WEYL SPECTRA OF OPERATOR MATRICES

WOO YOUNG LEE

(Communicated by David R. Larson)

Abstract. In this paper it is shown that if \( M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \) is a \( 2 \times 2 \) upper triangular operator matrix acting on the Hilbert space \( \mathcal{H} \oplus \mathcal{K} \) and if \( \omega(\cdot) \) denotes the “Weyl spectrum”, then the passage from \( \omega(A) \cup \omega(B) \) to \( \omega(M_C) \) is accomplished by removing certain open subsets of \( \omega(A) \cap \omega(B) \) from the former, that is, there is equality

\[
\omega(A) \cup \omega(B) = \omega(M_C) \cup \Theta,
\]

where \( \Theta \) is the union of certain of the holes in \( \omega(M_C) \) which happen to be subsets of \( \omega(A) \cap \omega(B) \).

Let \( \mathcal{H} \) and \( \mathcal{K} \) be Hilbert spaces, let \( \mathcal{L}(\mathcal{H}, \mathcal{K}) \) denote the set of bounded linear operators from \( \mathcal{H} \) to \( \mathcal{K} \), and abbreviate \( \mathcal{L}(\mathcal{H}, \mathcal{H}) \) to \( \mathcal{L}(\mathcal{H}) \). When \( A \in \mathcal{L}(\mathcal{H}) \) and \( B \in \mathcal{L}(\mathcal{K}) \) are given we denote by \( M_C \) an operator acting on \( \mathcal{H} \oplus \mathcal{K} \) of the form

\[
M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},
\]

where \( C \in \mathcal{L}(\mathcal{K}, \mathcal{H}) \). The invertibility and spectra of \( M_C \) were considered by Du and Jin [5]. In this paper we give some conditions for operators \( A \) and \( B \) to exist an operator \( C \) such that \( M_C \) is Weyl, and describe the Weyl spectra of \( M_C \).

Recall (\[7\], \[8\]) that an operator \( A \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) for Banach spaces \( \mathcal{X} \) and \( \mathcal{Y} \) is called regular if there is an operator \( A_0 \in \mathcal{L}(\mathcal{Y}, \mathcal{X}) \) for which \( A = AA_0 \); then \( A_0 \) is called a generalized inverse for \( A \). In this case, \( \mathcal{X} \) and \( \mathcal{Y} \) can be decomposed as follows (cf. \[8\] Theorem 3.8.2):

\[
A^{-1}(0) \oplus A'A(\mathcal{X}) = \mathcal{X} \quad \text{and} \quad A(\mathcal{X}) \oplus (AA')^{-1}(0) = \mathcal{Y}.
\]

It is familiar (\[6\], \[8\]) that \( A \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \) is regular if and only if \( A \) has closed range. An operator \( A \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \) is called relatively Weyl if there is an invertible operator \( A' \in \mathcal{L}(\mathcal{K}, \mathcal{H}) \) for which \( A = AA'A \). It is known (\[8\] Theorem 3.8.6]) that \( A \) is relatively Weyl if and only if \( A \) is regular and \( A^{-1}(0) \cong A(H)^\perp \), where \( \cong \) means a topological isomorphism between spaces. An operator \( A \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \) is called left-Fredholm if it is regular with finite dimensional null space and right-Fredholm if it is regular with its range of finite co-dimension. If \( A \) is both left- and right-Fredholm, we call it Fredholm. The index, \( \text{ind} \, A \), of a left- or right-Fredholm
operator $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is defined by $\text{ind} \; A = \dim A^{-1}(0) - \dim A(\mathcal{H})^\perp$. An operator $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is called Weyl if it is Fredholm of index zero. Thus a relatively Weyl operator with finite dimensional null space or its range of finite co-dimension is Weyl. If $A \in \mathcal{L}(\mathcal{H})$, then the left essential spectrum $\sigma_+^e(A)$, the right essential spectrum $\sigma_-^e(A)$, the essential spectrum $\sigma_e(A)$, and the Weyl spectrum $\omega(A)$ are defined by

$$
\sigma_+^e(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is not left-Fredholm} \};
\sigma_-^e(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is not right-Fredholm} \};
\sigma_e(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm} \};
\omega(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is not Weyl} \}.
$$

Evidently

$$\sigma_+^e(A) \cup \sigma_-^e(A) = \sigma_e(A) \subseteq \omega(A).$$

If we write iso $\mathcal{C}$ for the isolated points of $\mathcal{C} \subseteq \mathbb{C}$ and $\sigma(A)$ for the ordinary spectrum of $A$, then we define

$$\pi_{00}(A) := \{ \lambda \in \text{iso} \sigma(A) : 0 < \dim (A - \lambda I)^{-1}(0) < \infty \}$$

for the isolated eigenvalues of finite multiplicity.

