A MATRIX-VALUED CHOQUET–DENY THEOREM

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Abstract. Let $\sigma$ be a positive matrix-valued measure on a locally compact abelian group $G$ such that $\sigma(G)$ is the identity matrix. We give a necessary and sufficient condition on $\sigma$ for the absence of a bounded non-constant matrix-valued function $f$ on $G$ satisfying the convolution equation $f \ast \sigma = f$. This extends Choquet and Deny’s theorem for real-valued functions on $G$.

1. Introduction

Let $\sigma$ be a probability measure on a locally compact abelian group $G$. A real Borel function $f$ on $G$ is called $\sigma$-harmonic if it satisfies the integral equation

$$f(x) = (f \ast \sigma)(x) = \int_G f(x - y)d\sigma(y) \quad \text{(for all } x \in G).$$

A celebrated theorem of Choquet and Deny [2] asserts that every bounded $\sigma$-harmonic function on $G$ is constant if (and only if) $\sigma$ is adapted; that is, the support of $\sigma$ generates a dense subgroup of $G$. Choquet and Deny’s theorem plays an important role in probability theory and has been extended to various non-abelian groups (see, for example, [4, 5, 7, 8, 10]). Recently in [11], a vector-valued version of the Choquet–Deny theorem has been proved and used to obtain a vector-valued renewal theorem for the study of the $L^p$ dimension of some vector-valued self-similar measures. In a related paper [3], the equation $f \ast \sigma = f$ has been studied under the assumption that both $\sigma$ and $f$ are operator-valued, but with commuting ranges. In this paper, we remove the restriction of commuting ranges and prove a Choquet–Deny type theorem for matrix-valued functions defined on $G$. This theorem uses positive definite matrices and differs from that of [11] where matrices with non-negative entries are considered instead, and consequently different techniques are used.

Let $M_n$ be the $C^*$-algebra of $n \times n$ complex matrices. The pure states of $M_n$ are exactly the vector states $\rho(\cdot) = \langle \xi, \cdot \rangle$ where $\xi$ is a unit vector in $\mathbb{C}^n$. Let $M^+_n$ be the positive cone of $M_n$, consisting of all self-adjoint matrices with non-negative eigenvalues. An $M^+_n$-valued measure $\sigma$ on $G$ will be called a positive $M_n$-valued measure and its support is defined to be

$$\text{supp } \sigma = \{ x \in G : \sigma(V) \neq 0 \text{ for all open sets } V \text{ containing } x \}.$$ 

We say that $\sigma$ is adapted if $\rho \circ \sigma$ is adapted on $G$ for every pure state $\rho$ of $M_n$. We note that $\text{supp } (\rho \circ \sigma) \subseteq \text{supp } \sigma$. We can write $\sigma = (\sigma_{ij})$ where each $\sigma_{ij}$ is a...
complex-valued measure on $G$. A function $f : G \to M_n$ can also be denoted by $f = (f_{ij})$ where each $f_{ij}$ is a complex-valued function on $G$. The convolution $f * \sigma$ can be defined naturally in terms of $f_{ij} * \sigma_{ij}$ via matrix multiplication. Details are given later. An $M_n$-valued function $f$ on $G$ is called $\sigma$-harmonic if it satisfies the convolution equation $f * \sigma = f$. We can now state our main result.

**Theorem 1.** Let $\sigma$ be a positive $M_n$-valued measure on $G$ such that $\sigma(G)$ is the identity matrix. The following conditions are equivalent:

i) every bounded $\sigma$-harmonic $M_n$-valued function on $G$ is constant;

ii) $\sigma$ is adapted.

