WEYL-HEISENBERG FRAMES FOR SUBSPACES OF $L^2(R)$

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ABSTRACT. A Weyl-Heisenberg frame

$$\{E_{mk}T_{na}g\}_{m,n \in \mathbb{Z}} = \{e^{i\pi m(x-n)}g(x)\}_{m,n \in \mathbb{Z}}$$

for $L^2(R)$ allows every function $f \in L^2(R)$ to be written as an infinite linear combination of translated and modulated versions of the fixed function $g \in L^2(R)$. In the present paper we find sufficient conditions for $\{E_{mk}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame for $\overline{\text{span}}\{E_{mk}T_{na}g\}_{m,n \in \mathbb{Z}}$, which, in general, might just be a subspace of $L^2(R)$. Even our condition for $\{E_{mk}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame for $L^2(R)$ is significantly weaker than the previous known conditions. The results also shed new light on the classical results concerning frames for $L^2(R)$, showing for instance that the condition $G(x) := \sum_{n \in \mathbb{Z}} |g(x-na)|^2 > A > 0$ is not necessary for $\{E_{mk}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame for $\overline{\text{span}}\{E_{mk}T_{na}g\}_{m,n \in \mathbb{Z}}$.

Our work is inspired by a recent paper by Benedetto and Li, where the relationship between the zero-set of the function $G$ and frame properties of the set of functions $\{g(x-na)\}_{n \in \mathbb{Z}}$ is analyzed.

1. Preliminaries and notation

Let $H$ denote a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ linear in the first entry. Let $I$ denote a countable index set.

We say that $\{g_i\}_{i \in I} \subseteq H$ is a frame (for $H$) if there exist constants $A, B > 0$ such that

$$A \|f\|^2 \leq \sum_{i \in I} |\langle f, g_i \rangle|^2 \leq B \|f\|^2, \quad \forall f \in H.$$

In particular a frame for $H$ is complete, i.e., $\overline{\text{span}}\{g_i\}_{i \in I} = H$. In case $\{g_i\}_{i \in I}$ is not complete, $\{g_i\}_{i \in I}$ can still be a frame for the subspace $\overline{\text{span}}\{g_i\}_{i \in I}$; in that case we say that $\{g_i\}_{i \in I}$ is a frame sequence. The numbers $A, B$ that appear in the definition of a frame are called frame bounds.

Orthonormal bases and, more generally, Riesz bases are frames. Recall that $\{g_i\}_{i \in I}$ is a Riesz basis for $H$ if $\overline{\text{span}}\{g_i\}_{i \in I} = H$ and

$$\exists A, B > 0: A \sum_{i \in I} |c_i|^2 \leq \sum_{i \in I} |c_i g_i|^2 \leq B \sum_{i \in I} |c_i|^2, \quad \forall \{c_i\}_{i \in I} \in l^2(I).$$

If $\{g_i\}_{i \in I}$ is a Riesz basis for $\overline{\text{span}}\{g_i\}_{i \in I}$, we say that $\{g_i\}_{i \in I}$ is a Riesz sequence.
The present paper deals with frames having a special structure: all elements are translated and/or modulated versions of a single function. Let $L^2(R)$ denote the Hilbert space of functions on the real line which are square integrable with respect to the Lebesgue measure. First, define the following operators on functions $f \in L^2(R)$:

- Translation by $a \in R$: $$(T_a f)(x) = f(x-a), \quad x \in R.$$  
- Modulation by $b \in R$: $$(E_b g)(x) = e^{2\pi i b x} f(x), \quad x \in R.$$  

A frame for $L^2(R)$ of the form $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is called a Weyl-Heisenberg frame (or Gabor frame). For a collection of different papers concerning those frames we refer to the monograph [5].

Sufficient conditions for $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ to be a frame for $L^2(R)$ has been known for about 10 years. The basic insight was provided by Daubechies [3]. A slight improvement was proved in [6]:

**Theorem 1.1.** Let $g \in L^2(R)$ and suppose that

1. $\exists A, B > 0: \quad A \leq \sum_{n \in \mathbb{Z}} |g(x-na)|^2 \leq B$ for a.e. $x \in R$,

2. $$\lim_{b \to 0} \sum_{k \neq 0} \left\| \sum_{n \in \mathbb{Z}} T_{na} g T_{na + \frac{k}{b}} \right\|_\infty = 0.$$  

Then there exists $b_0 > 0$ such that $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is a Weyl-Heisenberg frame for $L^2(R)$ for all $b \in (0, b_0]$.

