CONTINUATION METHOD FOR $\alpha$-SUBLINEAR MAPPINGS

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Abstract. Let $B$ be a real Banach space partially ordered by a closed convex cone $P$ with nonempty interior $\overset{\circ}{P}$. We study the continuation method for the monotone operator $A : \overset{\circ}{P} \to \overset{\circ}{P}$ which satisfies

$$A(tx) \geq t^{\alpha(a,b)} A(x),$$

for all $x \in \overset{\circ}{P}, t \in [a, b] \subset (0, 1)$, where $\alpha(a, b) \in (0, 1)$. Thompson’s metric is among the main tools we are using.

1. Introduction

Let $B$ be a real Banach space partially ordered by a closed convex cone $P$ with nonempty interior, which is denoted by $\overset{\circ}{P}$. Suppose $A : \overset{\circ}{P} \to \overset{\circ}{P}$ is monotone, i.e., $Ax \geq Ay$ when $x \geq y$, and satisfies

$$A(tx) \geq \varphi(t) A(x),$$

where $t \in (0, 1)$ and $\varphi$ is a positive function on $(0, 1)$. The fixed points of this type of operator were much discussed under various assumptions on $\varphi$. Among them, M. A. Krasnosel’ski studied $\omega_0$-concave operator (5), where $\varphi(t) = [1 + \eta(x,t)]t$ with $\eta(x,t) > 0$, D. Guo established the existence of the unique fixed point for $\alpha$-concave operators (4), where $\varphi(t) = t^\alpha$ with $\alpha \in (0, 1)$, and U. Krause proved fixed point theorems for ascending operators (6), where $\varphi : [0, 1] \to [0, 1]$ is continuous and $\lambda < \varphi(\lambda)$ for $\lambda \in (0, 1)$. In [1], we investigated the mixed monotone counterpart of the monotone operator $A$ which satisfies

$$A(tx) \geq t^{\alpha(a,b)} A(x),$$

for all $x \in \overset{\circ}{P}, t \in [a, b] \subset (0, 1)$, where $\alpha(a, b) \in (0, 1)$. This class of operator includes Guo’s $\alpha$-concave operator and U. Krause’s ascending operator (see [1 Corollary 3.2]). We say that a monotone operator is $\alpha$-sublinear if it satisfies (2).

One important method for solving an operator equation $F(x) = 0$ is the continuation method, i.e., to continuously deform $F$ to a simpler operator $G$ such that $G(x) = 0$ is easily solved. In the present paper, we intend to discuss the continuation method for $\alpha$-sublinear mappings. Our work is motivated by a paper of A. Granas (3).
$x, y \in P - \{0\}$ are called comparable if there exist positive numbers $\lambda$ and $\mu$ such that $\lambda x \leq y \leq \mu x$. This defines an equivalent relationship, and splits $P - \{0\}$ into disjoint components of $P$. $\hat{P}$ is a component of $P$ if $\hat{P} \neq \emptyset$.

Unless specified otherwise, throughout this paper, we assume that the norm is monotone, i.e., $0 < x \leq y$ implies that $\|x\| \leq \|y\|$. Hence all the cones in this paper are normal, since $P$ is normal iff $B$ has an equivalent norm which is monotone.

Let $C$ be a component of $P$ and $x, y \in C$. Put

$$M(x/y) = \inf \{ \lambda : x \leq \lambda y \} \quad \text{and} \quad M(y/x) = \inf \{ \mu : y \leq \mu x \}.$$  

Thompson’s metric is defined by

$$\tilde{d}(x, y) = \ln \{ \max \{M(x/y), M(y/x)\} \}.$$  

$\tilde{d}(x, y)$ is a metric on $C$ and $C$ is complete with respect to $\tilde{d}$ under our assumption on $P$ ([7, Lemma 3]).

The following theorem is just the monotone operator version of Theorem 3.1 in [1], which was proved by appealing to Thompson’s metric.