2. Choquet–Deny type theorem

We need some vector measure preliminaries. Let $B$ be the algebra of Borel sets in $G$. By an $M_n$-valued measure on $G$, we mean a (norm) countably additive function $\sigma : B \to M_n$. If we use the matrix notation $\sigma = (\sigma_{ij})$, then each $\sigma_{ij}$ is a complex-valued measure on $G$. We adopt the definition of a complex measure in [13] and note that complex-valued measures are not only bounded, but also of bounded total variation [13, Theorem 6.4]. As in [6, p. 2], we define the semivariation of $\sigma$ to be a non-negative function $k_{\sigma}$ whose value on a set $S \in B$ is given by

$$k_{\sigma}(S) = \sup \{|\rho \circ \sigma|(S) : \rho \in M_n^*, \|\rho\| \leq 1\}$$

where $|\rho \circ \sigma|$ is the total variation of the complex measure $\rho \circ \sigma$. We will write $\rho \sigma$ for $\rho \circ \sigma$. We say that $\sigma$ is of bounded semivariation if $\|\sigma\|(G) < \infty$ which is equivalent to the condition that $\sigma(B)$ is norm-bounded in $M_n$ by [6, Proposition 11]. Given any $A = (a_{ij}) \in M_n$, we have

$$\|A\| \leq \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2} \leq n\|A\|.$$ 

The above inequality together with the boundedness of complex measures implies that every $M_n$-valued measure on $G$ is of bounded semivariation.

Given a Borel function $\lambda : G \to \mathbb{C}$, the vector integral $\int_G \lambda d\sigma$ is defined in the usual way (see [6, Definition 12]). If $\lambda$ is bounded, then we have

$$\left\|\int_G \lambda d\sigma\right\| \leq \|\lambda\|_{\infty} \|\sigma\|(G).$$

In general, we note that the bound of the above inequality cannot be sharpened to $\|\lambda\|_{\infty} \|\sigma(G)\|$, as the following example shows, although it can be if $\sigma$ is positive with commuting range.

**Example 1.** Let $\alpha = -\frac{\pi}{4}$ and $\beta = \frac{3\pi}{4}$. Let $\sigma$ be the following $M_2$-valued measure on $\mathbb{R}$,

$$\sigma = A\delta_{\alpha} + B\delta_{\beta}$$

where $\delta_x$ is the point mass at $x \in \mathbb{R}$,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$
and \( \sigma(\mathbb{R}) \) is the identity matrix. Let
\[
D = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad |D| = \sqrt{D^*D} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}
\]
and define \( \rho: M_2 \to \mathbb{C} \) by
\[
\rho(X) = \frac{1}{2} \text{trace}(DX)
\]
for \( X \in M_2 \). Then \( \|\rho\| = \frac{1}{2} \text{trace}(|D|) = 1 \). Therefore
\[
\|\sigma\|(|\mathbb{R}|) \geq |\rho\sigma|(|\mathbb{R}|) \geq |\rho(A)| + |\rho(B)| = \frac{\sqrt{8}}{2} + \frac{\sqrt{2}}{2} > 1 = \|\sigma(\mathbb{R})\|.
\]
Given \( \lambda(x) = e^{ix} \), we have
\[
\left\| \int_{\mathbb{R}} \lambda(x)d\sigma(x) \right\| \geq |e^{\alpha i}\rho(A) + e^{\beta i}\rho(B)| = \frac{\sqrt{8}}{2} + \frac{\sqrt{2}}{2} > 1 = \|\lambda\|_{\infty}\|\sigma(\mathbb{R})\|.
\]
As in [1, 3], using the natural bilinear map
\[
(A, B) \in M_n \times M_n \mapsto AB \in M_n,
\]
we can define the \( \sigma \)-integrable functions \( f: G \to M_n \) and the bilinear vector integrals \( \int_S f d\sigma \) for \( S \in \mathcal{B} \). For our purpose, we can simplify the construction in the following way. Given \( \sigma = (\sigma_{ij}) \) and \( f = (f_{ij}) \), we say that \( f \) is \( \sigma \)-integrable on \( S \) if the integral \( \int_S f_{ik} d\sigma_{kj} \) exists for all \( i, j, k \) in which case we define
\[
\int_S f d\sigma = \left( \sum_{k=1}^{n} \int_S f_{ik} d\sigma_{kj} \right) \in M_n.
\]
For example, if \( f_{ij} = \delta_{ij}\lambda \) where \( \delta_{ij} \) is the Kronecker delta, then we have the \((i, j)\)-th entry \( \left( \int_S f d\sigma \right)_{ij} = \int_S \lambda d\sigma_{ij} \) and \( \int_S f d\sigma = \int_S \lambda d\sigma \). We can also define the following convolution of \( f \) and \( \sigma \) if it exists:
\[
f \ast \sigma(x) = \int_G f(x - y)d\sigma(y).
\]
Given two matrix-valued measures \( \sigma = (\sigma_{ij}) \) and \( \gamma = (\gamma_{ij}) \), their convolution can be defined as
\[
\sigma \ast \gamma = \left( \sum_k \sigma_{ik} \ast \gamma_{kj} \right).
\]
Given a complex-valued Borel measure \( \mu \) on \( G \), the integral \( \int_S f d\mu \in M_n \) denotes the Bochner integral of \( f = (f_{ij}) \) whenever it is well defined (see [5] p. 44)).