The proof of Theorem 1.1 is based on the following identity, valid for all continuous functions $f$ with compact support whenever $g$ satisfies (1):

3. $$\sum_{m,n \in \mathbb{Z}} |(f, E_{mb} T_{na} g)|^2 = \frac{1}{b} \int |f(x)|^2 G(x) dx + \frac{1}{b} \sum_{k \neq 0} \int f(x) f(x-k/b) \sum_{n \in \mathbb{Z}} g(x-na) g(x-na-k/b) dx.$$  

An estimate of the second term in (3) now shows that $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is actually a frame for all values of $b$ for which

4. $$\sum_{k \neq 0} \left\| \sum_{n \in \mathbb{Z}} T_{na} g T_{na + \frac{k}{b}} \right\|_\infty < A.$$  

A more recent result can be found in [4]: in Theorem 2.3 it is proved that if (1) is satisfied and there exists a constant $D < A$ such that

5. $$\sum_{k \neq 0} \sum_{n \in \mathbb{Z}} |g(x-na) g(x-na-k/b)| \leq D$$  

then $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(R)$ with bounds $\frac{A-D}{b}$, $\frac{B+D}{b}$. The reader should observe that [4] does not provide us with a generalization of the results in [3], [6] in a strict sense: there are cases where (5) is satisfied but (4) is not, and vice versa. The main point is that other conditions (that are easy to check) for $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ to be a frame can be derived from (5); cf. Theorem 2.4 in [4].
Define the Fourier Transform $\mathcal{F}(f) = \hat{f}$ of $f \in L^1(\mathbb{R})$ by

$$\hat{f}(y) = \int f(x)e^{-2\pi iyx} \, dx.$$ \hfill (4)

As usual we extend the Fourier Transform to an isometry from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. We denote the inverse Fourier transformation of $g \in L^2(\mathbb{R})$ by $\mathcal{F}^{-1}g$ or $\hat{g}$. It is important to observe the following commutator relations, valid for all $a \in \mathbb{R}$:

$$\mathcal{F}T_a = E_{-a}\mathcal{F}, \quad \mathcal{F}E_a = T_a\mathcal{F}.$$ \hfill (5)

We need a result from [2]. The basic insight was provided by Benedetto and Li [1], who treated the case $a = 1$.

**Theorem 1.2.** Let $g \in L^2(\mathbb{R})$. Then $\{T_nag\}_{n \in \mathbb{Z}}$ is a frame sequence with bounds $A,B$ if and only if

$$0 < aA \leq \sum_{n \in \mathbb{Z}} |\hat{g}((x + n)/a)|^2 \leq aB \quad \text{for a.e. } x \quad \text{for which } \sum_{n \in \mathbb{Z}} |\hat{g}((x + n)/a)|^2 \neq 0.$$ \hfill (6)

In that case $\{T_nag\}_{n \in \mathbb{Z}}$ is a Riesz sequence if and only if the set of $x$ for which $\sum_{n \in \mathbb{Z}} |\hat{g}((x + n)/a)|^2 = 0$ has measure zero.

Theorem 1.2 leads immediately to an equivalent condition to (1). Define the function $G$ and its kernel $N_G$ by

$$G : \mathbb{R} \to [0,\infty], \quad G(x) := \sum_{n \in \mathbb{Z}} |g(x - na)|^2,$$

$$N_G = \{ x \in \mathbb{R} \mid G(x) = 0 \}.$$

**Corollary 1.3.** $\{E_zg\}_{n \in \mathbb{Z}}$ is a frame sequence with bounds $A,B$ if and only if

$$0 < \frac{A}{a} \leq \sum_{n \in \mathbb{Z}} |g(x - na)|^2 \leq \frac{B}{a} \quad \text{for a.e. } x \in \mathbb{R} - N_G.$$ \hfill (7)

In that case $\{E_zg\}_{n \in \mathbb{Z}}$ is a Riesz sequence if $N_G$ has measure zero.