**Theorem 1.1.** Let $C$ be a component of $P$, and $A : C \to C$ be $\alpha$-sublinear. Then $A$ has exactly one fixed point $x^*$ in $C$, and for any point $x_0 \in C$, we have $A^n(x_0) \to x^*$ as $n \to \infty$.

We also need the following two lemmas.

**Lemma 1.2** (Thompson [7]). If the norm is monotone, then

$$\|x - y\| \leq 3 b e^{\tilde{d}(x, y) - 1}$$

for all $x, y \in P$ with $\|x\| \leq b$ and $\|y\| \leq b$.

**Lemma 1.3.** Let $u \in \hat{P}$ and $B(u, r) \subset P$, where $B(u, r) = \{ x \in B : \|x - u\| < r \}$. Then

$$\tilde{d}(x, u) \leq \ln \{ \max \left( \frac{r + \|x - u\|}{r}, \frac{r}{r - \|x - u\|} \right) \}$$

for all $x \in B(u, r)$.

**Proof.** Without loss of generality, we assume $x \neq u$. Then $x \in B(u, r)$ implies that $u \pm \frac{r(x - u)}{\|x - u\|} \in P$. It follows that

$$x \leq \frac{r + \|x - u\|}{r} u \quad \text{and} \quad u \leq \frac{r}{r - \|x - u\|} x.$$  

Hence

$$\tilde{d}(x, u) \leq \ln \{ \max \left( \frac{r + \|x - u\|}{r}, \frac{r}{r - \|x - u\|} \right) \}.$$  

Let $(X, d)$ be a complete metric space and $D \subset X$ a closed subset. We say that $T : D \to X$ is a generalized contraction if for each $(a, b) \subset (0, \infty)$, there exists $L(a, b) \in (0, 1)$ such that

$$d(Tx, Ty) \leq L(a, b) d(x, y),$$

where $x, y \in D$ and $a \leq d(x, y) \leq b$. The following theorem is due to M. A. Krasnosel’skiï ([5, Theorem 34.5], see also [2, Theorem (1.3.3)]).
Theorem 2.1. Suppose \( T : X \to X \) is a generalized contraction, then there exists a unique fixed point \( x^* \) of \( T \), and for any point \( x \in X \), we have \( \lim_{n \to \infty} x_n = x^* \), where \( x_n = T^n x, n = 1, 2, \ldots \).

This paper is organized as follows. In Section 2, we generalize A. Granas’s main theorem in [3] to generalized contraction mappings. Section 3 discusses the continuation method for \( \alpha \)-sublinear mappings. An example of application is given in Section 4.

2. Topological transversality for generalized contraction mappings

In this section, \( U \) stands for a bounded open set of \( X \). Let \( G(U) \) be the set of all generalized contraction mappings \( T : \overline{U} \to X \), and \( G_0(U) = \{ T \in G(U) : (\text{Fix} T) \cap \partial U = \emptyset \} \), where \( \text{Fix} T = \{ x \in \overline{U} : x = Tx \} \). We denote \( \text{diam} U = \sup \{ \| x-y \| : x, y \in U \} \) and \( \text{dist}(A_1, A_2) = \inf \{ \| x_1-x_2 \| : x_1 \in A_1, x_2 \in A_2 \} \), where \( A_1 \) and \( A_2 \) are subsets of \( X \).

We say \( T \in G_0(U) \) is traverse or essential (cf. [2, pp. 58-60] and [3]) if \( T \) has a fixed point, i.e., the graph of \( T \) crosses or traverses the diagonal of \( U \times X \). The following theorem discusses the topological transversality for operators in \( G_0(U) \).

Theorem 2.1. Suppose \( \{ H_t \} \subset G_0(U), t \in [0, 1] \), satisfy:

1. \( \text{H1} \) For each \((a, b) \subset (0, \infty)\), there exists \( L(a, b) \in (0, 1) \) such that
   \[
   d(H_t(x_1), H_t(x_2)) \leq L(a, b) d(x_1, x_2)
   \]
   for all \( t \in [0, 1] \) and \( x_1, x_2 \in \overline{U} \) with \( a \leq d(x_1, x_2) \leq b \), where \( L(a, b) \) is independent of \( t \).

