A MATRIX-VALUED CHOQUET–DENY THEOREM

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Abstract. Let \( \sigma \) be a positive matrix-valued measure on a locally compact abelian group \( G \) such that \( \sigma(G) \) is the identity matrix. We give a necessary and sufficient condition on \( \sigma \) for the absence of a bounded non-constant matrix-valued function \( f \) on \( G \) satisfying the convolution equation \( f \ast \sigma = f \). This extends Choquet and Deny’s theorem for real-valued functions on \( G \).

1. Introduction

Let \( \sigma \) be a probability measure on a locally compact abelian group \( G \). A real Borel function \( f \) on \( G \) is called \( \sigma \)-harmonic if it satisfies the integral equation

\[
  f(x) = (f \ast \sigma)(x) = \int_G f(x-y)d\sigma(y) \quad \text{for all } x \in G.
\]

A celebrated theorem of Choquet and Deny [2] asserts that every bounded \( \sigma \)-harmonic function on \( G \) is constant if (and only if) \( \sigma \) is adapted; that is, the support of \( \sigma \) generates a dense subgroup of \( G \). Choquet and Deny’s theorem plays an important role in probability theory and has been extended to various non-abelian groups (see, for example, [4, 5, 7, 8, 10]). Recently in [11], a vector-valued version of the Choquet–Deny theorem has been proved and used to obtain a vector-valued renewal theorem for the study of the \( L^p \) dimension of some vector-valued self-similar measures. In a related paper [3], the equation \( f \ast \sigma = f \) has been studied under the assumption that both \( \sigma \) and \( f \) are operator-valued, but with commuting ranges. In this paper, we remove the restriction of commuting ranges and prove a Choquet–Deny type theorem for matrix-valued functions defined on \( G \). This theorem uses positive definite matrices and differs from that of [11] where matrices with non-negative entries are considered instead, and consequently different techniques are used.

Let \( M_n \) be the \( C^\ast \)-algebra of \( n \times n \) complex matrices. The pure states of \( M_n \) are exactly the vector states \( \rho(\cdot) = \langle \xi, \cdot \xi \rangle \) where \( \xi \) is a unit vector in \( \mathbb{C}^n \). Let \( M_n^+ \) be the positive cone of \( M_n \), consisting of all self-adjoint matrices with non-negative eigenvalues. An \( M_n^+ \)-valued measure \( \sigma \) on \( G \) will be called a positive \( M_n \)-valued measure and its support is defined to be

\[
  \text{supp } \sigma = \{ x \in G : \sigma(V) \neq 0 \text{ for all open sets } V \text{ containing } x \}.
\]

We say that \( \sigma \) is adapted if \( \rho \circ \sigma \) is adapted on \( G \) for every pure state \( \rho \) of \( M_n \). We note that \( \text{supp } (\rho \circ \sigma) \subseteq \text{supp } \sigma \). We can write \( \sigma = (\sigma_{ij}) \) where each \( \sigma_{ij} \) is a...
complex-valued measure on $G$. A function $f: G \to M_n$ can also be denoted by $f = (f_{ij})$ where each $f_{ij}$ is a complex-valued function on $G$. The convolution $f \ast \sigma$ can be defined naturally in terms of $f_{kl} \ast \sigma_{ij}$ via matrix multiplication. Details are given later. An $M_n$-valued function $f$ on $G$ is called $\sigma$-harmonic if it satisfies the convolution equation $f \ast \sigma = f$. We can now state our main result.

**Theorem 1.** Let $\sigma$ be a positive $M_n$-valued measure on $G$ such that $\sigma(G)$ is the identity matrix. The following conditions are equivalent:

i) every bounded $\sigma$-harmonic $M_n$-valued function on $G$ is constant;

ii) $\sigma$ is adapted.