In the sequel, we will always assume that all \( M_n \)-valued measures \( \sigma \) are regular which means that each \( \rho\sigma \) is a regular Borel measure on \( G \), for every \( \rho \in M_n^* \).

Let \( \sigma = (\sigma_{ij}) \) be an \( M_n \)-valued measure on \( G \). We define its Fourier transform \( \hat{\sigma} \) on the dual group \( \hat{G} \) by
\[
\hat{\sigma}(\lambda) = \int_G \lambda(-x)d\sigma(x) \in M_n
\]
for \( \lambda \in \hat{G} \). We also define the determinant \( \text{det} \sigma \) by convolution
\[
\text{det} \sigma = \sum_{\pi} \text{sgn}(\pi)\sigma_{1\pi(1)} \ast \cdots \ast \sigma_{n\pi(n)}
\]
where $\pi$ is a permutation. But for an $M_n$-valued function $f = (f_{ij})$ on $G$, we define the determinant $\det f$ by pointwise multiplication

$$\det f = \sum_{\pi} \text{sgn}(\pi) f_{1\pi(1)} \cdots f_{n\pi(n)}.$$ 

Using the above notation, we have

$$\det \tilde{\sigma}(\lambda) = \tilde{\det} \sigma(\lambda).$$

We also have

$$\rho(\tilde{\sigma}(\lambda)) = \int_G \lambda(-x) d\rho(x) = \rho \sigma(\lambda)$$

for $\rho \in M_n^*.$

The proof of Theorem 1 is achieved by writing the equation $f = f \ast \sigma$ in the form $f \ast \mu = 0$ and convolving it with a judiciously chosen $M_n$-valued measure which reduces the equation to simultaneous scalar convolution equations, but convolved by the same scalar measure, as in the proof of Lemma 3, to which one can apply the $L^1(G)$-Tauberian theorem [9, 39.27] and get the result readily. The setting where the Tauberian theorem applies is the following lemma which can be proved as in [12, Theorem 2].

**Lemma 2.** Let $\mu$ be a complex-valued measure on a locally compact abelian group $G$. The following conditions are equivalent:

- i) for every $f \in L^\infty(G)$, $f \ast \mu = 0$ implies that $f$ is constant;
- ii) $\mu(\lambda) \neq 0$ for $\lambda \in \hat{G} \setminus \{e\}$ where $e$ is the identity in $\hat{G}$.

We need to extend the above lemma to the following matrix-valued setting.

**Lemma 3.** Let $\mu$ be an $M_n$-valued measure on a locally compact abelian group $G$. The following conditions are equivalent:

- i) for every bounded $M_n$-valued function $f$ on $G$, $f \ast \mu = 0$ implies that $f$ is constant;
- ii) $\det \tilde{\mu}(\lambda) \neq 0$ for $\lambda \in \hat{G} \setminus \{e\}$.