2. \( \text{H2} \) For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( d(H_t(x), H_t(y)) \leq \varepsilon \) for all \( x, y \in \overline{U} \) and \( |t_1 - t_2| < \delta \), where \( \delta \) is independent of \( x \).

If \( H_0 \) has a fixed point in \( U \), then so does \( H_t \) for each \( t \in [0, 1] \).

Proof. Let \( \Lambda = \{ \lambda \in [0, 1] : x = H_\lambda(x) \text{ for some } x \in U \} \). \( \Lambda \neq \emptyset \) since \( 0 \in \Lambda \).

(i) \( \Lambda \) is closed in \([0, 1] \).

Let \( \lambda_n \to 0 \) with \( \lambda_n \in \Lambda \) and \( x_n \in U \) such that \( x_n = H_{\lambda_n} x_n \). Then

\[
\begin{align*}
d(x_n, x_m) &\leq d(H_{\lambda_n}(x_n), H_{\lambda_m}(x_n)) + d(H_{\lambda_m}(x_n), H_{\lambda_m}(x_m)).
\end{align*}
\]

We claim that \( \{ x_n \} \) is a Cauchy sequence. Otherwise, for any \( k > 0 \), there exist \( n_k, m_k > k \) such that \( d(x_{n_k}, x_{m_k}) \geq \delta \), where \( \delta \) is a positive constant. Let \( M = \text{diam} U \). (5) leads to

\[
\begin{align*}
d(x_{n_k}, x_{m_k}) &\leq d(H_{\lambda_{n_k}}(x_{n_k}), H_{\lambda_{m_k}}(x_{n_k})) + L(\delta, M) d(x_{n_k}, x_{m_k}),
\end{align*}
\]

and so

\[
\begin{align*}
\delta &\leq d(x_{n_k}, x_{m_k}) \leq \frac{d(H_{\lambda_{n_k}}(x_{n_k}), H_{\lambda_{m_k}}(x_{n_k}))}{1 - L(\delta, M)}.
\end{align*}
\]

By (H2), \( d(H_{\lambda_{n_k}}(x_{n_k}), H_{\lambda_{m_k}}(x_{n_k})) \to 0 \) as \( k \to \infty \). We reach a contradiction from (6). Hence there exists \( x_0 \in \overline{U} \) such that \( x_n \to x_0 \).

On the other hand,

\[
\begin{align*}
d(x_n, H_\lambda(x_0)) &\leq d(H_{\lambda_n}(x_n), H_\lambda(x_0)) + d(H_\lambda(x_0), H_\lambda(x_0)) \\
&\leq d(H_{\lambda_n}(x_n), H_\lambda(x_0)) + d(x_n, x_0) \to 0 \text{ as } n \to \infty.
\end{align*}
\]

Thus \( x_0 = H_\lambda(x_0) \). Since \( (\text{Fix} H_\lambda) \cap \partial U = \emptyset, x_0 \in U \) and \( \lambda_0 \in \Lambda \).
(ii) \( \Lambda \) is open in \([0, 1]\).

Let \( \lambda_0 \in \Lambda \) and \( x_0 = H_{\lambda_0}(x_0) \), where \( x_0 \in U \). Choose \( r > 0 \) such that \( r < \text{dist}(x_0, \partial U) \). There exists \( \varepsilon_1 > 0 \) so that \( d(H_\lambda(x_0), H_{\lambda_0}(x_0)) < \frac{r}{2} \) when \( \|\lambda - \lambda_0\| < \varepsilon_1 \). Hence for \( \lambda \in [0, 1] \cap (\lambda_0 - \varepsilon_1, \lambda_0 + \varepsilon_1) \) and \( x \in B(x_0, \frac{r}{2}) \),

\[
d(H_\lambda(x), x_0) \leq d(H_\lambda(x), H_{\lambda_0}(x_0)) + d(H_{\lambda_0}(x_0), x_0) \leq d(x, x_0) + \frac{r}{2} \leq r.
\]