Recall that a sequence of module-homomorphisms

$$
\begin{align*}
A_0 & \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n
\end{align*}
$$

is said to be exact if $\text{ran} \; f_i = \ker f_{i+1} \; (i = 1, \cdots, n-1)$.

We begin with:

**Lemma 1.** Suppose

$$
(1.1) \quad 0 \longrightarrow A_0 \xrightarrow{T_1} A_1 \xrightarrow{T_2} \cdots \xrightarrow{T_n} A_n \longrightarrow 0
$$

is an exact sequence of Banach spaces. If each $T_j \; (1 \leq j \leq n)$ is regular, then

$$
(1.2) \quad \begin{cases} 
\bigoplus_{i=0}^{n-1} A_{2i} \cong \bigoplus_{i=0}^{n-1} A_{2i+1} & \text{ (n even)}, \\
\bigoplus_{i=0}^{n} A_{2i} \cong \bigoplus_{i=0}^{n} A_{2i+1} & \text{ (n odd)}. 
\end{cases}
$$

Hence, in particular, if the sequence (1.1) is an exact sequence of Hilbert spaces, then (1.2) holds.

**Proof.** If $T_j = T_jT_j^\#T_j$ with $T_j^\# \in \mathcal{L}(A_j, A_{j-1}) \; (1 \leq j \leq n)$, then each space $A_j$ can be decomposed as follows:

$$
(1.3) \quad A_{j-1} = T_j^{-1}(0) \oplus T_j^\#T_j(A_{j-1}) \quad (1 \leq j \leq n+1),
$$

where $T_{n+1} : A_n \rightarrow 0$ is the zero operator. Since the given sequence is exact, we have that $T_j(A_{j-1}) = T_{j+1}^{-1}(0) \; (1 \leq j \leq n)$. Since the restriction $T_j^\#$ of $T_j$ to $T_j^\#T_j(A_{j-1})$ is one-one and $T_j^\#(T_j^\#T_j(A_{j-1})) = T_j(A_{j-1})$, it follows that $T_j^\# : T_j^\#T_j(A_{j-1}) \rightarrow T_{j+1}^{-1}(0)$ is an isomorphism, i.e.,

$$
(1.4) \quad T_j^\#T_j(A_{j-1}) \cong T_{j+1}^{-1}(0) \quad (1 \leq j \leq n).
$$

Now (1.2) follows from (1.3) and (1.4). The second assertion follows from the first together with the observation that the exactness of the sequence gives that $T_j$ has closed range, so that $T_j$ is regular for $1 \leq j \leq n$. 

From Lemma 1 we can see that if \( 0 \to \mathcal{A}_0 \to \mathcal{A}_1 \to \cdots \to \mathcal{A}_n \to 0 \) is an exact sequence of finite dimensional spaces, then \( \sum_{i=0}^{n} (-1)^i \dim(\mathcal{A}_i) = 0 \) (cf. [17, Theorem A.6]).

**Corollary 2.** Suppose \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \) are Hilbert spaces. If \( T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \), \( S \in \mathcal{L}(\mathcal{Y}, \mathcal{Z}) \) and \( ST \in \mathcal{L}(\mathcal{X}, \mathcal{Z}) \) have closed ranges, then there is isomorphism

\[
T^{-1}(0) \oplus S^{-1}(0) \oplus (ST\mathcal{X})^\perp \cong (ST)^{-1}(0) \oplus (T\mathcal{X})^\perp \oplus (S\mathcal{Y})^\perp.
\]

*Proof.* From the “one-diagram” proof of the index theorem due to Yang [16], we can see that the sequence

\[
0 \to T^{-1}(0) \to (ST)^{-1}(0) \to S^{-1}(0) \to (T\mathcal{X})^\perp \to (ST\mathcal{X})^\perp \to (S\mathcal{Y})^\perp \to 0
\]

is exact. Thus the result follows at once from Lemma 1.

