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A REMARK ON GENERALISED PUTNAM-FUGLEDE THEOREMS

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ABSTRACT. Given $A, B \in B(H)$, the algebra of operators on a Hilbert space H, define $\delta_{A,B} : B(H) \to B(H)$ and $\triangle_{A,B} : B(H) \to B(H)$ by $\delta_{A,B}(X) = AX - XB$ and $\triangle_{A,B}(X) = AXB - X$. Let P_1 and P_2 be two classes of operators strictly larger than the class of normal operators. Define $(P_1, P_2) \in PF(\delta)$ (resp., $PF(\triangle)$) if $ker\delta_{A,B} \subset ker\delta_{A^*,B^*}$ (resp., $ker\triangle_{A,B} \subset ker\triangle_{A^*,B^*}$) for all $A \in P_1$ and $B^* \in P_2$. This note shows that the equivalence $(P_1, P_2) \in PF(\delta) \Longleftrightarrow (P_1, P_2) \in PF(\triangle)$ holds for a number of the commonly considered classes of operators.

1. Introduction

Let H be an infinite dimensional complex Hilbert space, and let $B(H)$ denote the algebra of operators (= bounded linear transformations) on H. Let $\delta_{A,B} : B(H) \to B(H)$ denote the generalised derivation $\delta_{A,B}(X) = AX - XB$. The Putnam-Fuglede theorem [8, p. 104] says that if A, B^* are normal operators, then $ker\delta_{A,B} = ker\delta_{A^*,B^*}$. Define (the elementary operator) $\triangle_{A,B} : B(H) \to B(H)$ by $\triangle_{A,B}(X) = AXB - X$. Then the Putnam-Fuglede theorem has a (natural) $\triangle_{A,B}$ analogue: if A, B^* are normal, then $ker\triangle_{A,B} = ker\triangle_{A^*,B^*}$. A number of generalisations of the Putnam-Fuglede theorem, and its $\triangle_{A,B}$ analogue, are to be found in the extant literature, amongst them generalisations where the normal operators A, B^* are replaced by operators from a class (or classes) strictly larger than the class of normal operators. Here the particular classes which have drawn a lot of attention are those consisting of either subnormal or hyponormal or M-hypornomal or dominant or k-quasihyponormal operators, and it is known that $ker\delta_{A,B} \subset ker\delta_{A,B}$ ($ker\triangle_{A,B} \subset ker\triangle_{A^*,B^*}$) for A, B^* belonging to many a pair of these classes. (See [2], [5], [6], [7], [9], [10], [11], [12], [13], [14] and some of the references there.) Let us say that a pair of classes P_1 and P_2 ($\subset B(H)$) have the $PF(\delta)$ property, denoted $(P_1, P_2) \in PF(\delta)$, if $ker\delta_{A,B} \subset ker\delta_{A^*,B^*}$ for all $A \in P_1$ and $B^* \in P_2$. Define $(P_1, P_2) \in PF(\triangle)$ analogously. In this note we consider the equivalence $(P_1, P_2) \in PF(\delta) \Longleftrightarrow (P_1, P_2) \in PF(\triangle)$. Using a very simple argument we show that the mentioned equivalence holds for a large number of classes P (amongst them the classes of operators stated above).

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We shall denote the closure of the range, the orthogonal complement of the kernel and the restriction to an invariant subspace M of the operator X by $\overline{ranX}, ker^{\perp}X$ and $X|M$, respectively. The spectrum, the point spectrum, the approximate point spectrum and the residual spectrum of X will be denoted by $\sigma(X), \sigma_p(X), \sigma_a(X)$ and $\sigma_r(X); \sigma_{jp}(X)$ will denote the set of normal eigen-values of X and $\sigma_{ja}(X)$ will denote the joint approximate point spectrum of X ($i.e., \sigma_{ja}(X) = \{\lambda :$ there exists a sequence $\{x_n\}$ of unit vectors in H such that $(X-\lambda)x_n \to 0$ and $(X-\lambda)^*x_n \to 0$ as $n \to \infty\}$). We say that $X \in B(H)$ is a quasi-affinity if both X and X^* are injective.