**Proof.** i) $\Rightarrow$ ii). Suppose that $\det \tilde{\mu}(\lambda) = 0$ for some $\lambda \in \hat{G} \setminus \{e\}$. Then there exists $\xi \in \mathbb{C}^n \setminus \{0\}$ such that $\mu(\lambda)^T \xi = 0$, where $T$ denotes transpose. Define $f: G \to M_n$ by $f(x) = \lambda(x)(\zeta_i)$ where

$$(\zeta_{i1}, \zeta_{i2}, \ldots, \zeta_{in}) = (\xi_1, \xi_2, \ldots, \xi_n)$$

for all $i$. Then $f \ast \mu = 0$, but $f$ is not constant.

ii) $\Rightarrow$ i). Let $f = (f_{ij})$ be a bounded $M_n$-valued function such that $f \ast \mu = 0$. Let $\gamma = (\gamma_{ij})$ be the $M_n$-valued measure defined as the adjoint matrix of $\mu = (\mu_{ij})$, using convolution, so that

$$\mu \ast \gamma = \begin{pmatrix} \det \mu & 0 \\ \vdots & \ddots \\ 0 & \det \mu \end{pmatrix}.$$ 

Then we have

$$f \ast \begin{pmatrix} \det \mu & 0 \\ \vdots & \ddots \\ 0 & \det \mu \end{pmatrix} = f \ast \mu \ast \gamma = 0.$$
which gives

\[ f_{ij} \ast \det \mu = 0 \]

for all \( i, j \). Since \( \det \mu(\lambda) = \det \tilde{\mu}(\lambda) \neq 0 \) for \( \lambda \in \hat{G} \setminus \{i\} \), Lemma 2 implies that \( f_{ij} \) is constant.

**Lemma 4.** Let \( A \in M_n^+ \) be such that \( \langle A\xi, \xi \rangle = 0 \) for some \( \xi \in \mathbb{C}^n \). Then \( A\xi = 0 \).

**Proof.** We have \( A = B^2 \) for some \( B \in M_n^+ \). Hence \( \langle B^2\xi, \xi \rangle = 0 \) which gives \( B\xi = 0 \) and \( A\xi = 0 \). \( \square \)

We also need the following well-known result for Theorem 1.

**Lemma 5.** Let \( \nu \) be a probability measure on a locally compact abelian group \( G \). Then \( \nu \) is adapted if, and only if, \( \tilde{\nu}(\lambda) \neq 1 \) for every \( \lambda \in \hat{G} \setminus \{i\} \).

Let \( f \) be an \( M_n \)-valued function on \( G \) and let \( \Delta_e \) be the diagonal matrix in which each diagonal entry is the point mass \( \delta_e \) at the identity \( e \) of \( G \). Then we have \( f \ast \sigma = f \) if, and only if, \( f \ast (\sigma - \Delta_e) = 0 \). Given \( \lambda \in \hat{G} \), we have

\[ \left( \det (\sigma - \Delta_e) \right)(\lambda) = \det (\tilde{\sigma}(\sigma - \Delta_e)(\lambda)) = \det (\tilde{\sigma}(\lambda) - I_n). \]

Now we are ready to prove Theorem 1.

**Proof of Theorem 1.** \( i) \implies ii) \). Let \( \rho(\cdot) = \langle \cdot, \xi \rangle \) be a pure state of \( M_n \). We need to show that \( \rho \sigma \) is adapted. Suppose otherwise. By Lemma 5, there exists \( \lambda \in \hat{G} \setminus \{i\} \) such that \( \tilde{\rho}\sigma(\lambda) = 1 \); that is \( \langle \tilde{\sigma}(\lambda)\xi, \xi \rangle = 1 \). As we do not know if \( \|\tilde{\sigma}(\lambda)\| \) is at most 1 (see Example 1), we cannot conclude immediately that \( \tilde{\sigma}(\lambda)\xi = \xi \) although it is true but requires the following arguments. Since \( \rho \sigma \) is a probability measure, we have

\[ \rho \sigma \{x \in G : \lambda(-x) = 1\} = 1. \]