**Lemma 3.** For a given pair \((A, B)\) of operators, if \((A \ 0 \ B)\) is Weyl, then \( M_C \) is Weyl for every \( C \in \mathcal{L}(\mathcal{K}, \mathcal{H}) \). Hence, in particular, we have

\[
\omega(M_C) \subseteq \omega\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \subseteq \omega(A) \cup \omega(B).
\]

*Proof.* If \((A \ 0 \ B)\) is Weyl, then \( A \) and \( B \) are both Fredholm, and \( \text{ind } A + \text{ind } B = 0 \). Write

\[
M_C = \begin{pmatrix} I & 0 & C \\ 0 & I & 0 \\ A & 0 & I \end{pmatrix}.
\]

Since \((I \ C \ 0)\) is invertible for every \( C \in \mathcal{L}(\mathcal{K}, \mathcal{H}) \), and since \((I \ 0 \ B)\) and \((A \ 0 \ B)\) are both Fredholm, it follows that \( M_C \) is Fredholm. Furthermore we have that \( \text{ind } M_C = \text{ind } (I \ 0 \ B) + \text{ind } (A \ 0 \ B) = 0 \) and therefore \( M_C \) is Weyl for every \( C \in \mathcal{L}(\mathcal{K}, \mathcal{H}) \). The inclusions in (3.1) are evident from the first assertion.

The following lemma gives a necessary condition for \( M_C \) to be Weyl:

**Lemma 4.** If \( M_C \) is Weyl for some \( C \in \mathcal{L}(\mathcal{K}, \mathcal{H}) \), then \( A \in \mathcal{L}(\mathcal{H}) \) and \( B \in \mathcal{L}(\mathcal{K}) \) satisfy the following conditions:

(i) \( A \) is left-Fredholm,

(ii) \( B \) is right-Fredholm,

(iii) \( A^{-1}(0) \oplus B^{-1}(0) \cong A(\mathcal{H})^\perp \oplus B(\mathcal{K})^\perp \),

which in turn implies that \((A \ 0 \ B)\) is relatively Weyl.

*Proof.* From (3.2) we can see that if \( M_C \) is Fredholm, then \((A \ 0 \ B)\) is left-Fredholm and \((I \ C \ 0)\) is right-Fredholm, so that \( A \) is left-Fredholm and \( B \) is right-Fredholm. On the other hand since, evidently, \((I \ C \ 0)\) and \((A \ 0 \ B)\) have closed ranges, it follows from Corollary 2 that

\[
\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} I & C \\ 0 & B \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \left( \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix} \right)^\perp
\]

\[
\cong \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} I & C \\ 0 & B \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \left( \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix} \right)^\perp.
\]

Thus if \( M_C \) is Weyl, then (4.1) reduces to (iii). For the second assertion, noting that if the pair \((A, B)\) satisfies the conditions (i) and (ii), then \((A, B)\) has a pair of generalized inverses \((A', B')\), we have that \((A' \ 0 \ B')\) is a generalized inverse of \((A \ 0 \ B)\) and the condition (iii) is just the equivalence \((A \ 0 \ B)^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cong ((A \ 0 \ B) \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix})^\perp \), which implies that \((A \ 0 \ B)\) is relatively Weyl.
By the argument of Lemma 4, we can see that if any two of \( A, B, \) and \( \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \) are Fredholm, then so is the other and, in that case, \( \text{ind} \left( \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \right) = \text{ind} A + \text{ind} B \) (cf. [3, Lemma 5.2, 10]).

The first inclusion in (3.1) may be proper. However, we have a large class of operators for which the first inclusion in (3.1) is reversible. To see this recall ([3, Lemma 5.2], [10]).

Indeed the second inclusion in (6.1) follows from Lemma 3. For the first inclusion (6.1) holds.

**Corollary 5.** If either \( SP(A) \) or \( SP(B) \) has no pseudoholes, then, for every \( C \in \mathcal{L}(K, \mathcal{H}) \),

\[
(5.1) \quad \omega(M_C) = \omega \left( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right).
\]

Hence, in particular, if either \( A \in \mathcal{L}(\mathcal{H}) \) or \( B \in \mathcal{L}(K) \) is essentially normal (i.e., the self-commutator is a compact operator), then (5.1) holds.