For \( (1 - L(\frac{r}{2}, r)) \), there exists \( \varepsilon_2 > 0 \) such that \( d(H_\lambda(x_0), H_{\lambda_0}(x_0)) < (1 - L(\frac{r}{2}, r)) r \) when \( \|\lambda - \lambda_0\| < \varepsilon_2 \). Then for \( \lambda \in [0, 1] \cap (\lambda_0 - \varepsilon_2, \lambda_0 + \varepsilon_2) \) and \( x \in U \) with \( \frac{r}{2} \leq d(x, x_0) \leq r \),

\[
d(H_\lambda(x), x_0) \leq d(H_\lambda(x), H_{\lambda_0}(x_0)) + d(H_{\lambda_0}(x_0), x_0) \leq L(\frac{r}{2}, r) d(x, x_0) + (1 - L(\frac{r}{2}, r)) r \leq r.
\]

Put \( \varepsilon = \max\{\varepsilon_1, \varepsilon_2\} \). For all \( \lambda \in [0, 1] \cap (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \), we have \( H_\lambda : \overline{B}(x_0, r) \to \overline{B}(x_0, r) \). By the Generalized Contraction Principle, there exists \( x \in \overline{B}(x_0, r) \subseteq U \) such that \( H_\lambda(x) = x \). We conclude that \( \lambda \in \Lambda \), and \( [0, 1] \cap (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \subseteq \Lambda \).

Therefore \( \Lambda \neq \emptyset \) is both open and closed, and consequently \( \Lambda = [0, 1] \). \( \square \)

The following example illustrates that Theorem 2.1 is indeed more general than Theorem 3.1 in \( \mathbb{R}^3 \).

Example 2.2. Let \( X = [0, \infty) \), \( d(x, y) = \|x - y\| \) and \( G_t(x) = \frac{x}{1 + t} + t \), where \( x, y \in X \) and \( t \in [0, 1] \). Consider \( U = [0, 2] \). Then \( \partial U = \{2\} \). For each \( t \in [0, 1] \), we have

\[
d(G_t(x), G_t(y)) = \frac{\|x - y\|}{(1 + x)(1 + y)} \leq \frac{\|x - y\|}{1 + \|x - y\|} = \frac{1}{1 + d(x, y)} d(x, y).
\]

For \( 0 < a \leq d(x, y) \leq b < \infty \), we can put \( L(a, b) = \frac{1}{1 + a} \). Hence \( G_t \) is a generalized contraction, however it is not a contraction in the usual sense. Since \( 2 \) is not a fixed point for any \( G_t \) and \( G_0 \) has a fixed point \( 0 \in U \), we apply Theorem 2.1 to conclude that \( G_t \) has a fixed point in \( U = [0, 2] \) for each \( t \in [0, 1] \).

3. Continuation method for \( \alpha \)-sublinear mappings

In this section, we will use Thompson’s metric and Theorem 2.1 as tools to study \( \alpha \)-sublinear mappings.

Theorem 3.1. Let \( S_t : \hat{P} \to \hat{P} \) be monotone for each \( t \in [0, 1] \), and satisfy:

(H1) For each \( [a, b] \subset (0, 1) \), there exists \( \alpha(a, b) \in (0, 1) \) such that

\[
S_t(cx) \geq c^{\alpha(a,b)} S_t(x)
\]

for all \( x \in \hat{P} \) and \( c \in [a, b] \), where \( \alpha(a, b) \) is independent of \( t \in [0, 1] \).

(H2) There exists a bounded open set \( U \) with \( \overline{U} \subset \hat{P} \) and \( \text{dist}(\overline{U}, \partial P) = r > 0 \).