2. Results

Given $A \in B(H)$ there exists a Hilbert space $H^0 \supset H$ and an isometric *-isomorphism $A \to A^0$ preserving order such that $\sigma(A) = \sigma(A^0)$ and $\sigma_a(A) = \sigma_a(A^0) = \sigma_p(A^0)$. This is the Berberian extension theorem [1]; this theorem will play an important role in the sequel.

We say that $A \in B(H)$ belongs to the class P of operators if:

(i) $A \in P$, then the Berberian extension A^0 of A also $\in P$;

(ii) $A \in P$ is invertible, then $A^{-1} \in P$;

(iii) $A \in P$ and M is an invariant subspace of A, then $A|M \in P$;

(iv) $\sigma_a(A) = \sigma_{ja}(A)$.

Given P classes P_1 and P_2 we say that the pair $(P_1, P_2) \in PF(\delta)$ (resp., $(P_1, P_2) \in PF(\triangle)$) if $ker\delta_{A,B} \subset ker\delta_{A^*,B^*}$ (resp., $ker\triangle_{A,B} \subset ker\triangle_{A^*,B^*}$) for all $A \in P_1$ and $B^* \in P_2$.

Examples of P classes, and of P classes P_1 and P_2 such that $(P_1, P_2) \in PF(\delta)$ (and/or $PF(\triangle)$), abound. Recall that the operator T is said to be:

subnormal if T has a normal extension;

p-hyponormal, $0 < p \leq 1$, if $|T^*|^{2p} \leq |T|^{2p}$;

dominant if to each complex number λ there corresponds a real number M_λ such that $||(T-\lambda)^*x|| \leq M_\lambda ||(T-\lambda)x||$ for all $x \in H$;

M-hyponormal if it is a dominant operator for which there exists a number M such that $M_\lambda \leq M$ for all λ;

k-quasihyponormal, $k \geq 1$ some integer, if $||A^{*^k}Ax|| \leq ||A^{k+1}x||$ for all $x \in H$.

The classes consisting of operators T which are either subnormal or p-hyponormal or dominant or M-hyponormal or injective k-quasihyponormal are P classes; also, $ker\delta_{A,B} \subset ker\delta_{A^*,B^*}$ for all A and B^* in any combination of these classes except for when both A and B^* are dominant (see [5], [6]).

The following lemma is well known.

Lemma 1. *Let $A, B \in B(H)$.*

(i) *If $ker\delta_{A,B} \subset ker\delta_{A^*,B^*}$, then, for all $X \in ker\delta_{A,B}$, $\overline{ranX}$ reduces $A, ker^{\perp}X$ reduces B, and $A|\overline{ranX}$ and $B|ker^{\perp}X$ are unitarily equivalent normal operators.*

(ii) *If $ker\triangle_{A,B} \subset ker\triangle_{A,^*,B^*}$, then, for all $X \in ker\triangle_{A,B}, \overline{ranX}$ reduces A, $ker^{\perp}X$ reduces B, and $A|\overline{ranX}$ and $B|ker^{\perp}X$ are normal operators.*

The following theorem is the main result of this note. Since the class of normal operators is not a P class, the theorem does not cover the case in which one of the P_i classes is the class of normal operators. However the argument of the proof of the theorem extends to this case also, as we shall see in Corollary 3.