**Proof.** From Lemma 3, we have that \( \omega(M_C) \subseteq \omega \left( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right) \). For the reverse, observe that if \( SP(A) \) has no pseudoholes, then, for every \( \lambda \in \mathbb{C} \),

\[
(5.2) \quad A - \lambda I \text{ is left-Fredholm} \implies A - \lambda I \text{ is Fredholm}.
\]

Thus if \( \lambda \notin \omega(M_C) \), then by the remark after Lemma 4 and (5.2), \( A - \lambda I \) and \( B - \lambda I \) are both Fredholm. Further since \( \begin{pmatrix} A - \lambda I & 0 \\ 0 & B - \lambda I \end{pmatrix} \) is relatively Weyl we must have that \( \lambda \notin \omega \left( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right) \). If instead \( SP(B) \) has no pseudoholes, then the same argument gives the result.

The condition “either \( SP(A) \) or \( SP(B) \) has no pseudoholes” is essential in Corollary 5. For example consider the following operators on \( \ell_2 \otimes \ell_2 \):

\[
(5.3) \quad A = U \otimes 1, \quad B = U^* \otimes 1 \quad \text{and} \quad C = (1 - UU^*) \otimes 1,
\]

where \( U \) is the unilateral shift on \( \ell_2 \). Then \( \omega \left( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right) = \mathbb{D} \) and \( \omega \left( \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \right) = \mathbb{T} \).

The following is our main theorem. It says that the passage from \( \omega(A) \cup \omega(B) \) to \( \omega(M_C) \) is accomplished by removing certain open subsets of \( \omega(A) \cap \omega(B) \) from the former.

**Theorem 6.** For a given pair \( (A, B) \) of operators there is equality, for every \( C \in \mathcal{L}(K, \mathcal{H}) \),

\[
\omega(A) \cup \omega(B) = \omega(M_C) \cup \mathfrak{S},
\]

where \( \mathfrak{S} \) is the union of certain of the holes in \( \omega(M_C) \) which happen to be subsets of \( \omega(A) \cap \omega(B) \).

**Proof.** We first claim that, for every \( C \in \mathcal{L}(K, \mathcal{H}) \),

\[
(6.1) \quad \left( \omega(A) \cup \omega(B) \right) \setminus \left( \omega(A) \cap \omega(B) \right) \subseteq \omega(M_C) \subseteq \omega(A) \cup \omega(B).
\]

Indeed the second inclusion in (6.1) follows from Lemma 3. For the first inclusion suppose that \( \lambda \notin \omega(M_C) \). Then by Lemma 4, \( \begin{pmatrix} A - \lambda I & 0 \\ 0 & B - \lambda I \end{pmatrix} \) is relatively Weyl, so that by the remark after Lemma 4, \( A - \lambda I \) is Weyl if and only if \( B - \lambda I \) is Weyl.
Therefore if \( \lambda \in (\omega(A) \cup \omega(B)) \setminus \omega(M_C) \), then \( \lambda \in \omega(A) \cap \omega(B) \), which proves (6.1). We next claim that, for every \( C \in \mathcal{L}(\mathcal{K}, \mathcal{H}) \),

\[
\eta(\omega(M_C)) = \eta(\omega(A) \cup \omega(B)),
\]

(6.2) where \( \eta(\cdot) \) denotes the “polynomially convex hull” of the compact set \( \mathcal{C} \subseteq \mathbb{C} \). Since by (6.1), \( \omega(M_C) \subseteq \omega(A) \cup \omega(B) \) for every \( C \in \mathcal{L}(\mathcal{K}, \mathcal{H}) \), we need to show that \( \partial(\omega(A) \cup \omega(B)) \subseteq \partial\omega(M_C) \), where \( \partial \mathcal{C} \) denotes the topological boundary of the compact set \( \mathcal{C} \subseteq \mathbb{C} \). But since \( \text{int} \omega(M_C) \subseteq \text{int}(\omega(A) \cup \omega(B)) \), it suffices to show that \( \partial(\omega(A) \cup \omega(B)) \subseteq \omega(M_C) \). Indeed there are inclusions

\[
\partial(\omega(A) \cup \omega(B)) \subseteq \partial\omega(A) \cup \partial\omega(B) \subseteq \sigma^+_e(A) \cup \sigma^-_e(B) \subseteq \omega(M_C),
\]

where the last inclusion follows from Lemma 4 and the second inclusion follows from the punctured neighborhood theorem (\([8\, \text{Theorem 9.8.9}],\ [9]\) for every operator \( T \),

\[
\partial\omega(T) \subseteq \partial\sigma_e(T) \subseteq \sigma^+_e(T) \cap \sigma^-_e(T).
\]