For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
\|S_{t_1}(x) - S_{t_2}(x)\| < \varepsilon
\]

for all \( x \in \overline{U} \) and \( t_1, t_2 \in [0, 1] \) with \( |t_1 - t_2| < \delta \), where \( \delta \) is independent of \( x \).
Suppose \((\text{Fix} \, S_t) \cap \partial U = \emptyset\) for all \(t \in [0, 1]\). If \(S_0\) has a fixed point in \(U\), then so does \(S_t\) for each \(t \in [0, 1]\), and the sequence \(\{S^n_t(x)\}\) converges to the unique fixed point of \(S_t\) for any \(x \in P\).

**Proof.** Let \(x, y \in \overset济}{\overline{P}}\) with \(d(x, y) \in [-\ln b, -\ln a]\), where \([a, b] \subset (0, 1)\). Without loss of generality, assume \(M(x/y) \geq M(y/x)\). Then \(d(x, y) = \ln M(x/y)\) and \(\frac{1}{b} \leq M(x/y) \leq \frac{1}{a}\). Now
\[
S_t(x) \geq S_t(M(y/x)^{-1}y) \\
\geq S_t(M(x/y)^{-1}y) \\
\geq M(x/y)^{-\alpha(a,b)} S_t(y).
\]
Thus \(M(S_t(y)/S_t(x)) \leq M(x/y)^{\alpha(a,b)}\). On the other hand,
\[
S_t(y) \geq S_t(M(x/y)^{-1}x) \\
\geq M(x/y)^{-\alpha(a,b)} S_t(x)
\]
implies that \(M(S_t(x)/S_t(y)) \leq M(x/y)^{\alpha(a,b)}\). Hence
\[
d(S_t(x), S_t(y)) \leq \ln[M(x/y)^{\alpha(a,b)}] \\
= \alpha(a,b) \ln M(x/y) \\
= L(-\ln b, -\ln a) d(x, y),
\]
where \(L(-\ln b, -\ln a) = \alpha(a, b)\).

Lemma 1.2 and Lemma 1.3 imply that \(U\) is also open in Thompson’s metric, and its closure \(\overline{U}\) and boundary \(\partial U\) are identical in both the norm topology and Thompson’s metric topology.

Let \(\varepsilon > 0\) be given. There exists \(\varepsilon_1 \in (0, r)\) such that
\[
\ln\{\max\{\frac{r + \beta}{r}, \frac{r}{r - \beta}\}\} < \varepsilon
\]
for all \(\beta \in [0, \varepsilon_1]\). By (H2), there exists \(\delta > 0\) such that \(\|S_{t_1}(x) - S_{t_2}(x)\| < \varepsilon_1\) for all \(x \in \overline{U}\) and \(t_1, t_2 \in [0, 1]\) with \(|t_1 - t_2| < \delta\). Using Lemma 1.3, we have
\[
d(S_{t_1}(x), S_{t_2}(x)) \leq \ln\{\max\{\frac{r + \beta}{r}, \frac{r}{r - \beta}\}\} < \varepsilon
\]
for all \(x \in \overline{U}\) and \(t_1, t_2 \in [0, 1]\) with \(|t_1 - t_2| < \delta\). If \((\text{Fix} \, S_t) \cap \partial U = \emptyset\) for all \(t \in [0, 1]\) and \(S_0\) has a fixed point in \(U\), then we can apply Theorem 2.1 to conclude that \(S_t\) has a fixed point in \(U\) for each \(t \in [0, 1]\).

(H1) implies that the sequence \(\{S^n_t(x)\}\) converges to the unique fixed point of \(S_t\) for any \(x \in \overset济}{\overline{P}}\) by Theorem 1.1. \(\square\)

The following is a nonlinear alternative theorem for \(\alpha\)-sublinear mappings.

