

## COMPACT SETS OF COMPACT OPERATORS IN ABSENCE OF $l^1$

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ABSTRACT. We characterize the compactness of a subset of compact operators between Banach spaces when the domain space does not have a copy of  $l^1$ .

In a recent paper, F. Galaz-Fontes [6] characterizes the (relative) compactness of a set of compact operators from a reflexive and separable Banach space  $X$  into another Banach space  $Y$ . Here we prove that such characterization is also valid when  $X$  does not contain a copy of  $l^1$ . In order to clarify our terminology we say that a set of bounded linear operators  $H \subset L(X, Y)$  is *sequentially weak-norm equicontinuous* (called *uniformly  $w$ -continuous* in [6]) if for each weakly-null sequence  $(x_n)$  in  $X$  the sequences  $(h(x_n))$  converge in norm to 0 uniformly in  $h \in H$ , i.e.  $\sup \{\|h(x_n)\| : h \in H\}$  converges to 0.

**Theorem 1.** *Let  $X$  be a Banach space without a copy of  $l^1$  and let  $H$  be a subset of bounded operators from  $X$  to a Banach space  $Y$ . Then  $H$  is a relatively compact subset in the space of compact operators  $K(X, Y)$  in the uniform topology of operators if and only if it verifies the following two conditions:*

- (1)  *$H$  is pointwisely relatively compact, i.e. for each  $x \in X$  the set*

$$H(x) = \{h(x) : h \in H\}$$

*is relatively compact in  $Y$ .*

- (2)  *$H$  is sequentially weak-norm equicontinuous.*

Note that if  $X$  does not contain a copy of  $l^1$ , then by applying the Rosenthal-Dor Theorem (see [11] and [4] or [3, Ch. IX]) every bounded sequence in  $X$  has a weakly-Cauchy subsequence and, therefore, a bounded linear operator  $h : X \rightarrow Y$  is compact if and only if it is *completely continuous*, that is, if and only if it takes weakly convergent sequences in  $X$  to convergent ones in  $Y$ . This tells us that condition (2) applied to a singleton characterizes compact operators when the space  $X$  does not have a copy of  $l^1$  [7, 17.7].

Our notation is standard:  $L(X, Y)$  is the Banach space of bounded linear operators from the Banach space  $X$  to the Banach space  $Y$ , endowed with the topology of the uniform convergence on the unit ball  $B_X$  of  $X$ ,  $K(X, Y)$  is its closed subspace

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of compact operators,  $X^*$  is the dual of  $X$  and  $\sigma(X, X^*)$  denotes the weak topology on  $X$ .

For the sake of completeness we consider the definition of a precompact set in the setting of uniform spaces. A Hausdorff uniform space  $R$  is said to be *precompact* if its completion is compact (see [2, T.G. II. 29] or [8, p. 36]). Equivalently,  $R$  is *totally bounded*, that is, for every *vicinity*  $N$  of  $R$  there exists a covering of  $R$  by finitely many sets which are small of order  $N$ . Within the framework of topological vector spaces this reads as follows: for a subset  $R$  of a topological vector space  $E$  the following are equivalent: (a)  $R$  is precompact, (b)  $R$  is relatively compact in the completion of  $E$  and (c) for each 0-neighbourhood  $V$  in  $E$  there exists a finite subset  $M$  such that  $R \subset M + V$  [7, 3.5.1]. Our proof relies on the Rosenthal-Dor Theorem and on the following two results: a version of Ascoli's classical theorem (Theorem A) and a characterization of relative compact sets of compact operators between Banach spaces due to Palmer [10] and Anselone [1] (Theorem B).

**Theorem A** ([2, Th.2 on page T.G. X.17]). *Consider two uniform spaces  $X$  and  $Y$ , a cover  $\mathcal{G}$  of  $X$  formed by precompact subsets and a set  $H$  of functions from  $X$  to  $Y$  such that the restriction of each  $h \in H$  to each  $A \in \mathcal{G}$  is uniformly continuous. Then  $H$  is precompact in the topology (uniformity) of uniform convergence on members of  $\mathcal{G}$  if and only if it verifies the following two conditions:*

- (i)  $H$  is pointwisely precompact, i.e.  $H(x) := \{h(x) : h \in H\}$  is precompact in  $Y$ , for each  $x \in X$ .
- (ii) For each  $A \in \mathcal{G}$  the set  $H_A$  of restrictions to  $A$  of the functions  $h \in H$  is uniformly equicontinuous.