This proves (6.2). Consequently, (6.2) says that the passage from \( \omega(M_C) \) to \( \omega(A) \cup \omega(B) \) is the filling in certain of the holes in \( \omega(M_C) \). But since, by (6.1), \( (\omega(A) \cup \omega(B)) \setminus \omega(M_C) \) is contained in \( \omega(A) \cap \omega(B) \), it follows that the filling in certain of the holes in \( \omega(M_C) \) should occur in \( \omega(A) \cap \omega(B) \). This completes the proof. \( \square \)

**Corollary 7.** If \( \omega(A) \cap \omega(B) \) has no interior points, then, for every \( C \in \mathcal{L}(\mathcal{K}, \mathcal{H}) \),

\[
\omega(M_C) = \omega(A) \cup \omega(B).
\]

(7.1)

In particular if either \( A \in \mathcal{L}(\mathcal{H}) \) or \( B \in \mathcal{L}(\mathcal{K}) \) is a compact operator (more generally, a “Riesz operator”), then (7.1) holds.

**Proof.** The first assertion follows at once from Theorem 6. The second assertion follows from the fact that the Weyl spectrum of a Riesz operator is contained in \( \{0\} \). \( \square \)

Let \( r(\cdot) \) and \( r_\omega(\cdot) \) denote the spectral radius and the “Weyl spectral radius”, respectively. Du and Jin [5 Proposition 4] have shown that for a given pair \((A, B)\) of operators, \( r(M_C) \) is a constant. We also have an analogue for \( r_\omega(\cdot) \):

**Corollary 8.** For a given pair \((A, B)\) of operators, \( r_\omega(M_C) \) is a constant. Furthermore if \( \pi_{00}(A \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}) = 0 \), then, for every \( C \in \mathcal{L}(\mathcal{K}, \mathcal{H}) \),

\[
r \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = r \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = r_\omega \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = r_\omega \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.
\]

(8.1)

**Proof.** The first assertion follows at once from Theorem 6. For the second assertion we claim that

\[
\eta \sigma \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \setminus \eta \omega \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \subseteq \pi_{00} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.
\]

(8.2)

Indeed if \( \lambda \in \eta \sigma \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \setminus \eta \omega \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \), then there exists \( \epsilon > 0 \) such that \( \{ \mu : |\lambda - \mu| < \epsilon \} \cap \eta \omega \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \emptyset \), which forces that \( \lambda \in \text{iso} \sigma \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \) because if it were not so, then \( \lambda \) would be in \( \eta \omega \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \), a contradiction. This proves (8.2) and hence the second equality in (8.1). \( \square \)
Remark 9. If \((A C B)\) is normaloid (i.e., norm equals spectral radius) and if \(\pi_{00}(A 0 B)\) = 0, then
\[
\left\| \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \right\| \leq r \begin{pmatrix} A & C \\ D & B \end{pmatrix}
\]
for every compact operator \(D \in \mathcal{L}(\mathcal{H}, \mathcal{K})\).

We can also argue, by (8.1),
\[
\left\| \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \right\| = r \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = r \begin{pmatrix} A & C \\ D & B \end{pmatrix} \leq r \begin{pmatrix} A & C \\ D & B \end{pmatrix}.
\]

Note that (9.1) may, in general, fail for even finite dimensional matrices. For example,
\[
\left\| \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\| = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad r \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & 1 \end{pmatrix} = \frac{3}{2}.
\]

H. Weyl [15] has shown that every hermitian operator \(A \in \mathcal{L}(\mathcal{H})\) satisfies the equality
\[
\sigma(A) \setminus \omega(A) = \pi_{00}(A).
\]

