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COMPACT SETS OF COMPACT OPERATORS IN ABSENCE OF l^1

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ABSTRACT. We characterize the compactness of a subset of compact operators between Banach spaces when the domain space does not have a copy of l^1.

In a recent paper, F. Galaz-Fontes [6] characterizes the (relative) compactness of a set of compact operators from a reflexive and separable Banach space X into another Banach space Y. Here we prove that such characterization is also valid when X does not contain a copy of l^1. In order to clarify our terminology we say that a set of bounded linear operators $H \subset L(X,Y)$ is *sequentially weak-norm equicontinuous* (called *uniformly w-continuous* in [6]) if for each weakly-null sequence (x_n) in X the sequences $(h(x_n))$ converge in norm to 0 uniformly in $h \in H$, i.e. $\sup\{\|h(x_n)\| : h \in H\}$ converges to 0.

Theorem 1. *Let X be a Banach space without a copy of l^1 and let H be a subset of bounded operators from X to a Banach space Y. Then H is a relatively compact subset in the space of compact operators $K(X,Y)$ in the uniform topology of operators if and only if it verifies the following two conditions:*

(1) *H is pointwisely relatively compact, i.e. for each $x \in X$ the set*
$$H(x) = \{h(x) : h \in H\}$$
is relatively compact in Y.

(2) *H is sequentially weak-norm equicontinuous.*

Note that if X does not contain a copy of l^1, then by applying the Rosenthal-Dor Theorem (see [11] and [4] or [3, Ch. IX]) every bounded sequence in X has a weakly-Cauchy subsequence and, therefore, a bounded linear operator $h : X \longrightarrow Y$ is compact if and only if it is *completely continuous*, that is, if and only if it takes weakly convergent sequences in X to convergent ones in Y. This tells us that condition (2) applied to a singleton characterizes compact operators when the space X does not have a copy of l^1 [7, 17.7].

Our notation is standard: $L(X,Y)$ is the Banach space of bounded linear operators from the Banach space X to the Banach space Y, endowed with the topology of the uniform convergence on the unit ball B_X of X, $K(X,Y)$ is its closed subspace

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of compact operators, X^* is the dual of X and $\sigma(X, X^*)$ denotes the weak topology on X.

For the sake of completeness we consider the definition of a precompact set in the setting of uniform spaces. A Hausdorff uniform space R is said to be *precompact* if its completion is compact (see [2, T.G. II. 29] or [8, p. 36]). Equivalently, R is *totally bounded*, that is, for every *vicinity* N of R there exists a covering of R by finitely many sets which are small of order N. Within the framework of topological vector spaces this reads as follows: for a subset R of a topological vector space E the following are equivalent: (a) R is precompact, (b) R is relatively compact in the completion of E and (c) for each 0-neighbourhood V in E there exists a finite subset M such that $R \subset M + V$ [7, 3.5.1]. Our proof relies on the Rosenthal-Dor Theorem and on the following two results: a version of Ascoli's classical theorem (Theorem A) and a characterization of relative compact sets of compact operators between Banach spaces due to Palmer [10] and Anselone [1] (Theorem B).

Theorem A ([2, Th.2 on page T.G. X.17])**.** *Consider two uniform spaces X and Y, a cover $\mathcal{G}$ of X formed by precompact subsets and a set H of functions from X to Y such that the restriction of each $h \in H$ to each $A \in \mathcal{G}$ is uniformly continuous. Then H is precompact in the topology (uniformity) of uniform convergence on members of $\mathcal{G}$ if and only if it verifies the following two conditions:*

(i) *H is pointwisely precompact, i.e. $H(x) := \{h(x) : h \in H\}$ is precompact in Y, for each $x \in X$.*
(ii) *For each $A \in \mathcal{G}$ the set H_A of restrictions to A of the functions $h \in H$ is uniformly equicontinuous.*

Theorem B ([1] and [10])**.** *Let X and Y be normed linear spaces and H a set of linear operators on X into Y that satisfies*

(i) *$H(B_X) := \{h(x) : h \in H, x \in B_X\}$ is totally bounded,*
(ii) *$H^*y^* := \{h^*(y^*) : h^* \in H^*\}$ is totally bounded for each y^* in Y^*.*

Then H is totally bounded.