2. Choquet–Deny type theorem

We need some vector measure preliminaries. Let $\mathcal{B}$ be the algebra of Borel sets in $G$. By an $M_n$-valued measure on $G$, we mean a (norm) countably additive function $\sigma: \mathcal{B} \to M_n$. If we use the matrix notation $\sigma = (\sigma_{ij})$, then each $\sigma_{ij}$ is a complex-valued measure on $G$. We adopt the definition of a complex measure in [13] and note that complex-valued measures are not only bounded, but also of bounded total variation [13, Theorem 6.4]. As in [6, p. 2], we define the semivariation of $\sigma$ to be a non-negative function $k_\sigma$ whose value on a set $S \in \mathcal{B}$ is given by

$$k_\sigma(S) = \sup \{ |\rho \circ \sigma(S) : \rho \in M_n^*, \|\rho\| \leq 1 \}$$

where $|\rho \circ \sigma|$ is the total variation of the complex measure $\rho \circ \sigma$. We will write $\rho \sigma$ for $\rho \circ \sigma$. We say that $\sigma$ is of bounded semivariation if $\|\sigma\|(G) < \infty$ which is equivalent to the condition that $\sigma(\mathcal{B})$ is norm-bounded in $M_n$ by [6, Proposition 11]. Given any $A = (a_{ij}) \in M_n$, we have

$$\|A\| \leq \left( \sum_{i,j} |a_{ij}|^2 \right)^{1/2} \leq n\|A\|.$$ 

The above inequality together with the boundedness of complex measures implies that every $M_n$-valued measure on $G$ is of bounded semivariation.

Given a Borel function $\lambda: G \to \mathbb{C}$, the vector integral $\int_S \lambda d\sigma$ is defined in the usual way (see [6, Definition 12]). If $\lambda$ is bounded, then we have

$$\left\| \int_G \lambda d\sigma \right\| \leq \|\lambda\|_\infty \|\sigma\|(G).$$

In general, we note that the bound of the above inequality cannot be sharpened to $\|\lambda\|_\infty \|\sigma(G)\|$, as the following example shows, although it can be if $\sigma$ is positive with commuting range.

**Example 1.** Let $\alpha = -\frac{\pi}{4}$ and $\beta = \frac{3\pi}{4}$. Let $\sigma$ be the following $M_2$-valued measure on $\mathbb{R}$,

$$\sigma = A\delta_\alpha + B\delta_\beta$$

where $\delta_x$ is the point mass at $x \in \mathbb{R}$,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$
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and \( \sigma(\mathbb{R}) \) is the identity matrix. Let

\[
D = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad |D| = \sqrt{D^*D} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}
\]

and define \( \rho: M_2 \to \mathbb{C} \) by

\[
\rho(X) = \frac{1}{2} \text{trace}(DX)
\]

for \( X \in M_2 \). Then \( \|\rho\| = \frac{1}{2} \text{trace}(|D|) = 1 \). Therefore

\[
\|\sigma\|(|\mathbb{R}|) \geq |\rho\sigma|(|\mathbb{R}|) \geq |\rho(A)| + |\rho(B)| = \frac{\sqrt{8}}{2} + \frac{\sqrt{2}}{2} > 1 = \|\sigma(\mathbb{R})\|.
\]

Given \( \lambda(x) = e^{ix} \), we have

\[
\left\| \int_{\mathbb{R}} \lambda(x) d\sigma(x) \right\| \geq |e^{ix} \rho(A) + e^{iy} \rho(B)| = \frac{\sqrt{8}}{2} + \frac{\sqrt{2}}{2} > 1 = \|\lambda\| \|\sigma(\mathbb{R})\|.
\]

As in [1, 3], using the natural bilinear map

\[
(A, B) \in M_n \times M_n \mapsto AB \in M_n,
\]

we can define the \( \sigma \)-integrable functions \( f: G \to M_n \) and the bilinear vector integrals \( \int_S f d\sigma \) for \( S \in \mathcal{B} \). For our purpose, we can simplify the construction in the following way. Given \( \sigma = (\sigma_{ij}) \) and \( f = (f_{ij}) \), we say that \( f \) is \( \sigma \)-integrable on \( S \) if the integral \( \int_S f_{ik} d\sigma_{kj} \) exists for all \( i, j, k \) in which case we define

\[
\int_S f d\sigma = \left( \sum_{k=1}^n \int_S f_{ik} d\sigma_{kj} \right) \in M_n.
\]