Today we say that Weyl’s theorem holds for \(A \in \mathcal{L}(\mathcal{H})\) if \(A\) satisfies the equality (9.2). Weyl’s theorem has been extended from hermitian operators to hyponormal operators and to Toeplitz operators by L. Coburn [4], to several classes of operators including seminormal operators by S. Berberian [11, 2], and to a few classes of Banach space operators [12]. But Weyl’s theorem may or may not hold for a direct sum of operators for which Weyl’s theorem holds. For example, if \(U\) is the unilateral shift on \(\ell_2\), then Weyl’s theorem holds for both \(U\) and \(U^*\), while it does not hold for \(U \oplus U^*\). In this case note that \(\omega(U) \cup \omega(U^*) = \emptyset\) (the closed unit disk) and \(\omega(U \oplus U^*) = T\) (the unit circle). Recall (2) that an operator \(A \in \mathcal{L}(\mathcal{H})\) is called isoloid if every isolated point of \(\sigma(A)\) is an eigenvalue of \(A\). For example, every hyponormal operator is isoloid ([14, Theorem 2]). We then have:

**Lemma 10.** Suppose Weyl’s theorem holds for \(A \in \mathcal{L}(\mathcal{H})\) and \(B \in \mathcal{L}(\mathcal{K})\).

(a) If Weyl’s theorem holds for \((A 0 B)\), then
\[
\omega \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \omega(A) \cup \omega(B).
\]
(b) If \(A\) and \(B\) are isoloid, then the converse of (a) is true.

**Proof.** The proof of the statement (a) is known from [11, Theorem 4]. For the statement (b) observe that if \(A\) and \(B\) are isoloid, then
\[
\pi_{00} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = (\pi_{00}(A) \cap \rho(B)) \cup (\rho(A) \cap \pi_{00}(B)) \cup (\pi_{00}(A) \cap \pi_{00}(B)),
\]
where \(\rho(\cdot)\) denotes the resolvent set. If Weyl’s theorem holds for \(A\) and \(B\), then the right-hand side of (10.2) must be just the set \((\sigma(A) \cup \sigma(B)) \setminus (\omega(A) \cup \omega(B))\). Thus if (10.1) holds, then \(\pi_{00} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \setminus \omega \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\), which says that Weyl’s theorem holds for \((A 0 B)\). \(\square\)

The assumption “\(A\) and \(B\) are isoloid” is essential in the statement (b) of Lemma 10. For example if \(A, B : \ell_2 \to \ell_2\) are defined by
\[
A(x_1, x_2, \cdots) = (0, x_2, x_3, x_4, \cdots) \quad \text{and} \quad B(x_1, x_2, \cdots) = (0, x_1, \frac{1}{2} x_2, \frac{1}{3} x_3, \cdots),
\]
then we have that (i) Weyl's theorem holds for $A$ and $B$; (ii) $\omega(A) = \{1\}$ and $\omega(B) = \{0\}$; (iii) $\sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \{0, 1\}$; (iv) $\pi_{00}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \{0\}$; (v) $B$ is not isoloid.

**Corollary 11.** Suppose $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ are isoloid. If Weyl’s theorem holds for $A$ and $B$, and if $\omega(A) \cap \omega(B)$ has no interior points, then Weyl’s theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$.

**Proof.** This follows from Lemma 10 together with applying Corollary 7 with $C = 0$. \hfill \Box

It is familiar [6, p. 17]) that for given operators $A \in \mathcal{L}(\mathcal{H}), B \in \mathcal{L}(\mathcal{K})$, and $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, if the operator equation

\[(11.1) \quad AZ - ZB = C \quad \text{(where } Z \in \mathcal{L}(\mathcal{K}, \mathcal{H}) \text{ is the unknown)}\]

is solvable, then $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is similar to $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. In fact, $\left(\begin{smallmatrix} 1 & Z \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & -Z \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. Also it is known (cf. [6, Theorem I.4.1]) that if $\sigma(A) \cap \sigma(B) = \emptyset$, then the operator equation (11.1) is solvable. Thus if $\sigma(A) \cap \sigma(B) = \emptyset$, then for most of the familiar kinds of spectrum $\varpi$, there is equality

\[(11.2) \quad \varpi\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) = \varpi\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right).\]

Note that (11.2) for $\varpi = \omega$ is a special case of Corollary 7. However, evidently the condition “$\sigma(A) \cap \sigma(B)$ has no interior points” does not imply the solvability of the operator equation (11.1). For example, take $\mathcal{H} = \mathcal{K}, A = B = 0$ and $C = I$.

**Acknowledgements**

The author is grateful to the referee for valuable comments on this paper.

**References**


Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea
*E-mail address*: wylee@yurim.skku.ac.kr