**Proof.** The inequality

$$0 < \frac{A}{a} \leq \sum_{n \in \mathbb{Z}} |g(x - na)|^2 \leq \frac{B}{a} \quad \text{for a.e. } x \in \mathbb{R} - N_G$$

holds if and only if

$$0 < \frac{A}{a} \leq \sum_{n \in \mathbb{Z}} |g(x - na)|^2 \leq \frac{B}{a} \quad \text{for a.e. } x \in \mathbb{R} - N_G.$$ \hfill (8)

By Theorem 1.2, (8) is equivalent to $\{T_nag\}_{n \in \mathbb{Z}}$ being a frame sequence with bounds $A,B$. Applying the Fourier transformation this is equivalent to $\{E_zg\}_{n \in \mathbb{Z}}$ being a frame sequence with bounds $A,B$. \hfill $\square$

2. The results

From now on we concentrate on Weyl-Heisenberg frames $\{E_{mb}T_nag\}_{m,n \in \mathbb{Z}}$. Our first result gives a sufficient condition for $\{E_{mb}T_nag\}_{m,n \in \mathbb{Z}}$ to be a frame sequence. It can be considered as a “subspace version” of a result by Ron and Shen; cf. [7]. Our condition for $\{E_{mb}T_nag\}_{m,n \in \mathbb{Z}}$ to be a frame for $L^2(\mathbb{R})$ is significantly weaker than the conditions mentioned in section 1.

Let $L^2(\mathbb{R} - N_G)$ denote the set of functions in $L^2(\mathbb{R})$ that vanishes at $N_G$.  

Theorem 2.1. Let \( g \in L^2(R), \ a, b > 0 \) and suppose that

\[
A := \inf_{x \in [0, a] - N_G} \left[ \sum_{n \in \mathbb{Z}} |g(x - na)|^2 - \sum_{k \neq 0} \sum_{n \in \mathbb{Z}} g(x - na)g(x - na - \frac{k}{b}) \right] > 0,
\]

\[
B := \sup_{x \in [0, a]} \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} g(x - na)g(x - na - \frac{k}{b}) < \infty.
\]

Then \( \{E_{mb} T_{na} g\}_{m, n \in \mathbb{Z}} \) is a frame for \( L^2(R - N_G) \) with bounds \( \frac{A}{b}, \frac{B}{b} \).

Proof. First, observe that \( \overline{\text{span}}\{E_{mb} T_{na} g\}_{m, n \in \mathbb{Z}} \subseteq L^2(R - N_G) \). Now consider a function \( f \in L^2(R - N_G) \) which is bounded and has support in a compact set. The Heil-Walnut argument (3) is valid under the assumption (8) and it gives that

\[
\sum_{m, n \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2
= \frac{1}{b} \int |f(x)|^2 \sum_{n \in \mathbb{Z}} |g(x - na)|^2 dx
+ \frac{1}{b} \sum_{k \neq 0} \int f(x - k/b) \sum_{n \in \mathbb{Z}} g(x - na)g(x - na - k/b) dx.
\]

We want to estimate the second term above. For \( k \in \mathbb{Z} \), define

\[
H_k(x) := \sum_{n \in \mathbb{Z}} T_{na} g(x) T_{na + k/b} g(x).
\]

First, observe that

\[
\sum_{k \neq 0} |T_{-k/b} H_k(x)|
= \sum_{k \neq 0} |T_{-k/b} \sum_{n \in \mathbb{Z}} T_{na} g(x) T_{na + k/b} g(x)|
= \sum_{k \neq 0} \left| \sum_{n \in \mathbb{Z}} T_{na - k/b} g(x) \overline{T_{na} g(x)} \right|
= \sum_{k \neq 0} \left| \sum_{n \in \mathbb{Z}} T_{na + k/b} g(x) \overline{T_{na} g(x)} \right|
= \sum_{k \neq 0} \left| \sum_{n \in \mathbb{Z}} T_{na + k/b} g(x) T_{na} g(x) \right|
= \sum_{k \neq 0} |H_k(x)|.
\]
Now, by a slight modification of the argument in [1], Theorem 2.3,
\[
|\sum_{k \neq 0} \int f(x) |f(x - k/b)\sum_{n \in Z} g(x - na)g(x - na - k/b)dx| \\
\leq \sum_{k \neq 0} \int |f(x)\cdot |T_{k/b}f(x)\cdot |H_k(x)dx \\
= \sum_{k \neq 0} \int |f(x)|\sqrt{|H_k(x)|}\cdot |T_{k/b}f(x)|\sqrt{|H_k(x)|}dx \\
\leq \left( \sum_{k \neq 0} \int |f(x)|^2|H_k(x)|dx \right)^{1/2} \cdot \left( \sum_{k \neq 0} \int |T_{k/b}f(x)|^2|H_k(x)|dx \right)^{1/2} \\
= \left( \int |f(x)|^2 \sum_{k \neq 0} |H_k(x)|dx \right)^{1/2} \cdot \left( \int |f(x)|^2 \sum_{k \neq 0} |T_{-k/b}H_k(x)|dx \right)^{1/2} \\
= \int |f(x)|^2 \sum_{k \neq 0} |H_k(x)|dx.
\]