Theorem 2. *Given P classes P_1 and $P_2, (P_1, P_2) \in PF(\delta) \Longleftrightarrow (P_1, P_2) \in PF(\triangle)$.*

Proof. $\Rightarrow$. Suppose that $ker\delta_{A,B} \subset ker\delta_{A^*,B^*}$, where $A \in P_1$ and $B^* \in P_2$. Letting A^0, B^0 and X^0 denote the Berberian extensions of A, B and X, it follows that if $X \in ker\triangle_{A,B}$, then $\triangle_{A^0,B^0}(X^0) = 0$. This implies that $\overline{ranX^0}$ is invariant for A^0 and $ker^{\perp}X^0 (= \overline{ranX^{*0}})$ is invariant for B^{*0}. Let $A_0 = A^0|\overline{ranX^0}, B_0^* = B^{*0}|ker^{\perp}X^0$, and let $X_0 : ker^{\perp}X^0 \to \overline{ranX^0}$ be the quasi-affinity defined by setting $X_0x = X^0x$ for each $x \in ker^{\perp}X^0$. Clearly, $A_0 \in P_1, B_0^* \in P_2, \sigma_a(A_0) = \sigma_{ja}(A_0) = \sigma_p(A_0) = \sigma_{jp}(A_0)$ and $\sigma_a(B_0^*) = \sigma_{ja}(B_0^*) = \sigma_p(B_0^*) = \sigma_{jp}(B_0^*)$. Since $\triangle_{A_0,B_0}(X_0) = 0$, and X_0 is a quasi-affinity, $0 \notin \sigma(B_0^*)$ and $0 \notin \sigma(A_0)$.

Define the operators S and T by $S = A_0 \oplus I_{H_2}$ and $T^* = B_0^* \oplus I_{H_2}$,where I_{H_2} and I_{H_2} denote, respectively, the identity operators on $H_2 = H^0 \ominus \overline{ranX^0}$ and $H_2 = H^0 \ominus ker^{\perp}X^0$. It is then clear that $S \in P_1$ and $T^* \in P_2$. Furthermore, T^* is invertible and $\delta_{S,T^{-1}}(X^0) = 0$, where X^0 is the operator $X^0 = X_0 \oplus 0$ (see above). By hypothesis $(P_1, P_2) \in PF(\delta)$; hence $\delta_{S^*,T^{*-1}}(X^0) = 0$. This implies that $S^*X^0T^* = X^0$, or, $A_0^*X_0B_0^* = X_0$. Consequently, $\triangle_{A^{*0},B^{*0}}(X^0)$, and so also $\triangle_{A^*,B^*}(X) = 0$.

$\Leftarrow$. Arguing as above it is seen that if $X \in ker\delta_{A,B}$, then $\delta_{A_0,B_0}(X_0) = 0$. It is clear that if $0 \notin \sigma_p(A_0)$ (or $0 \notin \sigma_p(B_0^*)$), then A_0 and B_0 are invertible. If, on the other hand, $0 \in \sigma_p(A_0)$ (or $0 \in \sigma_p(B_0^*)$), then 0 is a joint eigen-value of both A_0 and B_0^* and there exist decompositions $A_0 = 0 \oplus A_{02}, X_0 = X_{01} \oplus X_{02}$ and $B_0 = 0 \oplus B_{02}$ such that $A_{02}X_{02} = X_{02}B_{02}$. The operators A_{02} and B_{02} are invertible, and it is seen that $\triangle_{0\oplus A_{02}\oplus I_{H_2},0\oplus B_{02}^{-1}\oplus I_{H_2}}(X_{01} \oplus X_{02} \oplus 0) = (X_{01} \oplus X_{02} \oplus 0)$. This, as above, implies that $A_0^*X_0 = X_0B_0^*$, and hence that $A^*X = XB^*$.

As stated earlier, even though the class **N** of normal operators is not a P class, Theorem 2 holds with either of the P classes P_i $(i = 1, 2)$ replaced by **N**.