**Theorem 3.2.** Let \(A : \overset济}{\overline{P}} \mapsto \overset济}{\overline{P}}\) be an \(\alpha\)-sublinear mapping and \(U\) be a nonempty open bounded subset with \(\overline{U} \subset \overset济}{\overline{P}}\) and \(\text{dist}(\overline{U}, \partial P) > 0\). If \(A(U)\) is bounded, then \(A\) has at least one of the following properties:

(i) \(A\) has a unique fixed point in \(\overline{U}\), and the sequence \(A^n(x)\) converges to that fixed point for any \(x \in \overset济}{\overline{P}}\).

(ii) \(A(\partial U)\) contains a point of some exterior ray, i.e., there exists \(x_0 \in U\) such that \(Ay_0 = x_0 + \tau(y_0 - x_0)\) for some \(\tau > 1\) and \(y_0 \in \partial U\).
Proof. Let \( x_0 \in U \) and consider \( S_t(x) = tAx + (1 - t)x_0 \). By the definition of \( \alpha \)-sublinear mapping, for each \( [a, b] \subset (0, 1) \), there exists \( \alpha(a, b) \in (0, 1) \) such that for all \( x \in \bar{P} \) and \( c \in [a, b] \),

\[
S_t(cx) = tA(cx) + (1 - t)x_0 \\
\geq t \alpha(a, b)Ax + (1 - t)x_0 \\
\geq \alpha(a, b)(tAx + (1 - t)x_0) \\
= \alpha(a, b) S_t(x).
\]

Let \( M = \sup\{\|y\| : y \in A(\bar{U})\} \). For any \( \varepsilon > 0 \), choose \( \delta = \frac{\varepsilon}{2 \max\{M, \|x_0\|\}} \). Then for \( x \in \bar{U} \) and \( t_1, t_2 \in [0, 1] \) with \( |t_1 - t_2| < \delta \),

\[
\|S_{t_1}(x) - S_{t_2}(x)\| = \|((t_1 - t_2)Ax - (t_1 - t_2)x_0)\| \\
\leq |t_1 - t_2| \|Ax\| + |t_1 - t_2| \|x_0\| \\
< \varepsilon.
\]

Note that \( S_0 \) has a fixed point \( x_0 \in U \). Assume that \( A \) does not have a fixed point in \( \bar{U} \), then by Theorem 3.1, there exists \( y_0 \in \partial U \) and \( t \in (0, 1) \) such that \( S_t(y_0) = y_0 \), i.e., \( t Ay_0 + (1 - t)x_0 = y_0 \). It follows that \( Ay_0 = x_0 + \tau(y_0 - x_0) \), where \( \tau = \frac{t}{1 - t} > 1 \).

Remark. The distinction between cases (i) and (ii) in Theorem 3.2 cannot be sharpened to a proper alternative. Let’s consider the so-called square root version of Fibonacci’s rabbit population model:

\[
\mathring{P} = \mathring{R}_+^2, \quad A(a, b) = (\sqrt{a} + \sqrt{b}, \sqrt{a}), \quad (a, b) \in \mathring{R}_+^2.
\]

Suppose \( U = (3, 4) \times (1, 2) \subset \mathring{R}_+^2 \). It is easy to check that \( A \) has a fixed point \((a^*, b^*) \approx (3.08, 1.75) \in U \) and \( A(\bar{U}) = [1 + \sqrt{3}, \ 2 + \sqrt{2}] \times [\sqrt{3}, 2] \). Take \( y_0 = (3, 1.44) \in \partial U \); then \( A(y_0) = (1.2 + \sqrt{3}, \ \sqrt{3}) \). Now there exists \( x_0 \in (4.8 - \sqrt{3}, \ 2.88 - \sqrt{3}) \in U \) such that \( A(y_0) = x_0 + \tau(y_0 - x_0) \) with \( \tau = 2 \). Hence cases (i) and (ii) in Theorem 3.2 are not mutually exclusive.

4. Example

The following example illustrates the application of Theorem 3.2 to the Dirichlet problem for a uniformly elliptic differential operator.