Note that \(\sum_{k \neq 0} |H_k(x)| = \sum_{k \neq 0} |\sum_{n \in Z} T_{na}g(x)T_{na+k/b}g(x)|\) is a periodic function with period \(a\). By (3) and the assumption (7) we now have
\[
\sum_{m,n \in Z} |(f, E_{mb}T_{na}g)|^2 \\
\geq \frac{1}{b} \int |f(x)|^2 \left[ \sum_{n \in Z} |g(x - na)|^2 - \sum_{k \neq 0} \sum_{n \in Z} g(x - na)g(x - na - k/b) \right] dx \\
\geq \frac{A}{b} \|f\|^2.
\]

Similarly, by (3) and (8),
\[
\sum_{m,n \in Z} |(f, E_{mb}T_{na}g)|^2 \\
\leq \frac{1}{b} \int |f(x)|^2 \left[ \sum_{n \in Z} |g(x - na)|^2 + \sum_{k \neq 0} \sum_{n \in Z} g(x - na)g(x - na - k/b) \right] dx \\
= \frac{1}{b} \int |f(x)|^2 \sum_{k \in Z} \sum_{n \in Z} g(x - na)g(x - na - k/b) \\
\leq \frac{B}{b} \|f\|^2.
\]

Since those two estimates holds on a dense subset of \(L^2(R - N_G)\), they hold on \(L^2(R - N_G)\). Thus \(\{E_{mb}T_{na}g\}_{m,n \in Z}\) is a frame for \(L^2(R - N_G)\) with the desired bounds. \(\square\)
The advantage of Theorem 2.1 compared to the results in section 1 is that we compare the functions \( \sum_{n \in \mathbb{Z}} |g(x - na)|^2 \) and \( \sum_{k \neq 0} |H_k(x)| \) pointwise rather than assuming that the supremum of \( \sum_{k \neq 0} |H_k(x)| \) is smaller than the infimum of \( \sum_{n \in \mathbb{Z}} |g(x - na)|^2 \). It is easy to give concrete examples where Theorem 2.1 shows that \( \{E_m T_n g\}_{m,n \in \mathbb{Z}} \) is a frame for \( L^2(R) \) but where the conditions in section 1 are not satisfied:

**Example.** Let \( a = b = 1 \) and define

\[
g(x) = \begin{cases} 1 + x & \text{if } x \in [0, 1[, \\ \frac{1}{2} x & \text{if } x \in [1, 2[, \\ 0 & \text{otherwise}. \end{cases}
\]

For \( x \in [0, 1] \) we have

\[
G(x) = \sum_{n \in \mathbb{Z}} |g(x - n)|^2 = g(x)^2 + g(x + 1)^2 = \frac{5}{4}(x + 1)^2
\]

and

\[
\sum_{k \neq 0} \sum_{n \in \mathbb{Z}} g(x - n)g(x - n - k) = (1 + x)^2,
\]

so by Theorem 2.1 \( \{E_m T_n g\}_{m,n \in \mathbb{Z}} \) is a frame for \( L^2(R) \) with bounds \( A = \frac{1}{4}, B = 9 \). But \( \inf_{x \in R} G(x) = \frac{1}{4} \) and

\[
\sum_{k \neq 0} \| \sum_{n \in \mathbb{Z}} T_n g T_{n+k} g \|_{\infty} = 4,
\]

so the condition (4) is not satisfied. (5) is not satisfied either.