**Theorem B** ([1] and [10]). *Let  $X$  and  $Y$  be normed linear spaces and  $H$  a set of linear operators on  $X$  into  $Y$  that satisfies*

- (i)  $H(B_X) := \{h(x) : h \in H, x \in B_X\}$  is totally bounded,
- (ii)  $H^*y^* := \{h^*(y^*) : h^* \in H^*\}$  is totally bounded for each  $y^*$  in  $Y^*$ .

*Then  $H$  is totally bounded.*

*Proof of Theorem 1.* We shall apply Ascoli's Theorem A to the uniform spaces  $(X, \sigma(X, X^*))$  and  $(Y, \|\cdot\|)$  and the family  $\mathcal{G}$  of weakly-Cauchy sequences in  $X$ . Note that  $\mathcal{G}$  is a cover of  $X$  formed by weakly-precompact sets and that each compact operator  $h : X \rightarrow Y$  is weak-to-norm continuous on bounded sets of  $X$  [5, Ths. VI. 5.6 and VI. 5.2], hence on each  $A$  in  $\mathcal{G}$ . With the topology  $\tau$  of the uniform convergence on the members of  $\mathcal{G}$  the space  $L(X, Y)$  is a locally convex space that we denote by  $L_\tau(X, Y)$  [7, 8.4].

( $\implies$ ) If  $H$  is relatively compact, then (1) follows from the continuity of the operator

$$T \in K(X, Y) \rightarrow T(x) \in Y$$

for each  $x$  in  $X$ . A standard argument leads us to condition (2) from the fact that each compact operator is completely continuous.

( $\impliedby$ ) Conversely, suppose that  $H$  verifies conditions (1) and (2). In particular condition (2) says that  $H \subset K(X, Y)$ . We shall prove that  $H$  is relatively compact in  $K(X, Y)$  in three steps:

*Step 1:*  $H$  is precompact in  $L_\tau(X, Y)$ . Applying Ascoli's Theorem, it is enough to verify that for each weakly-Cauchy sequence  $A$  in  $X$ , the set of restrictions  $H_A := \{h|_A : h \in H\}$  is uniformly equicontinuous when  $A$  is endowed with the

weak topology  $\sigma(X, X^*)$  and  $Y$  with the norm topology. It suffices to prove this for each weakly-null sequence in  $X$ . Let  $A := \{x_n : n \in \mathbb{N}\}$  be the set of terms of a weakly-null sequence in  $X$ . To prove that the set  $H_A$  of restrictions of  $H$  to  $A$  is uniformly equicontinuous we need to prove that, for each  $\varepsilon > 0$ , there exists a weak-neighbourhood  $U$  of 0 in  $X$  such that whenever  $x_n - x_m$  is in  $U$ , then  $\|h(x_n) - h(x_m)\| < \varepsilon$  for every  $h$  in  $H$ . We can suppose that  $x_n \neq x_m$  whenever  $n \neq m$  and that  $x_n \neq 0$  for every  $n$ . Condition (2) tells us that for a given  $\varepsilon > 0$  there exists  $N_1 \in \mathbb{N}$  for which  $\|h(x_n)\| < \varepsilon/2$  for every  $n > N_1$  and every  $h \in H$ . Then we have that  $\|h(x_n) - h(x_m)\| < \varepsilon$  for every  $n, m > N_1$  and  $h \in H$ . Now, let us consider an arbitrary weak 0-neighborhood  $V_1$  in  $X$ . Since  $(x_n)$  is a weakly null sequence, there exists  $N_2 \geq N_1$  in  $\mathbb{N}$  such that  $x_n \in \frac{1}{2}V_1$  for  $n > N_2$ . So, for  $n, m > N_2$  we have  $x_n - x_m \in V_1$  and  $\|h(x_n) - h(x_m)\| < \varepsilon$ . Since the elements  $x_1, \dots, x_{N_2}$  are different from each other and also different from 0 we can separate them in the weak topology, i.e. we can obtain a weak 0-neighbourhood  $V_2$  in  $X$  such that  $x_n$  and  $x_n - x_m$  are not in  $V_1$  if  $n \neq m$  are in  $\{1, 2, \dots, N_2\}$ . Now, it suffices to take  $U = V_1 \cap V_2$ .

*Step 2:  $H$  is relatively compact in  $(K(X, Y), \tau)$ .* The completion of  $L_\tau(X, Y)$  is a subspace of the space of all linear maps from  $X$  to  $Y$  [9, 39.7.(9)]. If  $R$  is an adherent point to  $(H, \tau)$  in its completion, then  $R : X \rightarrow Y$  is linear. Let us prove that  $R$  is completely continuous, hence compact as  $X$  does not have a copy of  $l^1$ . For a given weakly-null sequence  $(x_n)$  in  $X$  we can obtain for each  $\varepsilon > 0$  an element  $h_\varepsilon \in H$  such that

$$\sup \{\|(R - h_\varepsilon)(x_n)\| : n\} < \varepsilon.$$

Taking a null sequence  $(\varepsilon_m)$  and having in mind that  $(h_\varepsilon(x_n))$  is norm null in  $Y$  for each  $\varepsilon > 0$ , we obtain that  $(R(x_n))$  is norm null.