For example, if \( f_{ij} = \delta_{ij} \lambda \) where \( \delta_{ij} \) is the Kronecker delta, then we have the \((i, j)\)-th entry \( (\int_S f d\sigma)_{ij} = \int_S \lambda d\sigma_{ij} \) and \( \int_S f d\sigma = \int_S \lambda d\sigma \). We can also define the following convolution of \( f \) and \( \sigma \) if it exists:

\[
f \ast \sigma(x) = \int_G f(x - y) d\sigma(y).
\]

Given two matrix-valued measures \( \sigma = (\sigma_{ij}) \) and \( \gamma = (\gamma_{ij}) \), their convolution can be defined as

\[
\sigma \ast \gamma = \left( \sum_k \sigma_{ik} \ast \gamma_{kj} \right).
\]

Given a complex-valued Borel measure \( \mu \) on \( G \), the integral \( \int_S f d\mu \in M_n \) denotes the Bochner integral of \( f = (f_{ij}) \) whenever it is well defined (see [5, p. 44]).

In the sequel, we will always assume that all \( M_n \)-valued measures \( \sigma \) are regular which means that each \( \rho \sigma \) is a regular Borel measure on \( G \), for every \( \rho \in M_n^* \).

Let \( \sigma = (\sigma_{ij}) \) be an \( M_n \)-valued measure on \( G \). We define its Fourier transform \( \hat{\sigma} \) on the dual group \( \hat{G} \) by

\[
\hat{\sigma}(\lambda) = \int_G \lambda(-x) d\sigma(x) \in M_n
\]

for \( \lambda \in \hat{G} \). We also define the determinant \( \text{det} \sigma \) by convolution

\[
\text{det} \sigma = \sum_{\pi} \text{sgn}(\pi) \sigma_{1\pi(1)} \ast \cdots \ast \sigma_{n\pi(n)}
\]
where $\pi$ is a permutation. But for an $M_n$-valued function $f = (f_{ij})$ on $G$, we define the determinant $\det f$ by pointwise multiplication

$$\det f = \sum_{\pi} \text{sgn}(\pi)f_{1\pi(1)} \cdots f_{n\pi(n)}.$$ 

Using the above notation, we have

$$\det \sigma(\lambda) = \overline{\det \sigma(\lambda)}.$$ 

We also have

$$\rho(\overline{\sigma(\lambda)}) = \int_G \lambda(-x)d\rho(x) = \overline{\rho(\sigma(\lambda))}$$

for $\rho \in M_n^*$. 

The proof of Theorem 1 is achieved by writing the equation $f = f \ast \sigma$ in the form $f \ast \mu = 0$ and convolving it with a judiciously chosen $M_n$-valued measure which reduces the equation to simultaneous scalar convolution equations, but convolved by the same scalar measure, as in the proof of Lemma 3 (ii) $\Rightarrow$ i), to which one can apply the $L^1(G)$-Tauberian theorem [10, 39.27] and get the result readily. The setting where the Tauberian theorem applies is the following lemma which can be proved as in [12, Theorem 2].

**Lemma 2.** Let $\mu$ be a complex-valued measure on a locally compact abelian group $G$. The following conditions are equivalent:

i) for every $f \in L^\infty(G)$, $f \ast \mu = 0$ implies that $f$ is constant;

ii) $\overline{\mu}(\lambda) \neq 0$ for $\lambda \in \hat{G} \setminus \{i\}$ where $i$ is the identity in $\hat{G}$.

We need to extend the above lemma to the following matrix-valued setting.

**Lemma 3.** Let $\mu$ be an $M_n$-valued measure on a locally compact abelian group $G$. The following conditions are equivalent:

i) for every bounded $M_n$-valued function $f$ on $G$, $f \ast \mu = 0$ implies that $f$ is constant;

ii) $\text{det} \overline{\mu}(\lambda) \neq 0$ for $\lambda \in \hat{G} \setminus \{i\}$.