Proof of Theorem 1. We shall apply Ascoli's Theorem A to the uniform spaces $(X, \sigma(X, X^*))$ and $(Y, \|\cdot\|)$ and the family $\mathcal{G}$ of weakly-Cauchy sequences in X. Note that $\mathcal{G}$ is a cover of X formed by weakly-precompact sets and that each compact operator $h : X \longrightarrow Y$ is weak-to-norm continuous on bounded sets of X [5, Ths. VI. 5.6 and VI. 5.2], hence on each A in $\mathcal{G}$. With the topology τ of the uniform convergence on the members of $\mathcal{G}$ the space $L(X, Y)$ is a locally convex space that we denote by $L_\tau(X, Y)$ [7, 8.4].

($\Longrightarrow$) If H is relatively compact, then (1) follows from the continuity of the operator

$$T \in K(X, Y) \to T(x) \in Y$$

for each x in X. A standard argument leads us to condition (2) from the fact that each compact operator is completely continuous.

($\Longleftarrow$) Conversely, suppose that H verifies conditions (1) and (2). In particular condition (2) says that $H \subset K(X, Y)$. We shall prove that H is relatively compact in $K(X, Y)$ in three steps:

Step 1: H is precompact in $L_\tau(X, Y)$. Applying Ascoli's Theorem, it is enough to verify that for each weakly-Cauchy sequence A in X, the set of restrictions $H_A := \{h\,|_A : h \in H\}$ is uniformly equicontinuous when A is endowed with the

weak topology $\sigma(X, X^*)$ and Y with the norm topology. It suffices to prove this for each weakly-null sequence in X. Let $A := \{x_n : n \in \mathbb{N}\}$ be the set of terms of a weakly-null sequence in X. To prove that the set H_A of restrictions of H to A is uniformly equicontinuous we need to prove that, for each $\varepsilon > 0$, there exists a weak-neighbourhood U of 0 in X such that whenever $x_n - x_m$ is in U, then $\|h(x_n) - h(x_m)\| < \varepsilon$ for every h in H. We can suppose that $x_n \neq x_m$ whenever $n \neq m$ and that $x_n \neq 0$ for every n. Condition (2) tells us that for a given $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ for which $\|h(x_n)\| < \varepsilon/2$ for every $n > N_1$ and every $h \in H$. Then we have that $\|h(x_n) - h(x_m)\| < \varepsilon$ for every $n, m > N_1$ and $h \in H$. Now, let us consider an arbitrary weak 0-neighborhood V_1 in X. Since (x_n) is a weakly null sequence, there exists $N_2 \geq N_1$ in $\mathbb{N}$ such that $x_n \in \frac{1}{2}V_1$ for $n > N_2$. So, for $n, m > N_2$ we have $x_n - x_m \in V_1$ and $\|h(x_n) - h(x_m)\| < \varepsilon$. Since the elements $x_1, ..., x_{N_2}$ are different from each other and also different from 0 we can separate them in the weak topology, i.e. we can obtain a weak 0-neighbourhood V_2 in X such that x_n and $x_n - x_m$ are not in V_1 if $n \neq m$ are in $\{1, 2, ..., N_2\}$. Now, it suffices to take $U = V_1 \cap V_2$.

Step 2: H is relatively compact in $(K(X, Y), \tau)$. The completion of $L_\tau(X, Y)$ is a subspace of the space of all linear maps from X to Y [9, 39.7.(9)]. If R is an adherent point to (H, τ) in its completion, then $R : X \to Y$ is linear. Let us prove that R is completely continuous, hence compact as X does not have a copy of l^1. For a given weakly-null sequence (x_n) in X we can obtain for each $\varepsilon > 0$ an element $h_\varepsilon \in H$ such that

$$\sup\{\|(R - h_\varepsilon)(x_n)\| : n\} < \varepsilon.$$

Taking a null sequence (ε_m) and having in mind that $(h_\varepsilon(x_n))$ is norm null in Y for each $\varepsilon > 0$, we obtain that $(R(x_n))$ is norm null.