**Remark.** It is well known that \( G \) being bounded below is a necessary condition for \( \{E_m T_n g\}_{m,n \in \mathbb{Z}} \) to be a frame for \( L^2(R) \); cf. [3]. Theorem 2.1 shows that this condition is not necessary for \( \{E_m T_n g\}_{m,n \in \mathbb{Z}} \) to be a frame sequence. However, it is implicit in (7) that \( G \) has to be bounded below on \( R - N_G \) in order for Theorem 2.1 to work, and an easy modification of the proof in [3] shows that this is actually a necessary condition for \( \{E_m T_n g\}_{m,n \in \mathbb{Z}} \) to be a frame for \( L^2(R - N_G) \). We shall later give examples of frame sequences for which \( G \) is not bounded below on \( R - N_G \).

In case \( g \) has support in an interval of length \( \frac{1}{b} \) an equivalent condition for \( \{E_m T_n g\}_{m,n \in \mathbb{Z}} \) to be a frame sequence can be given. First, observe that by (3) this condition on \( g \) implies that for all continuous functions \( f \) with compact support, we have

\[
\sum_{m,n \in \mathbb{Z}} |\langle f, E_m T_n g \rangle|^2 = \frac{1}{b} \int |f(x)|^2 G(x) dx.
\]

It is not hard to show that this actually holds for all \( f \in L^2(R) \); cf. [6].

**Corollary 2.2.** Suppose that \( g \in L^2(R) \) has compact support in an interval \( I \) of length \( |I| \leq 1/b \). Then \( \{E_m T_n g\}_{m,n \in \mathbb{Z}} \) is a frame sequence with bounds \( A,B \) if and only if

\[
0 < bA \leq \sum_{n \in \mathbb{Z}} |g(x - na)|^2 \leq bB, \text{ for a.e. } x \in R - N_G.
\]

In that case \( \{E_m T_n g\}_{m,n \in \mathbb{Z}} \) is actually a frame for \( L^2(R - N_G) \).
Proof. Suppose that $g$ has support in an interval $I$ of length $|I| \leq \frac{1}{b}$. If $0 < bA \leq G(x) \leq bB$ for a.e. $x \in R - N_G$, it follows from Theorem 2.1 that $\{E_{mb T_{na}g}\}_{m,n \in Z}$ is a frame sequence with the desired bounds. Now suppose that $\{E_{mb T_{na}g}\}_{m,n \in Z}$ is a frame sequence with bounds $A, B$. Then, for every interval $I$ of length $|I| = 1/b$ and every function $f \in L^2(I)$,

$$\sum_{m,n} |\langle f, E_{mb T_{na}g} \rangle|^2 = \frac{1}{b} \int_R |f(x)|^2 G(x) dx \leq B\|f\|^2.$$ 

But this is clearly equivalent to

$$G(x) = \sum_{n \in Z} |g(x-na)|^2 \leq Bb \text{ a.e.}$$

To prove the lower bound for $G$ we proceed by way of contradiction. Suppose that for some $\epsilon > 0$ we have $0 < G(x) \leq (1-\epsilon)Ab$ on a set of positive measure. In this case there is a set $\Delta$ of positive measure and supported in an interval of length $\leq \frac{1}{\epsilon}$ so that $0 < G(x) \leq (1-\epsilon)Ab$ on $\Delta$. Then, for any function $f \in L^2(R)$ supported on $\Delta$, we have

$$\sum_{m,n} |\langle f, E_{mb T_{na}g} \rangle|^2 = \frac{1}{b} \int_R |f(x)|^2 G(x) dx \leq \frac{(1-\epsilon)Ab}{b} \int_R |f(x)|^2 dx = (1-\epsilon)A\|f\|^2.$$

Since $G(x) > 0$ on $\Delta$, there is a $k \in Z$ so that $\chi_{\Delta}T_{ka}g$ is not the zero function. With $\Delta' := \Delta \cap \text{Supp}(T_{ka}g)$ we have

$$f := \chi_{\Delta'T_{ka}g} \in \text{span} \{E_{mb T_{ka}g}\}_{m \in Z} \subseteq \text{span} \{E_{mb T_{na}g}\}_{m,n \in Z},$$

so the above calculation shows that the lower bound for $\{E_{mb T_{na}g}\}_{m,n \in Z}$ is at most $(1-\epsilon)A$, which is a contradiction. Thus

$$G(x) \geq bA \text{ for a.e. } x \in R - N_G.$$ 

In case the condition in Corollary 2.2 is satisfied, it follows from Theorem 2.1 that $\{E_{mb T_{na}g}\}_{m,n \in Z}$ is a frame for $L^2(R - N_G)$. 