Let \( \Omega \) be a bounded convex domain in \( \mathring{R}^n \) (\( n \geq 2 \)), whose boundary \( \partial \Omega \) belongs to \( C^{2+\mu} \) (\( 0 < \mu < 1 \)) and consider the Dirichlet problem

\[
\begin{cases}
Lu = f(x, u), \\
u|_{\partial \Omega} = 0,
\end{cases}
\]

where

\[
Lu = -\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u
\]

is a uniformly elliptic differential operator, i.e., there exists \( \nu > 0 \) such that

\[
\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \nu \|\xi\|^2, \quad x \in \Omega, \quad \xi = (\xi_1, \cdots, \xi_n) \in \mathring{R}^n.
\]
and \( a_{ij}(x) = a_{ji}(x), \ c(x) \geq 0, \) all coefficients \( a_{ij}, b_i, c \in C^\infty(\bar{\Omega}) \).

Suppose \( f(x, u) > 0 \) is continuous for all \( x \in \bar{\Omega} \) and \( u \geq 0 \). The solution of (8) is equivalent to the fixed point of the integral operator

\[
Au(x) = \int_{\bar{\Omega}} G(x, y) f(y, u(y)) \, dy,
\]

where \( G(x, y) \) is the corresponding Green function which satisfies

\[
0 < G(x, y) < \begin{cases} k_0 |x - y|^{2-n}, & n > 2, \\ k_0 \ln |x - y|, & n = 2, \end{cases}
\]

where \( x, y \in \Omega \) and \( x \neq y \).

It is well known that \( A \) is monotone and completely continuous from \( P \) into \( P \) (see [1], pp. 60-62), where \( P = \{ u \in C(\bar{\Omega}) \mid u(x) \geq 0, \ \forall x \in \bar{\Omega} \} \). The sup norm of \( C(\bar{\Omega}) \) is monotone in the partial order introduced by cone \( P \). Note that \( \hat{P} = \{ u \in C(\bar{\Omega}) \mid u(x) > 0, \ \forall x \in \bar{\Omega} \} \), and it is easy to see \( A : \hat{P} \to \hat{P} \). Let \( U = \{ u \in C(\bar{\Omega}) \mid m < u(x) < M, \ \forall x \in \bar{\Omega} \} \), where \( m \) and \( M \) are positive constants. Then \( \overline{U} \subset \hat{P} \) and \( A(\overline{U}) \) is bounded due to the complete continuity of \( A \).

If there exists a lower semicontinuous function \( \phi : (0, 1) \to (0, 1) \) such that \( \phi(r) > r \) and

\[
f(x, tu) \geq \phi(t) f(x, u),
\]

then \( A(tu) \geq \phi(t) A(u) \). This implies that \( A \) is \( \alpha \)-linear by observing \( \phi(t) = t^{\log_i(\phi(t))} \) and \( \log_i \phi(t) \) attains its maximum \( \alpha(a, b) \) on each \( [a, b] \subset (0, 1) \) due to the lower semicontinuity of \( \phi \). Applying Theorem 3.2, we have at least one of the following:

(i) \( A \) has a unique fixed point \( u_0 \in \overline{U} \) and the sequence

\[
u_{n+1}(x) = \int_{\bar{\Omega}} G(x, y) f(y, u_n(y)) \, dy, \quad n = 1, 2, \ldots,
\]

converges to \( u_0(x) \) in sup norm for any initial function \( u_1 \in C(\bar{\Omega}) \) with \( u_1(x) > 0 \) for all \( x \in \Omega \).

(ii) \( A(\partial \overline{U}) \) contains a point of some exterior ray, i.e., there exists \( u_0 \in C(\bar{\Omega}) \) with \( m < u(x) < M, \ x \in \bar{\Omega} \), such that

\[
\int_{\bar{\Omega}} G(x, y) f(y, v_0(y)) \, dy = u_0 + \tau (v_0(x) - u_0(x))
\]

for some \( \tau > 1 \) and \( v_0 \in \partial \overline{U} \).

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**References**


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