Phi(\mu_n), \Phi(\mu)) = \frac{1}{2} \log K_0(\nu_n) = \frac{1}{2} \log \frac{1 + k}{1 - k}.
\]

So \( \Phi(k_0(\mu_n)|\psi_n|/\psi_n) = \Phi(\mu_n) \) does not converge to \( \Phi(\mu) \) in the Teichmüller metric.

Noting that \( k_0(\mu_n) = k + o(1) \) as \( n \to \infty \), we conclude that, for any \((k_n)\) with \( k_n \in (0, 1) \), \( \Phi(k_n|\psi_n|/\psi_n) \) does not converge to \( \Phi(\mu) \) in the Teichmüller metric.

Now the proof of Theorem 2 is complete.

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Theorem 2 says that the converse of Theorem 1 is not true. On the other hand, by the density of Strebel points (see [4]), for any point \( \tau \) in the Teichmüller space \( T(R) \), there exist a sequence \((k_n)\) in \((0, 1)\) and a sequence \((\phi_n)\) in \( SQ(R) \) such that \( \Phi(k_n|\phi_n|/\phi_n) \) converges in the Teichmüller metric to \( \tau \). Now Theorem 1 implies that \((\phi_n)\) is a Hamilton sequence for any extremal Beltrami coefficient \( \mu \) in the class \( \tau \). We state this as

**Proposition 3.** Any point \( \tau \) in the Teichmüller space \( T(R) \) possesses a sequence \((\phi_n)\) in \( SQ(R) \) such that \((\phi_n)\) is a Hamilton sequence for any extremal Beltrami coefficient \( \mu \) in the class \( \tau \) and \( \Phi(k_n|\phi_n|/\phi_n) \) converges in the Teichmüller metric to \( \Phi(\mu) \) for some sequence \((k_n)\) in \((0, 1)\).

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