*Step 3:  $H$  is relatively compact in  $K(X, Y)$  (equivalently in  $L(X, Y)$ ).* Let us verify the conditions in Theorem B:

(i) Let  $(h_n(x_n))$  be a sequence in  $H(B_X)$ . Extracting a subsequence if necessary we can suppose that  $(x_n)$  is weakly-Cauchy. Let  $R \in K(X, Y)$  be a  $\tau$ -adherent point to  $(h_n)$  and consider a null sequence  $(\varepsilon_m)$  of positive scalars. Then for each  $k \in \mathbb{N}$  there exists  $m_k \in \mathbb{N}$  such that

$$\sup \{\|(R - h_{m_k})(x_n)\| : n\} < \varepsilon_k.$$

Since  $(R(x_n))$  is norm convergent in  $Y$ , we have that  $(h_{m_k}(x_{m_k}))$  is also norm convergent in  $Y$ .

(ii) To prove that  $H^*y^* := \{h^*(y^*) : h^* \in H^*\}$  is relatively compact in  $X^*$  let us consider the set of index  $I := H^*y^*$  and the operator

$$S : x \in X \rightarrow S(x) := (\langle x, x^* \rangle)_{x^* \in I} \in l_I^\infty.$$

Because of condition (b)  $S$  is a completely continuous, hence compact, operator. Therefore its adjoint  $S^*$  is also compact. For each  $z^* \in I$  denote by  $e_{z^*}$  the continuous linear form

$$(\alpha_{x^*}) \in l_I^\infty \rightarrow \alpha_{z^*} \in \mathbb{C} \text{ or } \mathbb{R};$$

then  $\{e_{z^*} : z^* \in I\}$  lies in the unit ball of the dual  $(l_I^\infty)^*$  and  $S^*(e_{z^*}) = z^*$ . Therefore

$$\{S^*(e_{z^*}) : z^* \in I\} = I = H^*y^*$$

is relatively compact in  $X^*$ . □

**Corollary 2.** *Let  $X, Y$  be two Banach spaces and  $H \subset L(X, Y)$ . Then  $H$  is a relatively compact subset of  $K(X, Y)$  in the uniform topology of operators if and only if it verifies the following three conditions:*

- (1)  *$H$  is pointwisely relatively compact.*
- (2)  *$H$  is sequentially weak-norm equicontinuous.*
- (3) *The restriction of the elements of  $H$  to each copy  $Z$  of  $l^1$  in  $X$  is relatively compact in  $K(Z, Y)$ .*

*Proof.* Only the if part needs proof. The conditions (2) and (3) guarantee that each  $h \in H$  is a compact operator since by the Rosenthal-Dor Theorem ([11] and [4]), each bounded sequence  $(x_n)$  in  $X$  has a subsequence, say  $(z_n)$ , which is either weakly-Cauchy, and then  $(h(z_n))$  is norm convergent by applying (2), or  $(z_n)$  is equivalent to the canonical basis of  $l^1$ , and then  $(h(z_n))$  is relatively norm-compact by applying (3). On the other hand, if  $H$  is not relatively compact in  $K(X, Y)$ , equivalently if  $H$  is not precompact, then there exist  $\varepsilon > 0$  and a sequence  $(h_n)$  in  $H$  such that  $\|h_n - h_m\| > \varepsilon$  for  $n \neq m$ . Let us consider a sequence  $(x_{n,m})$  in  $B_X$  such that for  $n \neq m$  we have

$$(*) \quad \|(h_n - h_m)(x_{n,m})\| > \varepsilon.$$

Applying the Rosenthal-Dor Theorem  $(x_{n,m})$  has some subsequence  $(z_k)$  which either is weakly-Cauchy, and inequality (\*) contradicts (2), or  $(z_k)$  is equivalent to the canonical basis of  $l^1$ , and then (\*) contradicts (3).  $\square$

*Remark 3.* Galaz-Fontes's proof is essentially different from ours in the sense that it relies on the fact that if  $X$  is a reflexive and separable Banach space, then  $B_X$  is metrizable and compact for the weak topology, something that we do not need in our proof. Of course both proofs coincide in the fact that they use Ascoli's Theorem.

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