Step 3: H is relatively compact in $K(X, Y)$ (equivalently in $L(X, Y)$). Let us verify the conditions in Theorem B:

(i) Let $(h_n(x_n))$ be a sequence in $H(B_X)$. Extracting a subsequence if necessary we can suppose that (x_n) is weakly-Cauchy. Let $R \in K(X, Y)$ be a τ-adherent point to (h_n) and consider a null sequence (ε_m) of positive scalars. Then for each $k \in \mathbb{N}$ there exists $m_k \in \mathbb{N}$ such that

$$\sup\{\|(R - h_{m_k})(x_n)\| : n\} < \varepsilon_k.$$

Since $(R(x_n))$ is norm convergent in Y, we have that $(h_{m_k}(x_{m_k}))$ is also norm convergent in Y.

(ii) To prove that $H^*y^* := \{h^*(y^*) : h^* \in H^*\}$ is relatively compact in X^* let us consider the set of index $I := H^*y^*$ and the operator

$$S : x \in X \to S(x) := (\langle x, x^* \rangle)_{x^* \in I} \in l_I^\infty.$$

Because of condition (b) S is a completely continuous, hence compact, operator. Therefore its adjoint S^* is also compact. For each $z^* \in I$ denote by e_{z^*} the continuous linear form

$$(\alpha_{x^*}) \in l_I^\infty \to \alpha_{z^*} \in \mathbb{C} \text{ or } \mathbb{R};$$

then $\{e_{z^*} : z^* \in I\}$ lies in the unit ball of the dual $(l_I^\infty)^*$ and $S^*(e_{z^*}) = z^*$. Therefore

$$\{S^*(e_{z^*}) : z^* \in I\} = I = H^*y^*$$

is relatively compact in X^*. □

Corollary 2. *Let X, Y be two Banach spaces and $H \subset L(X,Y)$. Then H is a relatively compact subset of $K(X,Y)$ in the uniform topology of operators if and only if it verifies the following three conditions:*

(1) *H is pointwisely relatively compact.*
(2) *H is sequentially weak-norm equicontinuous.*
(3) *The restriction of the elements of H to each copy Z of l^1 in X is relatively compact in $K(Z,Y)$.*

Proof. Only the if part needs proof. The conditions (2) and (3) guarantee that each $h \in H$ is a compact operator since by the Rosenthal-Dor Theorem ([11] and [4]), each bounded sequence (x_n) in X has a subsequence, say (z_n), which is either weakly-Cauchy, and then $(h(z_n))$ is norm convergent by applying (2), or (z_n) is equivalent to the canonical basis of l^1, and then $(h(z_n))$ is relatively norm-compact by applying (3). On the other hand, if H is not relatively compact in $K(X,Y)$, equivalently if H is not precompact, then there exist $\varepsilon > 0$ and a sequence (h_n) in H such that $\|h_n - h_m\| > \varepsilon$ for $n \neq m$. Let us consider a sequence $(x_{n,m})$ in B_X such that for $n \neq m$ we have

$$(*) \qquad \|(h_n - h_m)(x_{n,m})\| > \varepsilon.$$

Applying the Rosenthal-Dor Theorem $(x_{n,m})$ has some subsequence (z_k) which either is weakly-Cauchy, and inequality (*) contradicts (2), or (z_k) is equivalent to the canonical basis of l^1, and then (*) contradicts (3). □

Remark 3. Galaz-Fontes's proof is essentially different from ours in the sense that it relies on the fact that if X is a reflexive and separable Banach space, then B_X is metrizable and compact for the weak topology, something that we do not need in our proof. Of course both proofs coincide in the fact that they use Ascoli's Theorem.

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