For functions $g$ with the property that the translates $T_{na}g, n \in Z$, have disjoint support we can give an equivalent condition for $\{E_{mb T_{na}g}\}_{m,n \in Z}$ to be a frame sequence. Define the function

$$\tilde{G}(x) : R \rightarrow [0, \infty], \quad \tilde{G}(x) = \sum_{m \in Z} |g(x + \frac{m}{b})|^2.$$ 

Proposition 2.3. Let $g \in L^2(R), a, b > 0$ and suppose that

$$(9) \quad \text{supp}(g) \cap \text{supp}(T_{na}g) = \emptyset, \ \forall n \in Z - \{0\}.$$

Then $\{E_{mb T_{na}g}\}_{m,n \in Z}$ is a frame sequence with bounds $A, B$ if and only if there exist $A, B > 0$ such that

$$bA \leq \sum_{m \in Z} |g(x + \frac{m}{b})|^2 \leq bB \text{ for a.e. } x \in R - N_G.$$ 

In that case, $\{E_{mb T_{na}g}\}_{m,n \in Z}$ is a Riesz sequence iff $N_G$ has measure zero.
Proof. Because of the support condition (9), it is clear that \( \{E_{mb}g\}_{m \in \mathbb{Z}} \) is a frame sequence iff \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) is a frame sequence, in which case the sequences have the same frame bounds. But by Corollary 1.3 \( \{E_{mb}g\}_{m \in \mathbb{Z}} \) is a frame sequence with bounds \( A, B \) iff

\[
ba \leq \sum_{m \in \mathbb{Z}} |g(x + \frac{m}{b})|^2 \leq bB \text{ for a.e. } x \in R - N_G.
\]

Also, \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) is a Riesz sequence iff \( \{E_{mb}g\}_{m \in \mathbb{Z}} \) is a Riesz sequence, which, by Corollary 1.3, is the case iff \( N_G \) has measure zero.

We are now ready to show that \( G \) being bounded below on \( R - N_G \) (by a positive number) is not a necessary condition for \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) to be a frame sequence.

Example. Let \( a, b > 0 \) and suppose that \( \frac{1}{ab} \notin \mathbb{N} \). Chose \( \epsilon > 0 \) such that

\[
[0, \epsilon] + na \cap \left[ \frac{1}{b}, \frac{1}{b} + \epsilon \right] = \emptyset, \forall n \in \mathbb{Z}.
\]

This implies that \( \epsilon < \min(a, \frac{1}{b}) \). Define

\[
g(x) := \begin{cases} x & \text{if } x \in [0, \epsilon], \\ \sqrt{1 - (x - \frac{1}{b})^2} & \text{if } x \in \left[ \frac{1}{b}, \frac{1}{b} + \epsilon \right], \\ 0 & \text{otherwise.} \end{cases}
\]

Then the condition (9) in Proposition 2.3 is satisfied. Also, for \( x \in [0, \epsilon] \),

\[
\tilde{G}(x) = \sum_{m \in \mathbb{Z}} |g(x + \frac{m}{b})|^2 = g(x)^2 + g(x + 1)^2 = 1
\]

and for \( x \in [\epsilon, \frac{1}{b}] \), we have \( \tilde{G}(x) = 0 \). Thus, by Proposition 2.3 \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) is a frame sequence. But for \( x \in [0, \epsilon] \),

\[
G(x) = \sum_{n \in \mathbb{Z}} |g(x - na)|^2 = x^2.
\]

Thus \( G \) is not bounded below by a positive number on \( R - N_G \). By the remark after Theorem 2.1 this implies that \( \sup \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \neq L^2(R - N_G) \).