**Proof.** i) $\Rightarrow$ ii). Suppose that $\text{det} \overline{\mu}(\lambda) = 0$ for some $\lambda \in \hat{G} \setminus \{i\}$. Then there exists $\xi \in \mathbb{C}^n \setminus \{0\}$ such that $\overline{\mu}(\lambda)^T \xi = 0$, where $T$ denotes transpose. Define $f: G \to M_n$ by $f(x) = \lambda(x)(\zeta_i)$ where

$$(\zeta_{i1}, \zeta_{i2}, \ldots, \zeta_{in}) = (\xi_1, \xi_2, \ldots, \xi_n)$$

for all $i$. Then $f \ast \mu = 0$, but $f$ is not constant.

ii) $\Rightarrow$ i). Let $f = (f_{ij})$ be a bounded $M_n$-valued function such that $f \ast \mu = 0$. Let $\gamma = (\gamma_{ij})$ be the $M_n$-valued measure defined as the adjoint matrix of $\mu = (\mu_{ij})$, using convolution, so that

$$\mu \ast \gamma = \begin{pmatrix} \text{det} \mu & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \text{det} \mu & 0 \end{pmatrix}.$$ 

Then we have

$$f \ast \begin{pmatrix} \text{det} \mu & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \text{det} \mu & 0 \end{pmatrix} = f \ast \mu \ast \gamma = 0.$$
which gives
\[ f_{ij} \ast \text{det} \mu = 0 \]
for all \( i, j \). Since \( \text{det} \mu(\lambda) = \hat{\mu}(\lambda) = 0 \) for \( \lambda \in \hat{G} \setminus \{ e \} \), Lemma 2 implies that \( f_{ij} \) is constant.

**Lemma 4.** Let \( A \in M_n^+ \) be such that \( \langle A\xi, \xi \rangle = 0 \) for some \( \xi \in \mathbb{C}^n \). Then \( A\xi = 0 \).

**Proof.** We have \( A = B^2 \) for some \( B \in M_n^+ \). Hence \( \langle B^2 \xi, \xi \rangle = 0 \) which gives \( B\xi = 0 \) and \( A\xi = 0 \).

We also need the following well-known result for Theorem 1.

**Lemma 5.** Let \( \nu \) be a probability measure on a locally compact abelian group \( G \). Then \( \nu \) is adapted if, and only if, \( \hat{\nu}(\lambda) \neq 1 \) for every \( \lambda \in \hat{G} \setminus \{ e \} \).

Let \( f \) be an \( M_n \)-valued function on \( G \) and let \( \Delta_e \) be the diagonal matrix in which each diagonal entry is the point mass \( \delta_e \) at the identity \( e \) of \( G \). Then we have \( f \ast \sigma = f \) if, and only if, \( f \ast (\sigma - \Delta_e) = 0 \). Given \( \lambda \in \hat{G} \), we have
\[ \text{det} (\sigma - \Delta_e)(\lambda) = \text{det} (\sigma - \Delta_e(\lambda)) = \text{det} (\hat{\sigma} - I_n). \]

Now we are ready to prove Theorem 1.