Corollary 3. *Given a P class $P_1, (P_1, \mathbf{N}) \in PF(\delta) \Longleftrightarrow (P_1, \mathbf{N}) \in PF(\triangle)$.*

Proof. We consider the implication '$\Rightarrow$' only. Suppose that $ker\delta_{A,B} \subset ker\delta_{A^*,B^*}$, where $A \in P_1$ and $B^* \in \mathbf{N}$. Proceeding as in the proof of Theorem 2 one has that $\triangle_{A_0,B_0}(X_0) = 0$, where B_0^* is an invertible subnormal operator. Let N^* denote the minimal normal extension of B_0^* (on $K \supset ker^{\perp}X^0$, say); then N^* is invertible. Letting $C = A_0 \oplus 0$ and $Y = X_0 \oplus 0$ we have that $\delta_{C,N^{-1}}(Y) = 0$. Since $C \in P_1$, this implies that $\delta_{C^*,N^{*-1}}(Y) = 0$, or $\triangle_{A_0^*,B_0^*}(X_0) = 0$. The proof now follows.

We prove now an asymptotic version of Theorem 1. Given a natural number $n > 1$, let $\delta^n_{A,B}(\triangle^n_{A,B})$ denote an n-times application of $\delta_{A,B}$ (resp., $\triangle_{A,B}$).

Lemma 4. *Given P classes P_1 and P_2, suppose that $(P_1, P_2) \in PF(\delta)$ (or $(P_1, P_2) \in PF(\triangle)$). If $A \in P_1$ and $B^* \in P_2$, then*

$$ker\delta^n_{A,B} = ker\delta_{A,B} \text{ and } ker\triangle^n_{A,B} = ker\triangle_{A,B}$$

for all integers $n > 1$.

Proof. We consider $ker\triangle^n_{A,B}$; the case of $ker\delta^n_{A,B}$ is similarly dealt with.

Let $X \in ker\triangle^n_{A,B}$, and let $\triangle^{n-1}_{A,B}(X) = Y$. Since either of the hypotheses $(P_1, P_2) \in PF(\delta)$ or $(P_1, P_2) \in PF(\triangle)$ implies $\triangle_{A,B}(Y) = 0 = \triangle_{A^*,B^*}(Y)$, it follows that $A_1 = A|\overline{ranY}$ and $B_1^* = B^*|ker^{\perp}Y$ are normal operators. Let $X : ker^{\perp}Y \oplus kerY \to \overline{ranY} \oplus (\overline{ranY})^{\perp}$ have the representation $X = [X_{ij}]^2_{i,j=1}$. Letting

$A = A_1 \oplus A_2$ and $B^* = B_1^* \oplus B_2^*$, it then follows that $\triangle^n_{A_1,B_1}(X_{11}) = 0$ and

$$Y = \triangle^{n-1}_{A,B}(X) = [\triangle^{n-1}_{A_i,B_j}(X_{ij})]^2_{i,j=1} = \triangle^{n-1}_{A_1,B_1}(X_{11}) \oplus 0.$$

Since the operator $\triangle_{A_1,B_1}$ is normal, it follows that $\triangle_{A_1,B_1}(X_{11}) = 0$. Hence $Y = \triangle^{n-1}_{A,B}(X) = 0$. A finite induction argument now shows that $\triangle_{A,B}(X) = 0$, i.e., $ker\triangle^n_{A,B} \subset ker\triangle_{A,B}$. The reverse inclusion being obvious, the proof is complete.

Remark. Given P classes P_1 and P_2, let $(P_1, P_2) \in PF(\delta)$ or $PF(\triangle)$. Suppose that $A \in P_1$, $B^* \in P_2$ and $\delta^n_{A,B}(X)$ is compact for some integer $n > 1$. If $\pi(A) \in P_1$ and $\pi(B^*) \in P_2$, where $\pi : B(H) \to B(H)\backslash K(H)$ is the Calkin map, then $\delta_{A,B}(X)$ and $\triangle_{A,B}(X)$ are compact. This is seen as follows. Clearly, $\pi(\delta^n_{A,B}(X)) = \delta^n_{\pi(A),\pi B}(\pi(X))$, and so, since $\delta^n_{A,B}(X)$ is compact, $\delta^n_{\pi(A),\pi B}(\pi(X)) = 0$. By hypothesis $(P_1, P_2) \in PF(\delta)$ (or $PF(\triangle)$); also $\pi(A) \in P_1$ and $\pi(B^*) = \pi(B)^* \in P_2$. Hence, since

$$ker\delta^n_{\pi(A),\pi(B)} = ker\delta_{\pi(A),\pi(B)} = ker\triangle_{\pi(A),\pi(B)},$$

we have that $\delta_{\pi(A),\pi B}(\pi(X)) = 0 = \triangle_{\pi(A),\pi B}(\pi(X))$, i.e., $\delta_{A,B}(X)$ and $\triangle_{A,B}(X)$ are compact. This result has been proved for the case in which A,B^* are normal in [15] using a different argument.