Write \( V = \{x \in G : \lambda(-x) \neq 1\} \); then

\[ \langle \sigma(V)\xi, \xi \rangle = \rho \sigma(V) = 0. \]

It follows from Lemma 5 that \( \sigma(V)\xi = 0 \) and hence also

\[ \sigma(G \setminus V)\xi = \xi. \]

Thus

\[ \tilde{\sigma}(\lambda)\xi = \left( \int_G \lambda(-x) \, d\sigma(x) \right)\xi = \left( \int_{G \setminus V} 1 \, d\sigma(x) \right)\xi + \left( \int_V \lambda(x) \, d\sigma(x) \right)\xi = \sigma(G \setminus V)\xi = \xi. \]

Therefore \( \det (\tilde{\sigma}(\lambda) - I_n) = 0 \). By Lemma 5 there is a non-constant bounded \( M_n \)-valued function \( f \) such that \( f \ast (\sigma - \Delta_e) = 0 \) which contradicts condition \( i) \).

\( ii) \implies i) \). By Lemma 5 it suffices to show that \( \det (\tilde{\sigma} - \Delta_e)(\lambda) \neq 0 \) for all \( \lambda \in \hat{G} \setminus \{i\} \). Suppose otherwise, so that \( \det (\tilde{\sigma}(\lambda) - I_n) = 0 \) for some \( \lambda \neq i \). Then there is a unit vector \( \xi \in \mathbb{C}^n \) such that \( (\tilde{\sigma}(\lambda) - I_n)\xi = 0 \); that is, \( \tilde{\sigma}(\lambda)\xi = \xi \). Let
\[ \rho(\cdot) = \langle \cdot, \xi, \xi \rangle. \] Then we have
\[ \hat{\rho} \sigma(\lambda) = \rho(\hat{\sigma}(\lambda)) = \langle \hat{\sigma}(\lambda) \xi, \xi \rangle = 1. \]

Therefore, by Lemma 5, \( \rho \sigma \) is not adapted, contradicting condition \( ii \).

We end with an example which shows that condition \( ii \) in Theorem 1 cannot be replaced by the condition that \( \text{supp} \sigma \) generates a dense subgroup of \( G \).

**Example 2.** Let \( \nu \) be any adapted probability measure on \( \mathbb{R} \) with \( \nu\{0\} \geq \frac{1}{2} \) and let
\[ \sigma = \begin{pmatrix} \nu & \delta_0 - \nu \\ \delta_0 - \nu & \nu \end{pmatrix}. \]
Then \( \sigma \) is a positive \( M_2 \)-valued measure on \( \mathbb{R} \) such that \( \sigma(\mathbb{R}) \) is the identity matrix and \( \text{supp} \sigma = \text{supp} \nu \) generates a dense subgroup of \( \mathbb{R} \). A direct calculation reveals that every \( M_2 \)-valued function \( f = (f_{ij}) \), with \( f_{11} = f_{12} \) and \( f_{21} = f_{22} \), is \( \sigma \)-harmonic and need not be constant.

In fact under the change of coordinates \( u = x + y \) and \( v = x - y \), the measure \( \sigma \) is transformed to
\[ \begin{pmatrix} \delta_0 & 0 \\ 0 & 2\nu - \delta_0 \end{pmatrix}. \]
The bounded solutions of the convolution equation with this measure are then of the form
\[ \begin{pmatrix} g & \alpha \\ h & \beta \end{pmatrix}, \]
with \( \alpha \) and \( \beta \) constants, \( g \) and \( h \) any bounded functions. Undoing the change of coordinates we see that all bounded solutions of \( f = f \ast \sigma \) are the bounded functions \( f = (f_{ij}) \), with \( f_{11} = f_{12} \) and \( f_{21} = f_{22} \) both constant.

**References**


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