For \( ab > 1 \) it is even possible to construct an orthonormal sequence having all the features of the above example. For example, let \( a = 2, b = 1 \) and

\[
g(x) := \begin{cases} x & \text{if } x \in [0, 1], \\ \sqrt{2x - x^2} & \text{if } x \in [1, 2], \\ 0 & \text{otherwise.} \end{cases}
\]

Since

\[
\sum_{m \in \mathbb{Z}} |g(x + \frac{m}{b})|^2 = 1, \ \forall x,
\]

it follows by Proposition 2.3 that \( \{E_{m}T_{2n}g\}_{m,n \in \mathbb{Z}} \) is a Riesz sequence with bounds \( A = B = 1 \), which implies that \( \{E_{m}T_{2n}g\}_{m,n \in \mathbb{Z}} \) is an orthonormal sequence. But \( G(x) = \sum_{n \in \mathbb{Z}} |g(x - 2n)|^2 \) is not bounded below on \( R - N_G \).

\( G \) being bounded above is still a necessary condition for \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) to be a frame sequence (repeat the argument in Corollary 2.2). \( \tilde{G} \) also has to be bounded above.
Proposition 2.4. If \( \{ E_m T_{na} g \}_{m,n \in \mathbb{Z}} \) is a frame sequence with upper bound \( B \), then
\[
\sum_{m \in \mathbb{Z}} |g(x + \frac{m}{b})|^2 \leq B \quad \text{a.e.}
\]

Proof. If \( \{ E_m T_{na} g \}_{m,n \in \mathbb{Z}} \) is a frame sequence, then \( \{ \mathcal{F}^{-1} E_m T_{na} g \}_{m,n \in \mathbb{Z}} = \{ T_{mb} E_n \hat{g} \}_{m,n \in \mathbb{Z}} \) is a frame sequence with the same bounds. In particular the sequence \( \{ T_{mb} \hat{g} \}_{m,n \in \mathbb{Z}} \) has the upper frame bound \( B \). By Theorem 1.2 (or, more precisely, the proof of it in [2]) it follows that
\[
\sum_{m \in \mathbb{Z}} |g(\frac{x + m}{b})|^2 \leq B \quad \text{for a.e. } x.
\]

It follows that \( \sum_{m \in \mathbb{Z}} |g(x + \frac{m}{b})|^2 \leq B \) a.e. \( \Box \)

Remark. Recall that a wavelet frame for \( L^2(\mathbb{R}) \) has the form
\[
\{ \frac{1}{a^{m/2}} g(\frac{x}{a^m} - mb) \}_{m,n \in \mathbb{Z}},
\]
where \( a > 1, b > 0 \) and \( g \in L^2(\mathbb{R}) \) are fixed.

As well as Weyl-Heisenberg frames, wavelet frames play a very important role in applications. The theory for the two types of frames was developed at the same time, with the main contribution due to Daubechies. Several results for Weyl-Heisenberg frames have counterparts for wavelet frames. For example, Theorem 5.1.6 in [6] gives sufficient conditions for \( \{ \frac{1}{a^{m/2}} g(\frac{x}{a^m} - mb) \}_{m,n \in \mathbb{Z}} \) to be a frame based on a calculation similar to (3).

Also our results for Weyl-Heisenberg frames have counterparts for wavelet frames. The ideas in the proof of Theorem 2.1 can be used to modify [6], Theorem 5.1.6, which leads to the following:

Theorem 2.5. Let \( a > 1, b > 0 \) and \( g \in L^2(\mathbb{R}) \) be given. Let
\[
N := \left\{ \gamma \in [1, a] \left| \sum_{n \in \mathbb{Z}} |\hat{g}(a^n \gamma)|^2 = 0 \right. \right\}
\]

and suppose that
\[
A := \inf_{|\gamma| \notin [1, a] - N} \left[ \sum_{n \in \mathbb{Z}} |\hat{g}(a^n \gamma)|^2 - \sum_{k \neq 0, n \in \mathbb{Z}} |\hat{g}(a^n \gamma) \hat{g}(a^n \gamma + k/b)| \right] > 0,
\]
\[
B := \sup_{|\gamma| \in [0, a]} \sum_{k, n \in \mathbb{Z}} |\hat{g}(a^n \gamma) \hat{g}(a^n \gamma + k/b)| < \infty.
\]

Then \( \{ \frac{1}{a^{m/2}} g(\frac{x}{a^m} - mb) \}_{m,n \in \mathbb{Z}} \) is a frame sequence with bounds \( \frac{A}{B}, \frac{B}{A} \).

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