**Proof of Theorem 1.**

1. \( \implies \) ii). Let \( \rho(\cdot) = \langle \cdot, \xi \rangle \) be a pure state of \( M_n \). We need to show that \( \rho\sigma \) is adapted. Suppose otherwise. By Lemma 5, there exists \( \lambda \in \hat{G} \setminus \{ e \} \) such that \( \hat{\rho\sigma}(\lambda) = 1 \); that is \( \langle \hat{\rho\sigma}(\lambda)\xi, \xi \rangle = 1 \). As we do not know if \( \|\hat{\rho\sigma}(\lambda)\| \) is at most 1 (see Example 1), we cannot conclude immediately that \( \hat{\rho\sigma}(\lambda)\xi = \xi \) although it is true but requires the following arguments. Since \( \rho\sigma \) is a probability measure, we have
\[ \rho\sigma \{ x \in G : \lambda(-x) = 1 \} = 1. \]
Write \( V = \{ x \in G : \lambda(-x) \neq 1 \} \); then
\[ \langle \sigma(V)\xi, \xi \rangle = \rho\sigma(V) = 0. \]
It follows from Lemma 3 that \( \sigma(V)\xi = 0 \) and hence also
\[ \sigma(G \setminus V)\xi = \xi. \]
Thus
\[ \hat{\sigma}(\lambda)\xi = \left( \int_G \lambda(-x) \, d\sigma(x) \right) \xi = \left( \int_{G \setminus V} 1 \, d\sigma(x) \right) \xi + \left( \int_V \lambda(x) \, d\sigma(x) \right) \xi = \sigma(G \setminus V)\xi = \xi. \]
Therefore \( \text{det}(\hat{\sigma}(\lambda) - I_n) = 0 \). By Lemma 3, there is a non-constant bounded \( M_n \)-valued function \( f \) such that \( f \ast (\sigma - \Delta_e) = 0 \) which contradicts condition i).

ii) \( \implies \) i). By Lemma 3, it suffices to show that \( \text{det}(\sigma - \Delta_e)(\lambda) \neq 0 \) for all \( \lambda \in \hat{G} \setminus \{ e \} \). Suppose otherwise, so that \( \text{det}(\hat{\sigma}(\lambda) - I_n) = 0 \) for some \( \lambda \neq e \). Then there is a unit vector \( \xi \in \mathbb{C}^n \) such that \( (\hat{\sigma}(\lambda) - I_n)\xi = 0 \); that is, \( \hat{\sigma}(\lambda)\xi = \xi \). Let
\( \rho(\cdot) = \langle \cdot, \xi, \xi \rangle \). Then we have

\[
\tilde{\rho} \sigma(\lambda) = \rho(\tilde{\sigma}(\lambda)) = \langle \tilde{\sigma}(\lambda)\xi, \xi \rangle = 1.
\]

Therefore, by Lemma 5, \( \rho \sigma \) is not adapted, contradicting condition ii).

We end with an example which shows that condition ii) in Theorem 1 cannot be replaced by the condition that \( \text{supp} \sigma \) generates a dense subgroup of \( G \).

**Example 2.** Let \( \nu \) be any adapted probability measure on \( \mathbb{R} \) with \( \nu\{0\} \geq \frac{1}{2} \) and let

\[
\sigma = \begin{pmatrix}
\nu & \delta_0 - \nu \\
\delta_0 - \nu & \nu
\end{pmatrix}.
\]

Then \( \sigma \) is a positive \( M_2 \)-valued measure on \( \mathbb{R} \) such that \( \sigma(\mathbb{R}) \) is the identity matrix and \( \text{supp} \sigma = \text{supp} \nu \) generates a dense subgroup of \( \mathbb{R} \). A direct calculation reveals that every \( M_2 \)-valued function \( f = (f_{ij}) \), with \( f_{11} = f_{12} \) and \( f_{21} = f_{22} \), is \( \sigma \)-harmonic and need not be constant.

In fact under the change of coordinates \( u = x + y \) and \( v = x - y \), the measure \( \sigma \) is transformed to

\[
\begin{pmatrix}
\delta_0 & 0 \\
0 & 2\nu - \delta_0
\end{pmatrix}.
\]

The bounded solutions of the convolution equation with this measure are then of the form

\[
\begin{pmatrix} g & \alpha \\ h & \beta \end{pmatrix},
\]

with \( \alpha \) and \( \beta \) constants, \( g \) and \( h \) any bounded functions. Undoing the change of coordinates we see that all bounded solutions of \( f = f \ast \sigma \) are the bounded functions \( f = (f_{ij}) \), with \( f_{11} = f_{12} \) and \( f_{21} = f_{22} \) both constant.

**References**


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