The inspiration for the following theorem comes from [11, Theorem 2′].

Theorem 5. *Given P classes P_1 and P_2, suppose that $A \in P_1, B^* \in P_2$ and either $(P_1, P_2) \in PF(\delta)$ or $(P_1, P_2) \in PF(\triangle)$. Let n be a fixed natural number, and let $k > 0$. Then, for all given $\epsilon > 0$, there exists a $d = d(k, n, \epsilon) > 0$ such that:*
(a) $||\delta_{A,B}(X)|| + ||\delta_{A^*,B^*}(X)|| < \epsilon$ *whenever* $||X|| < k$ *and* $||\delta^n_{A,B}(X)|| < d$;
(b) $||\triangle_{A,B}(X)|| + ||\triangle_{A*,B^*}(X)|| < \epsilon$ *whenever* $||X|| < k$ *and* $||\triangle^n_{A,B}(X)|| < d$.

Proof. The proof is by contradiction. We consider case (b): case (a) is similarly dealt with.

Suppose that there exists an $\epsilon_0 > 0$ and, for each integer $m \geq 1$, an operator X_m such that $||X_m|| \leq k, ||\triangle^n_{A,B}(X_m)|| < \frac{1}{m}$ and either $||\triangle_{A,B}(X_m)|| > \epsilon_0$ or $||\triangle_{A^*,B^*}(X_m)|| > \epsilon_0$. Considering subsequences if need be, we may assume without loss of generality that either $||\triangle_{A,B}(X_m)|| > \epsilon_0$ or $||\triangle_{A^*,B^*}(X_m)|| > \epsilon_0$ for all integers $m \geq 1$. Now choose for each m a unit vector $x_m \in H$ such that either $||\triangle_{A,B}(X_m)x_m|| \geq \epsilon_{0/2}$ or $||\triangle_{A^*,B^*}(X_m)x_m|| \geq \epsilon_{0/2}$ for all m.

Since $||X^0_m|| \leq k$, the hypotheses imply that $||\triangle^n_{A^0,B^0}(X^0_m)|| = 0$, where $A^0 \in P_1, B^{0*} \in P_2$ and either $(P_1, P_2) \in PF(\delta)$ or $(P_1, P_2) \in PF(\triangle)$. Lemma 4 applies, and we conclude that $\triangle_{A^0,B^0}(X^0_m) = 0 = \triangle_{A^{0*},B^{0*}}(X^0_m)$. Hence

$$\text{glim}||\triangle_{A,B}(X_m)x_m|| = 0 = \text{glim } ||\triangle_{A^*,B^*}(X_m)x_m||.$$

(Here "glim" is a Banach generalised limit on the set of all bounded numerical sequences [1].) This is a contradiction.

We conclude this note with the remark that it is not necessary for P_1 and P_2 to be P classes for the equivalence $(P_1, P_2) \in PF(\delta) \Longleftrightarrow (P_1, P_2) \in PF(\triangle)$ to hold. Thus if P_1 is the class of contractions with $C_{\cdot 0}$ completely non-unitary part and P_2 is the P class consisting of isometries, then the said equivalence holds; again, if we restrict ourselves to only those $X \in ker\delta_{A,B}$ (and $X \in ker\triangle_{A,B}$) which are compact, then the equivalence holds for the case in which P_1 is the class of contractions and P_2 is the P class consisting of isometries. (The proof of these assertions is left to the reader, but see [3, Theorem 8] and [4, Theorem 2].)

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